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# Deforming varieties of k-planes of projective complete intersections

by Ciprian Borcea

We consider the variety F of k-dimensional linear projective subspaces lying on a generic projective complete intersection S. Under general assumptions involving k, the multidegree and the dimension of S, we prove that F is connected, smooth, and its local deformations come from deformations of S.

Introduction. Linear varieties lying on a projective variety have been considered in several contexts.

A classical instance, going back to Cayley [6], is that of a smooth cubic surface. There are twenty-seven lines on such a surface, and, as observed later, the incidence preserving permutations of this set of lines form a group isomorphic to the Weyl group of a root system of type $E_6$. It is also the monodromy group of the global family of smooth cubics and the Galois group of the corresponding enumerative problem (see [12]).

Similar results (involving the root system $D_{2k+3}$) hold for the k-planes contained in a smooth 2k-dimensional intersection of two quadrics ([14], [16]).

Beyond the enumerative level, and besides homogeneous-rational varieties such as Grassmannians or linear spaces lying on a smooth quadric, a first example should be the Fano surface of lines contained in a cubic threefold ([11]). The Abel-Jacobi map induces an isomorphism from the Albanese variety of the

Fano surface to the intermediate Jacobian of the cubic threefold and one has a global Torelli theorem ([7],[19]).

With planes instead of lines, but generically this time, the analogous statements hold true for cubic fivefolds ([8],[10]).

Nor should be cubic fourfolds be neglected here: their varieties of lines are irreducible symplectic projective fourfolds ([3]) which play an important role in the proof of the global Torelli theorem ([20]).

We also mention the variety of k-planes contained in a smooth (2k+1)-dimensional intersection of two quadrics: it is an Abelian variety isomorphic with the intermediate Jacobian of the given intersection of quadrics ([9],[16]).

All these varieties may be realized as zero loci of sections of certain homogeneous vector bundles over Grassmannians ([1],[18]). This circumstance makes the Schubert calculus relevant, for instance, in computing Chern numbers; it also reduces questions about connectivity, regularity, etc., as well as deformations to questions about the cohomology of homogeneous vector bundles.

Our main concern will be to set up a general framework for a calculus with weights, such that the theorem of Bott [5] become expressive in this context - a perspective we initially used in [4].

Specific computations enabled Wehler to deal with small deformations of Fano surfaces: he showed, namely, that all of them are induced by deformations of the corresponding cubic threefolds ([21]). This result is here extended to a large class (Theorem 5.3). Similarly (Theorem 4.1), we extend (and give an alternative proof for) the connectedness result of Barth and Van de Ven concerning lines on hypersurfaces ([2]).

## §1. Varieties of k-planes

We shall consider projective k-planes contained in a complete intersection $S = S_n(d)$ of dimension n and multidegree $d=(d_1,\dots,d_r)$ in the projective space $P = P_{n+r}$ over the complex field C.

Let $\mathcal{O}_P(m)$ denote the $m^{th}$ tensor power of the hyperplane line bundle on P and let S be given as the variety of zeroes $Z(s)=S$ of a section $s \in H^0(P,E)$, where $E = \bigoplus_{t=1}^{r} \mathcal{O}_P(d_t)$.

Denote by $G = G(k+1,n+r+1)$ the Grassmann variety of projective k-planes in P i.e. (k+1)-planes in $C^{n+r+1}$ and let $\Gamma \subset P\times G$ be the subvariety defined by the incidence relation $\Gamma = \{(x,\pi) \mid x \in \pi\}$, with canonical projections:

$$\begin{array}{ccc} & \Gamma & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle q} \\ P & & G \end{array}$$

p represents $\Gamma$ as a G(k,n+r)-bundle over P and q represents $\Gamma$ as a $P_k$-bundle over G. Accordingly, we have isomorphisms:

$$H^0(P,E) \xrightarrow{\sim} H^0(\Gamma, p^*E) \xrightarrow{\sim} H^0(G, q_*p^*E).$$

If $0 \to \mathcal{T} = \mathcal{T}_{k+1} \to G\times C^{n+r+1} \to Q = Q_{n+r-k} \to 0$ denotes the canonical exact sequence of vector bundles over the Grassmannian G, we have a natural identification:

$q_*p^*\mathcal{O}_P(m) = S^m(\mathcal{T}^*)$ = the $m^{th}$ symmetric tensor power of the dual tautological bundle.

Put $\mathcal{E} = q_*p^*E$ .

Let $\Phi$ be the isomorphism indicated above:

$$\Phi : H^0(P,E) \xrightarrow{\sim} H^0(G,\mathcal{E}) = \bigoplus_{t=1}^{r} H^0(G, S^{d_t}(\tau^*)) .$$

To $s \in H^0(P,E)$, defining the variety $Z(s)=S$, we thus associate $\Phi(s) \in H^0(G,\mathcal{E})$, defining the variety of zeroes $Z(\Phi(s)) = F_k(S) = F$, which consists of all k-planes contained in $S \subset P$ .

