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SHEAF PROPERTIES AND COHOMOLOGY IN STANDARD H-CONES OF FUNCTIONS

Abstract. In the framework of the theory of H-cones of functions on a set X, for every natural open subset U of X we consider the cones $\mathcal{H}_o(U)$, $\mathcal{H}(U)$, $\mathcal{P}(U)$ of the elements of "harmonic type" and of "potential type" of $\mathcal{S}(U)$. Assuming a local property for the initial cone $\mathcal{S}$ we investigate the sheaf properties of the mappings

$$U \to \mathcal{H}_o(U), \quad U \to \mathcal{H}(U), \quad U \to \mathcal{P}(U)$$

We prove that the vector space $\mathcal{R}(U)$ of the elements which are locally differences of super-harmonic functions is an algebra. This allows us to find partition of unity in the sheaves $U \to \mathcal{R}(U)$ and $U \to \tilde{\mathcal{P}}(U)$ ($\tilde{\mathcal{P}}$ beeing the sheaf associated to the presheaf $U \to \mathcal{P}(U) - \mathcal{P}(U)$). Finally we compute the cohomology groups for the sheaf of functions which are locally differences of harmonic functions. We constat the pleasant fact that some results from potential theory associated with a harmonic space (see [7],[8] [9]) can be generalised in our framework.

For the general theory of standard H-cones of functions on a set we refer to [2] and for the basic notions of sheaf theory which are used to [1], [6].

I want to express my gratitude to Prof. N.Boboc for his valuable suggestions during the achievement of this work.

1. PRELIMINARIES

Let $(\mathcal{S}, X)$ be a standard H-cone of functions on a set X. On X we consider the natural topology (i.e. the coarsest topology on X which makes continuous the universally continuous elements of $\mathcal{S}$). All over in this paper "open" set or "closed" set means with respect to the natural topology.

We suppose that <u>X is semisaturated</u> (i.e. for every $\mu\in\mathcal{S}^*$, the dual of $\mathcal{S}$, $\mu\leq\nu$, ν H-measure on X it follows that μ is a H-measure on X) and that <u>$\mathcal{S}$ satisfies the natural sheaf property</u> - (i.e. $U\to\mathcal{S}(U)$, U open subset of X, is a sheaf).

$\mathcal{S}(U)$ is the localisation of $\mathcal{S}$ to U i.e.

$\mathcal{S}(U)=\{f:U\to\overline{R}_+ \mid f<\infty$ on a dense subset of U and there exist $s_n\in\mathcal{S}, s_n<\infty$ such that $(s_n-B^{X\setminus U}s_n)|_U\nearrow f\}$. $B^{X\setminus U}$ is the balayage on the set $X\setminus U$ in the cone $\mathcal{S}$. It is known that $\mathcal{S}(U)$ is a standard H-cone of functions and that U is semisaturated with respect to $\mathcal{S}(U)$ (see [3]). It is also known that the natural sheaf property implies that the D_o axiom is valid in $(\mathcal{S}, X)$ (i.e. for every G_1, G_2 open subsets of X such that $G_1\cup G_2=X$ it follows $B^{G_1}B^{G_2}=B^{G_2}B^{G_1}$).

For an open set $U\subset X$, $s\in\mathcal{S}(U)$, $A\subset U$ we denote ${}^UB^A$ the balayage on A in the standard H-cone $\mathcal{S}(U)$ i.e. ${}^UB^A_s=\wedge\{t\in\mathcal{S}(U)\mid t\geq s$ on $A\}$ ("$\wedge$" mean the infimum in the cone $\mathcal{S}(U)$)

$\mathcal{V}_x$ = the set of neighbourhoods of x.

$$\mathrm{carr}_U s=\{x\in U\mid \forall\, V\in\mathcal{V}_x,\ V\subset U,\ {}^UB^{U\setminus V}s\neq s\}.$$

We'll use the following important remark: since $\mathcal{S}$ satisfies the D_o axiom we have: ${}^UB^{U\setminus V}(s|_U)=B^{X\setminus V}s|_U$ on V for all $s\in\mathcal{S}$ (see [4]) and hence:

$$\mathrm{carr}_U s=\{x\in U\mid \forall\, V\in\mathcal{V}_x,\ V\subset U,\ B^{X\setminus V}s\neq s\}$$

Denoting by "$\preceq$" the specific order in $\mathcal{S}(U)$ we define the following convex subcones of $S(U)$:

$H_o(U): =\{s\varepsilon S(U) | \mathrm{carr}_U s = \emptyset\}$

$P(U):=\{s\varepsilon S(U) | h\preccurlyeq s,\ h\varepsilon S(U), \mathrm{carr}_U h=\emptyset \Rightarrow h=0\}$

$H(U):=\{s\varepsilon S(U) | p\varepsilon P(U),\ p\preccurlyeq s \Rightarrow p=0\}$.

It is easy to see that $H(U)$ coincides with the band generated by $H_o(U)$ i.e.

$H(U)=\{\sum_n h_n | h_n \varepsilon H_o(U)\}$.

For $V\subset U\subset X$, V,U open subsets of X we define

$r_{UV}: H_o(U)\to H_o(V);\ r_{UV}(h)=h|_V$

$\rho_{UV}: P(U)\to P(V);\ \rho_{UV}(p)=p|_V - \curlyvee\{h\varepsilon H(H) | h \underset{S(V)}{\preccurlyeq} p|_V\}$

("$\curlyvee$" means the specific supremum in the cone $S(V)$).

<u>Remark</u>. $\rho_{UV}(p)$ means in fact the potential part of the canonical decomposition of $p|_V$ into the harmonic part and the potential part.

We'll prove that ρ_{UV} is well defined. Let $h\varepsilon H_o(U)$ and $x\varepsilon V$. Since $\mathrm{carr}_U h=\emptyset$, there exist $W\varepsilon V_x$ such that $x\varepsilon W\subset\overline{W}\subset V$ and ${}^U B^{U\setminus W} h=h$. Since S verifies the D_o axiom:

${}^V B^{V\setminus W}(h|_V)={}^U B^{U\setminus W}(h)|_V=h|_V$ hence $\mathrm{carr}_V(h|_V)=\emptyset$

i.e. $h|_V\varepsilon H_o(V)$.

Now r_{UV} can be extended in an obvious manner to $H(U)$ and this extension will be denoted again by r_{UV}.

2. SHEAF PROPERTIES

Proposition 1. i) $(\mathcal{P}(U), \rho_{UV})_{V\subset U}$ is a presheaf of convex cones.

ii) $(\mathcal{H}_o(U), r_{UV})_{V\subset U}$ is a sheaf of convex cones.

Proof. i) Obviously $\rho_{UU}=\mathrm{Id}$.

Let $W\subset V\subset U$ open subsets of X. Have to show that: $\rho_{VW}\circ\rho_{UV}=\rho_{UW}$.

Let $p\varepsilon\mathcal{P}(U)$. Denote $\rho_{UV}(p)=p_1$. Then on V: $p=p_1+h_1$ where $h_1\varepsilon\mathcal{H}(V)$.

Denote: $\rho_{VW}(p_1)=p_2$. Then $p_1=p_2+h_2$ on W with $h_2\varepsilon\mathcal{H}(W)$

$\rho_{UW}(p)=p_3$. Then $p=p_3+h_3$ on W with $h_3\varepsilon\mathcal{H}(W)$.

We obtain $p=h_1+h_2+p_2=h_3+p_3$ on W and by the unicity of the decomposition it follows $p_2=p_3$ i.e.

$$\rho_{UW}(p)=\rho_{VW}(p_1)=\rho_{VW}\rho_{UV}(p)$$

ii) It is clear that $U\to\mathcal{H}_o(U)$ is a presheaf. Let $U=\bigcup_{i\varepsilon I} U_i$, $U_i\subset X$ open subsets and $h_i\varepsilon\mathcal{H}_o(U_i)$ such that

$h_i|_{U_i\cap U_j}=h_j|_{U_i\cap U_j}$ for every i,j such that $U_i\cap U_j\neq\emptyset$. Since $U\to\mathcal{S}(U)$ is a sheaf, there exist $h\varepsilon\mathcal{S}(U)$ such that $h|_{U_i}=h_i$ for every $i\varepsilon I$.

For every $i\varepsilon I$, let $x\varepsilon U_i$. There exist $W\varepsilon\mathcal{V}_x$ such that $x\varepsilon W\subset\overline{W}\subset U_i$ and ${}^{U_i}B^{U_i\setminus W}h_i=h_i$ on U_i. Since ${}^{U_i}B^{U_i\setminus W}h_i= {}^{U}B^{U\setminus W}(h)|_{U_i}$ (by axiom D_o) it follows ${}^{U}B^{U\setminus W}(h)=h_i$ on U_i hence ${}^{U}B^{U\setminus W}(h)=h$ on U. The last equality implies $\mathrm{carr}_U h\subset U\setminus U_i$ $\forall$ $i\varepsilon I$, hence $\mathrm{carr}_U h=\emptyset$ i.e. $h\varepsilon\mathcal{H}_o(U)$.

Denote $\mathcal{P}_c(U)=\{p\varepsilon\mathcal{P}(U) \mid p \text{ continuous}\}$

$\mathcal{H}_c(U)=\{h\varepsilon\mathcal{H}(U) \mid h \text{ continouous}\}$.

Now we intend to find an other description of the linkage mappings ρ_{UV}. For that we need the specific multiplication in $\mathcal{S}(U)$ and we briefly recall this operation.

For $f\varepsilon\mathcal{B}_b^+(U)$ (measurable bounded functions on U) and $p\varepsilon\mathcal{P}(U)$ we define $f\overset{*}{_U}p$ the specific multiplication of p and f in $\mathcal{S}(U)$. (By theorem 3.4.6 of [2] there exist a kernel $V_{P,U}:\mathcal{B}_b^+(U)\to$ $\to\mathcal{P}(U)$ uniquely determined by the properties $V_{P,U}(1)=p$ and $carr_U Vf\subset\overline{\{f>0\}}$ for every $f\varepsilon\mathcal{B}_b^+(U)$. We define: $f\overset{*}{_U}p:=V_{P,U}(f)$).

<u>Lemma 2.</u> Let $g\varepsilon\mathcal{B}_b^+(U)$, $V\subset U$, V,U open subsets of X; g=0 on V, $p\varepsilon\mathcal{P}(U)$. Then $\rho_{UV}(g\overset{*}{_U}p)=0$.

<u>Proof.</u> Let $x\varepsilon V$ and $W\varepsilon\mathcal{V}_x$ such that $x\varepsilon W\subset\overline{W}\subset V$. By axiom D_o:

$${}^V B^{V\setminus W}(g\overset{*}{_U}p|_V)={}^U B^{U\setminus W}(g\overset{*}{_U}p)=g\overset{*}{_U}p \text{ on } V.$$

(the last equality holds since: $carr_U(g\overset{*}{_U}p)\subset supp\, g\subset U\setminus V\subset U\setminus\overline{W}$ hence: ${}^U B^{U\setminus W}(g\overset{*}{_U}p)\geq{}^U B^{U\setminus\overline{W}}(g\overset{*}{_U}p)=g\overset{*}{_U}p$. Hence: $carr_V((g\overset{*}{_U}p|_V)=\emptyset$ i.e. $\rho_{UV}(g\overset{*}{_U}p)=0$.

<u>Proposition 3.</u> $\rho_{UV}(p)=(p-{}^U B^{U\setminus V})|_V$ for every $p\varepsilon\mathcal{P}_c(U)$, $V\subset U$, V,U open subsets of X.

<u>Proof.</u> We remark that $carr_V({}^U B^{U\setminus V}p)=\emptyset$. Indeed: let $x\varepsilon V$ and $W\varepsilon\mathcal{V}_x$ such that $x\varepsilon W\subset\overline{W}\subset V$. It follows: ${}^U B^{U\setminus W}({}^U B^{U\setminus V}p)\geq$ $\geq{}^U B^{U\setminus\overline{W}}({}^U B^{U\setminus V}p)={}^U B^{U\setminus V}p$ on U (since $U\setminus\overline{W}$ is open, $U\setminus\overline{W}\supset U\setminus V$ and $\mathcal{S}$ satisfies axiom D_o imply ${}^U B^{U\setminus\overline{W}}({}^U B^{U\setminus V})={}^U B^{U\setminus V}$) i.e. $carr_V({}^U B^{U\setminus V}p)=$ $=\emptyset$. We obtain $\rho_{UP}p\preccurlyeq p-{}^U B^{U\setminus V}p$ on V.

For the converse inequality it is enough to show that $(p-{}^{U}B^{U\setminus V}p)|_{V}\in P(V)$.

Let $q\in S(V)$, $q\preccurlyeq p-{}^{U}B^{U\setminus V}p$ on V with $carr_{V}q=\emptyset$. Then $q=p'-{}^{U}B^{U\setminus V}p$ with $p'\preccurlyeq p$ (see [3]). Since: $carr_{V}q=\emptyset$, for every $x\in V$ there exist $W\in\mathcal{V}_{x}$ such that $x\in W\subset\overline{W}\subset V$ and:

$$ {}^{V}B^{V\setminus W}(p'-{}^{U}B^{U\setminus V}p)=p'-{}^{U}B^{U\setminus V}p \text{ on } V. $$

$$ \text{Since: } {}^{V}B^{V\setminus W}\left({}^{U}B^{U\setminus V}p\right)\geq {}^{V}B^{V\setminus \overline{W}}\left({}^{U}B^{U\setminus V}p\right)={}^{U}B^{U\setminus V}p $$

(because $V\setminus\overline{W}\supset U\setminus V$ and S satisfies axiom D_0) we get: ${}^{U}B^{U\setminus W}p'-{}^{U}B^{U\setminus V}p=p'-{}^{U}B^{U\setminus V}p$ on V hence: ${}^{U}B^{U\setminus W}p'=p'$ on V. Then: ${}^{U}B^{U\setminus W}p'=p'$ on U i.e. $carr_{U}(p')\subset U\setminus V$. Since $p'\preccurlyeq p$, $p\in P_{c}(U)$, p' is continuous hence there exist a decreasing sequence of open subsets of U, $(\Gamma_{n})_{n}$ such that $\Gamma_{n}\supset U\setminus V$ and $\bigwedge_{n}{}^{U}B^{\Gamma_{n}}p'={}^{U}B^{U\setminus V}p'$. Since $\Gamma_{n}\supset U\setminus V\supset carr_{U}(p')$ it follows: ${}^{U}B^{\Gamma_{n}}p'=p'$ for every $n\in N$. It follows: ${}^{U}B^{U\setminus V}p'=p'$ i.e. $q=0$ and hence: $(p-{}^{U}B^{U\setminus V}p)|_{V}\in P(V)$ and we are done. This description of the mappings ρ_{UV} will be very usefull in the sequel.

<u>Exemple.</u> Consider S the standard H cone associated to the heat equation on $X=R^{n}\times R$, $n\geq 2$. Let $\Omega=\sum_{i=1}^{n}\frac{\partial}{\partial x_i^2}-\frac{\partial}{\partial t}$ be the heat operator.

Then: $H_{0}(U)=\{s\in S(U)|\,B^{X\setminus V}s=s$ for every V open, respectively compact subset of U, $\overline{V}\subset U\}=\{h|\Omega h=0\}$.

Since the Doob property holds in this case $H(U)=H_{0}(U)$. Formula (*) $B^{X\setminus U}p|_{U}=\curlyvee\{h\in H(U)|h\preccurlyeq p$ on $U\}$ for every open subset U of X and every $p\in P_{c}(X)$ finite can be checked using the Perron-Wienner-Brelot method for the solution of the Dirichlet problem (see the results from [5]).

Proposition 4. $(\mathcal{H}(U), r_{UV})_{V\subset U}$ is a sheaf of convex cones.

Proof. It is obvious that it is a presheaf.

Let $U=\bigcup_{i\in I} U_i$, $h_i \in \mathcal{H}(U_i)$ such that $h_i|_{U_i\cap U_j} = h_j|_{U_i\cap U_j}$ for $U_i\cap U_j\neq\emptyset$.

Since $\mathcal{S}$ has the natural sheaf property, there exist $h\in\mathcal{S}(U)$ such that $h_{|U_i}=h_i$.

Let $q\in\mathcal{P}(U)$, $q\lessapprox h$ on U. We have to show that $q=0$. We have: $q_{|U_i}\preccurlyeq h_i$ on U_i. But $h_i=\sum_{n\in N} h_i^n$ on U_i with $h_i^n\in\mathcal{S}(U_i)$, $\mathrm{carr}_{U_i}(h_i^n)=\emptyset$. Let $k\in N$ and denote $t_k=\sum_{n=1}^{k} h_i^n$, $u_k=\sum_{n>k} h_i^n$ hence $q_{|U_i}\preccurlyeq t_k+u_k$ on U_i. By the Riesz decomposition theorem $q_{|U_i}=q_i'+q_i''$ on U_i with $q_i'\preccurlyeq t_k$ on U_i, $q_i''\preccurlyeq u_k$ on U_i, $\mathrm{carr}_{U_i}(q_i')\subseteq \bigcup_{i=1}^{k}\mathrm{carr}_{U_i}(h_i^n)=\emptyset$ i.e. $q_i'\in \mathcal{H}_0(U_i)\subset\mathcal{H}(U_i)$. But $q_{|U_i}=h_i'+\rho_{UU_i}(q)$ with $h_i'\in\mathcal{H}(U_i)$. We obtain: $h_i'\succcurlyeq q_i'$ on U_i hence $\rho_{UU_i}(q)\preccurlyeq q_i''\preccurlyeq u_k$ on U_i. Since: $\lim_{k\to\infty} u_k=0$ it follows $\rho_{UU_i}(q)=0$ i.e. $q_{|U_i}={}^{U}B^{U\setminus U_i}(q)|_{U_i}$. Let $x\in U_i$, there exist $W\in\mathcal{V}_x$ such that $x\in W\subset\overline{W}\subset U_i$, then: ${}^{U}B^{U\setminus W}q|_{U_i}\geq {}^{U}B^{U\setminus U_i}q|_{U_i}= q_{|U_i}$ i.e. ${}^{U}B^{U\setminus W}q=q$ on U_i. It follows: ${}^{U}B^{U\setminus W}q=q$ on U, $\mathrm{carr}_U q\subset U\setminus U_i$, $i\in I$; hence $\mathrm{carr}_U q=0$. Since $q\in\mathcal{P}(U)$ we obtain $q=0$.

Example: For the standard H cone associated to the heat equation: $U\to\mathcal{H}(U)=\{h\in\mathcal{S}(U)\,|\,\Omega h=0\}$ is obviously a sheaf.

Proposition 5. $\rho_{UV}(f^*_U p)=f|_{V}{}^{*}_{V}\rho_{UV}(p)$ for every $p\in\mathcal{P}_c(U)$ and $f\in\mathcal{B}_b^*(U)$

i.e. the following diagram is commutative

$$\begin{array}{ccc} \mathcal{P}_c(U) & \xrightarrow{\alpha_U} & \mathcal{P}_c(U) \\ {\scriptstyle \rho_{UV}}\downarrow & & \downarrow{\scriptstyle \rho_{UV}} \\ \mathcal{P}_c(V) & \xrightarrow{\alpha_V} & \mathcal{P}_c(V) \end{array}$$

where $\alpha_U(p)=f^*_U p$

Proof. Since $\rho_{UV}(f\overset{*}{U}p)=\rho_{UV}(f|_{V}\overset{*}{U}p)+\rho_{UV}(f|_{U\setminus V}\overset{*}{U}p)$
$=\rho_{UV}(f|_{V}\overset{*}{U}p)=\rho_{UV}(f|_{V}\overset{*}{U}(1\overset{*}{V}p))$ (we have used Lemma 2) and
$\rho_{UV}(p)=\rho_{UV}(1_V\overset{*}{U}p)+\rho_{UV}(1_{U\setminus V}\overset{*}{U}p)=\rho_{UV}(1_V\overset{*}{U}p)$ in the formula we have to prove we may suppose that supp $f\subset V$ and $carr_U\, p\subset V$.

By proposition 3 we have to prove that:

$$(f\overset{*}{U}p-{}^{U}B^{U\setminus V}(f\overset{*}{U}p))|_V=f|_V\overset{*}{V}(p-{}^{U}B^{U\setminus V}p)|_V$$

Since: $1_{U\setminus V}\overset{*}{U}p\in\mathcal{P}(U)$ and $carr_U(1_{U\setminus V}\overset{*}{U}p)\subset(U\setminus V)\cap V=\emptyset$ it results $1_{U\setminus V}\overset{*}{U}p = 0$.

For $f\in\mathcal{B}_b^+(V)$ let $\tilde{f}(x)=\begin{cases} f(x), & x\in V\\ 0\ , & x\in U\setminus V\end{cases}$. It is enough to prove that $\tilde{f}\to(\tilde{f}\overset{*}{U}p-{}^{U}B^{U\setminus V}(\tilde{f}\overset{*}{U}p))|_V$ has the properties of the specific multiplication in $\mathcal{S}(V)$ which corresponds to the potential $(p-{}^{U}B^{U\setminus V}p)|_V$.

For $f=1$; $\tilde{f}=1_V$ hence:

$$(1_V\overset{*}{U}p-{}^{U}B^{U\setminus V}(1_V\overset{*}{U}p))|_V=(1_U\overset{*}{U}p-{}^{U}B^{U\setminus V}(1_U\overset{*}{U}p))|_V=(p-{}^{U}B^{U\setminus V}p)|_V;$$

$$carr_V(\tilde{f}\overset{*}{U}p-{}^{U}B^{U\setminus V}(\tilde{f}\overset{*}{U}p))|_V\subset carr_V(\tilde{f}\overset{*}{U}p)\subset supp_V\tilde{f}$$

and we are done.

3. THE ALGEBRA $\mathcal{R}(U)$

Denote $\mathcal{S}_{c,b}(U)=\{s\in\mathcal{S}(U)\,|\,s$ continuous, bounded$\}$.
$[\mathcal{S}(U)]:=\mathcal{S}(U)-\mathcal{S}(U)$ and:

$\mathcal{R}(U)=\{f:U\to\mathbb{R}$ continuous $|$ for every $x\in U$ there exist $V\in\mathcal{V}_x$, $V\subset U$ such that $f_{|V}\in[\mathcal{S}_{c,b}(V)]\}$. If we define for $V\subset U$ open subsets of X:

$$s_{UV}:\mathcal{R}(U)\to\mathcal{R}(V) \qquad s_{UV}(f)=f|_V$$

it is obvious that $(\mathcal{R}(U),\ s_{UV})$ is a sheaf of vector spaces.

Now our aim is to show that (with the usual multiplication of functions) $\mathcal{R}(U)$ is an algebra such that every strictly positive element of $\mathcal{R}(U)$ has its inverse in $\mathcal{R}(U)$.

We shall use the following results of [4]. The hypothesis of them are obviously fulfilled in our context.

Proposition 6

$(\mathcal{S},\ X)$ has the natural sheaf property $\Leftrightarrow$ $\mathcal{S}$ has the axiom D_o and for every open subset U of X there exist $p\in\mathcal{P}(U)$, $p>0$.

Proposition 7. If X is semi saturated and exist $p\in\mathcal{P}(X)$, $p>0$; $f:X\to R_+$ lower semi-continuous. Then $f\in\mathcal{S}(U)\Leftrightarrow$ for every $x\in U$, for every open neighbourhood D of x (with respect to the natural topology on U denoted $\tau_o(U)$) there exist a neighbourhood W in the fine topology, of x such that $W\subset D$ and:

$$\varepsilon_x^{U\setminus W,U}(f)=\varepsilon_x^{X\setminus W}(f)\leq f(x).$$

Denote $\mathcal{V}_x^{fine}=\{V \mid V$ is a neighbourhood of x in the fine topology on $X\}$.

Remarks: i) Here $\varepsilon_x^{U\setminus W,U}$ is the balayed of the Dirac measure ε_x on the set $U\setminus W$ in the cone $\mathcal{S}(U)$ and $\varepsilon_x^{U\setminus W,U}=\varepsilon_x^{X\setminus W}$ since the axiom D_o is satisfied.

ii) By the caracterisation of Proposition 7 we can call the functions in $\mathcal{S}(U)$ "superharmonic functions" on U.

$\widetilde{\mathcal{S}}(U)=\{f:U\to R_+$ upper semi-continuous$\mid \forall\ x\in U, \forall\ D\in\mathcal{V}_{\tau_o|_U}(x)$, there

there exist $W\in\mathcal{V}_x^{fine}$, $W\subset D$ such that $\varepsilon_x^{X\setminus W}(f)\geq f(x)\}$.

The following properties are easy to check:

1) $\widetilde{S}(U)$ is a convex cone of functions on U.

2) Denoting $\widetilde{r}_{UV}:S(U)\to S(V)$, $V\subset U$, $\widetilde{r}_{UV}(f)=f_{|V}$ then $(\widetilde{S}(U),\widetilde{r}_{UV})_{V\subset U}$ is a presheaf of convex cones.

3) If $f\in\widetilde{S}(U)$, $0<f\leq M$ then $M-f\in S(U)$ and hence $f\in[S(U)]$ (since $f=M-(M-f)$).

4) If $(f_n)_n\in\widetilde{S}(U)$ is an increasing sequence, $f_n\leq M$ for every $n\in N$ then $f\overset{\det}{=}\check{\sup_n} f_n\in[S(U)]$ ($\check{\sup_n} f_n$ denotes the upper semi-continuous regularisation of $\sup_n f_n$).

Lemma 8. If $u\in S(V)$, $v\in S(V)$, $u>0$, $v\geq 0$ then for every $n\geq 1$ $v^n u^{1-n}\in S(V)$.

Proof. Obviously $v^n u^{1-n}$ is upper semi-continuous. Let $x\in V$ and $D\in\mathcal{V}_{\tau_o|V}(x)$. Since $v\in S(V)$ there exist $W\in\mathcal{V}_x^{fine}$, $W\subset D$ such that $\varepsilon_x^{X\setminus W}(v)\geq u(x)$.

Using Hölder inequality we obtain:

$$v^n(x)\leq(\varepsilon_x^{X\setminus W}(v))^n=\left(\int \frac{v}{u^{\frac{n-1}{n}}}\, u^{\frac{n-1}{n}}\, d\varepsilon_x^{X\setminus W}\right)^n\leq$$

$$\leq\left(\int \frac{v^n}{u^{n-1}}d\varepsilon_x^{X\setminus W}\right)\left(\int u\, d\varepsilon_x^{X\setminus W}\right)^{n-1}\leq\left(\int \frac{v^n}{u^{n-1}}d\varepsilon_x^{X\setminus W}\right)u^{n-1}(x)$$

hence $\frac{v^n}{u^{n-1}}(x)\leq\int \frac{v^n}{u^{n-1}}d\varepsilon_x^{X\setminus W}$. By proposition 7 we obtain $v^n u^{1-n}\in S(V)$.

Proposition 9. If $u\in S(V)$, $v\in S(V)$ are bounded and there exist $0<\alpha<1$ such that $0\leq v<\alpha u$ then $\frac{u^2}{u+v}\in[S(V)]$.

Proof. $\frac{u^2}{u+v}=u(1+\frac{v}{u})^{-1}=u\sum_{n\geq 1}(-1)^n(\frac{v}{u})^n=$

$= \sum_{m\geq 1} \frac{v^{2m}}{u^{2m-1}} - \sum_{m\geq 1} \frac{v^{2m+1}}{u^{2m}} = w_1 - w_2$. Since the series are uniformely and absolutely convergent $w_1 \varepsilon S(V)$, $w_2 \varepsilon S(V)$ and

$w_1 \leq u \cdot \sum_{m\geq 1} \alpha^{2m} = \frac{u}{1-\alpha^2} \leq M_1$. Similarly $w_2 \leq M_2$.

By the above property 3 we obtain that $w_1 \varepsilon [S(V)]$, $w_2 \varepsilon [S(V)]$ hence $w_1 - w_2 \varepsilon [S(V)]$.

From now on we make the following additional assumption.

(*) | for every $x \varepsilon X$ there exist $h_o \varepsilon \tilde{S}_c(X)$ bounded such that $h_o(x) > 0$. |

Remark. The hypothesis (*) is equivalent with:

(**) for every $x \varepsilon X$ there exist $h \varepsilon S_c(X) \cap \tilde{S}_c(X)$ bounded such that $h(x) > 0$.

Indeed if we take h_o like in (*) then we'll show that Rh_o (the reduite of h_o in S) satisfies (**).

Let $x \varepsilon X, D \varepsilon \mathcal{V}_{\tau_o|_V}(x)$, $W \varepsilon \mathcal{V}^{fine}(x)$, $W \subset D$ such that $\varepsilon_x^{X \setminus W}(h_o) \geq$ $\geq h_o(x)$. Take $t \varepsilon S$, $t \geq Rh_o$ on $X \setminus W$. Then for every $y \varepsilon W$ there exist $W' \varepsilon \mathcal{V}^{fine}(y)$, $W' \subset W$ such that $t(y) \geq \varepsilon_y^{X \setminus W'}(t) \geq \varepsilon_y^{X \setminus W}(h_o) \geq h_o(y)$ hence $t \geq Rh_o$ on X it follows: $B^{X \setminus W} Rh_o(x) = Rh_o(x)$ i.e. $Rh_o \varepsilon S_c(X) \cap \tilde{S}_c(X)$.

Proposition 10. If $u \varepsilon S(V)$ then for every $f, g \varepsilon R(V)$, $\frac{f \cdot g}{u} \varepsilon R(V)$.

Proof. Let $x \varepsilon V$ and $W \varepsilon \mathcal{V}_x$ such that $x \varepsilon W \subset \overline{W} \subset V$, $f|W = s_1 - s_2$, $g|W = t_1 - t_2$ with $s_1, s_2, t_1, t_2 \varepsilon S_{c,b}(W)$. Denote $\alpha = \max(\sup_W s_1, \sup_W s_2)$, $\beta = \max(\sup_W t_1, \sup_W t_2)$. Take h_o given by hypothesis (*) and construct

$$v_i := \alpha h_o - s_i, \quad w_i := \beta h_o - t_i \quad ; \quad i = 1, 2$$

Then: $f|_W=v_2-v_1$, $g|_W=w_2-w_1$ and

$$\frac{f.g}{u}=\frac{v_1w_1+v_2w_2}{u}-\frac{v_1w_2+v_2w_1}{u} \text{ on } W.$$

But $v_i\varepsilon\widetilde{S}_{c,b}(W)$, $w_i\varepsilon\widetilde{S}_{c,b}(W)$, $v_i\geq 0$, $w_i\geq 0$.

Applying Lemma 8 (with n=2) it follows:

$$\frac{v_i^2}{u}, \frac{w_j^2}{u}, \frac{(v_i+w_j)^2}{u}\ \varepsilon\ \widetilde{S}_{c,b}(W)\subset[S_{c,b}(W)],\quad i=1,2$$

$$\frac{v_iw_j}{u}=\frac{1}{2}\left[\frac{(v_i+w_j)^2}{u}-\frac{v_i^2}{u}-\frac{w_j^2}{u}\right]\varepsilon[S_{c,b}(W)]$$

Then: $\frac{f.g}{u}\varepsilon[S_{c,b}(W)]$ i.e. $\frac{f.g}{u}\varepsilon R(V)$.

<u>Theorem 11.</u> If condition (*) is fulfilled then

i) $f,g,h\varepsilon R(U)$, $h>0$ on U imply: $\frac{f.g}{h}\varepsilon R(U)$

ii) $R(U)$ is an algebra of functions and $f\varepsilon R(U)$, $f>0$ imply $\frac{1}{f}\varepsilon R(U)$.

<u>Proof.</u> ii) is obvious from i) since $1\varepsilon S$

i) Let $x\varepsilon U$ and $V\varepsilon \mathcal{V}_x$ such that $h|_V=u_1-u_2$ with u_1, $u_2\varepsilon S_{c,b}(V)$. Let $\lambda:=\frac{1}{3}u_1(x)+\frac{2}{3}u_2(x)$ and $v:=\lambda-u_2$, $u:=u_1-\lambda$.

We remark that $v(x)=\frac{1}{3}h(x)>0$, $u(x)=\frac{2}{3}h(x)>0$. Consider $\frac{1}{2}<\alpha<1$ such that $v(x)<\alpha u(x)$. By the continuity of v and u there exist $W\varepsilon\mathcal{V}_x$ such that $v>0$ on W, $u>0$ on W and $v<\alpha u$ on W.

But $v|_W\varepsilon\widetilde{S}(W)$, $u|_W\varepsilon S(W)$ and (proposition 10) $\frac{u^2}{u+v}\varepsilon[S_{c,b}(W)]$.

Since $u+v=u_1-u_2=h$ on W we obtain

$\frac{f.g}{h}=\frac{f.g}{u+v}=\frac{f.g}{u}\cdot\frac{u^2}{u+v}\cdot\frac{1}{u}$ on W and once again by proposition 10

$\frac{f.g}{h}\varepsilon R(W)$

Hence: $\frac{f.g}{h}\varepsilon R(U)$.

4. COHOMOLOGY FUR STANDARD H-CONES OF FUNCTIONS

Lemma 12. For every open subset U of X there exist $f\varepsilon[S(X)]$, f continuous such that $f>0$ on U and $f=0$ on U.

Proof. By Lemma 3.1.2 of [2] for every $x\varepsilon U$ there exist $p_x, q_x\varepsilon S$, continuous, bounded such that $0\leq p_x-q_x\leq 1$, $p_x(x)=$ $=q_x(x)+1$ and $p_x=q_x$ on $X\setminus U$. Let $M=\max(\sup_x p_x, \sup_x q_x)$. Replacing p_x and q_x by $\frac{p_x}{M}$ respectively $\frac{q_x}{M}$ we may suppose: $p_x\leq 1$, $q_x\leq 1$. (condition $p_x(x)=q_x(x)+1$ becomes $p_x(x)-q_x(x)>0$ but this is all we need).

Let $G_x=\{p_x-q_x>0\}$. Then $U\subset\bigcup_{x\varepsilon U}G_x$ and hence there exist $(x_n)_n\subset X$ such that $U\subset\bigcup_n G_{x_n}$.

Let $G_{x_n}=\{p_{x_n}-q_{x_n}>0\}$. Define: $f:=\sum_{n\geq 1}\frac{1}{2^n}p_{x_n}-\sum_{n\geq 1}\frac{1}{2^n}q_{x_n}$

and it is easy to see that f verifies the request of the lemma. We recall now:

Definition 13. A sheaf $\mathcal{F}$ on a paracompact space X is called "fine sheaf" it for every locally finite covering of X there exist a partition of unity in $\mathcal{F}$ subordinated to the cover.

WARNING. The expression "fine sheaf" have nothing in common with the fine topology of the standard H cone S. This terminology is used in Sheaf Theory and here it is not danger of confusion since the only topology on (S, X) we are dealing with is the natural one.

Since for S the space X is metrisable (in the natural topology) hence paracompact it makes sense to ask if the sheaf $(\mathcal{R}(U), s_{UV})_{V\subset U}$ is "fine". Fortunately the answer is affirmative.

Theorem 14. The sheaf $(\mathcal{R}(U), s_{UV})_{V\subset U}$ is "fine".

Proof. Let $(U_i)_{i\in I}$ a locally finite covering of X. Let $(V_i)_{i\in I}$ a locally finite covering of X such that $\overline{V_i}\subset U_i$. For every $i\in I$ denote by f_i the function corresponding to V_i given by Lemma 12.

Let $f := \sum_{i\in I} f_i \in \mathcal{R}$ (the sum is locally finite). Obviously f is continuous and $f>0$ on X. By Theorem 11: $g_i := \frac{f_i}{f} \in \mathcal{R}$ and obvious $\sum_{i\in I} g_i = 1$, $g_i = 0$ on $X\setminus\overline{V_i} \supset X\setminus U_i$.

Now, the partition of unity subordonated to $(U_i)_{i\in I}$ is given by:

$$x\in X, \quad \lambda_{i,x} : \mathcal{R}\to\mathcal{R}, \quad \lambda_{i,x}(f_x) :\overset{\text{def}}{=} (g_i f)_x$$

(f_x denotes the germe of f in $x\in X$).

□

Consider the presheaf $(\mathcal{P}_c(U), \rho_{UV})_{V\subset U}$. We extend ρ_{UV} in an obvious manner to $[\mathcal{P}_c(U)]$ and denote again by ρ_{UV} this extension.

Denote by $\tilde{\mathcal{P}}$ the sheaf associated to the presheaf $([\mathcal{P}_c(U)], \rho_{UV})_{V\subset U}$.

<u>Theorem 15</u>. The sheaf $\widetilde{P}$ is "fine".

<u>Proof</u>. Let $(U_i)_{i\varepsilon I}$ a locally finite covering of X.
Let $(V_i)_{i\varepsilon I}$ a locally finite covering of X such that $V_i\subset\overline{V}_i\subset U_i$.
Let $E_i=V_i\setminus\bigcup_{j<i}V_j$. Define:

$$(\lambda_{E_i})_U:P(U)\to P(U) \quad (\lambda_{E_i})_U(p):=1_{E_i\cap U}\overset{*}{U}\, p$$

and extend $(\lambda_{E_i})_U$ to $[P(U)]$.

By Proposition 5: $((\lambda_{E_i})_U)_U$ is an endomorphism of the presheaf $([P_c(U)], \rho_{UV})_{V\subset U}$.

Denoting $\widetilde{P}_x$ the fiber in $x\varepsilon X$ of $\widetilde{P}$ and $\rho_{Ux}:\widetilde{P(U)}\to\widetilde{P}_x$ the canonical map and defining

$$(\lambda_{Ei})_x:\widetilde{P}_x\to\widetilde{P}_x$$

$$(\lambda_{E_i})_x(a)=\rho_{Ux}((\lambda_{E_i})_U(p)) \text{ where } a=\rho_{Ux}(p)$$

with $p\varepsilon[P(U)]$

then λ_{E_i} is well defined, is an endomorphism of the sheaf $\widetilde{P}$ and it is readely seen that gives the partition of unity subordinated to the cover $(U_i)_{i\varepsilon I}$. □.

For U open subset of X we construct the sheaf of functions which are locally differences of continuous, bounded, harmonic functions, namely:

$$\widetilde{H}(U)=\{f:U\to R \text{ continuous} \mid \forall x\varepsilon U,\ \exists V\varepsilon V_x,\ V\subset U \text{ such that } f|_V\varepsilon[H_{c,b}(V)]\}.$$

For $V\subset U$, $r_{UV}:\widetilde{H}(U)\to\widetilde{H}(V)$, $r_{UV}(f):=f|_V$. Obviously $\widetilde{H}=\{\widetilde{H}(U), r_{UV}\}_{V\subset U}$ is a sheaf of vector spaces.

Now we are in position to construct an exact sequence of sheaves.

Proposition 16. $0 \to \tilde{H} \to R \xrightarrow{\Delta} \tilde{P} \to 0$ is an exact sequence of sheaves.

Proof. First of all we have to define Δ

For $f \in R$ and $x \in V$ there exist $V \in V_x$ such that $f_{|V} = s_1 - s_2$

Let $h_i = \gamma\{t \in H(V) \mid t \lesssim s_i|_V$ on $V\}$ $i=1,2$. Define

$$(\Delta f)_x := \rho_{Vx}((s_1 - h_1) - (s_2 - h_2))$$

Then it is easy to verify that Δ is well defined, is a surjective morphism of sheaves and that. Ker $\Delta = \tilde{H}$ hence the sequence is exact.

The above results allows us to compute the cohomology groups for H.

Proposition 17. If condition (*) is fulfilled then: $H^q(X, \tilde{H}) = 0$ for $q \geq 2$ and $H^1(X, \tilde{H}) \simeq \Gamma(X, \tilde{P}) / \Delta\Gamma(X, R)$.

Proof. We use the exact sequence of proposition 16, we write the cohomology long sequence associated to it and take into account that since R and $\tilde{P}$ are "fine sheaves", $H^q(X, R) = 0$ $q \geq 1$ and $H^q(X, \tilde{P}) = 0$, $q \geq 1$ and we are done.

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