Remark 1.1. The rank of $\mathcal{E}$ is $\sum_{t=1}^{r} \binom{d_t+k}{k}$ , and we expect F to be non-empty for $\dim G - \mathrm{rk}\,\mathcal{E} \geqslant 0$ i.e. for:

$$(k+1)(n+r-k) - \sum_{t=1}^{r} \binom{d_t+k}{k} \geqslant 0 \qquad (A_0)$$

This will presently be seen to be true, provided S is not a quadric, in which case the assumption $n \geqslant 2k$ is needed. Note that, if S is neither a quadric, nor a linear space, condition $(A_0)$ already implies $n > 2k$.

## §2. Dimension and smoothness in the generic case

Let $V = H^0(P,E)$ and consider the subvariety $I \subset G\times V$ defined by: $I = \{(s,\Pi) \mid s|_{\Pi}=0\}$, with projections:

$$\begin{array}{ccc} & I & \\ {\scriptstyle\alpha}\swarrow & & \searrow{\scriptstyle\beta} \\ G & & V \end{array}$$

$\alpha$ represents I as a sub-vector-bundle of $G\times V \to G$, which shows that I is smooth, while $\beta$ is proper and the fibre over $s \in V$ is precisely $Z(\Phi(s))$.

Confirming our remark 1.1, we have:

Proposition 2.1 If $\dim G - \mathrm{rk}\,\mathcal{E} \geqslant 0$, $\beta$ is onto, provided $n \geqslant 2k$ in the case of quadrics.

Proof: If we find a k-plane $\Pi$ in S, with S smooth along $\Pi$, and such that the normal bundle $N_{\Pi/S}$ has $H^1(\Pi, N_{\Pi/S})=0$, the proposition will follow from Kodaira's criterion for stability of compact submanifolds [15].

We consider the exact sequence:

$$0 \longrightarrow N_{\Pi/S} \longrightarrow N_{\Pi/P} \longrightarrow N_{S/P}\big|_{\Pi} \longrightarrow 0 \tag{1}$$

We have: $N_{\Pi/P} = \bigoplus^{n+r-k} \mathcal{O}_{\Pi}(1)$ and $N_{S/P}\big|_{\Pi} \cong \bigoplus_{t=1}^{r} \mathcal{O}_{\Pi}(d_t)$ .

Let $\Pi$ be given by $x_{k+1}= \dots =x_{n+r}=0$ , for homogeneous coordinates $(x_0: \dots :x_{n+r})$, so that $s \in H^0(P,E)$, $s\big|_{\Pi}=0$ will be given by r homogeneous polinomials $(s_1, \dots ,s_r)$ of the form:

$$s_t = \sum_{i=k+1}^{n+r} x_i \cdot p_t^{(i)} + r_t \tag{2}$$

where:

$$p_t^{(i)} = \sum_{\mu} c_{t\mu}^{(i)} \cdot x^{\mu} \tag{3}$$

$$\mu=(\mu_0, \dots ,\mu_k), \quad x^{\mu} = x_0^{\mu_0}\dots x_k^{\mu_k}, \quad |\mu| = \mu_0+\dots+\mu_k = d_t-1$$

and every monomial in $r_t$ contains a product $x_i x_j$ with $i \geqslant j > k$ .

Since we may suppose $n \geqslant 2k$, the condition that S be smooth along $\Pi$ is satisfied for generic s. (For example, the following matrix of partial derivatives $\left(\frac{s_t}{x_i}(x)\right)_{i \geqslant k+1}$ , $x \in \Pi$ may be produced:

$$\begin{pmatrix} x_0^{d_1-1} & \dots\dots & x_k^{d_1-1} & 0 & 0 \dots\dots\dots \\ 0 & x_0^{d_2-1} & \dots\dots & x_k^{d_2-1} & 0 \dots\dots\dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 \dots 0 & x_0^{d_r-1} \dots\dots & & x_k^{d_r-1} \dots \end{pmatrix}).$$

We represent a global section of $N_{\pi/P}$ by a matrix $A = (a_{ij})_{0 \leq j \leq k < i \leq n+r}$ , so that the map $H^0(N_{\pi/P}) \xrightarrow{\sigma} H^0(N_{S/P}|_{\pi})$ induced from (1) is described by:

$$A \longrightarrow \left( \sum_{j \leq k < i} a_{ij} \cdot p_t^{(i)} \cdot x_j \right)_{1 \leq t \leq r} \in H^0\left( \bigoplus_{t=1}^{r} \mathcal{O}_{\pi}(d_t) \right) \qquad (4).$$

Looking at monomial coefficients in (4) and using (3), one obtaines that $\sigma$ is a surjection if and only if the linear system (with indeterminates $a_{ij}$):

$$\sum_{j \leq k < i} a_{ij} \cdot c^{(i)}_{t,\nu(j)} = 0 \qquad t = 1,\ldots,r \qquad (5)$$
$$\nu = (\nu_0,\ldots,\nu_k)$$
$$|\nu| = d_t$$

where: $\nu(j) = \nu - (0,\ldots,\underset{j}{1},0,\ldots,0)$ and $c^{(i)}_{t,\nu(j)} = 0$ for $\nu(j)$ improper;

has maximal rank, namely $\sum_{t=1}^{r} \binom{d_t+k}{k} = \mathrm{rk}\,\mathcal{E} = R$.

For generic s, this is actually the case. To see it, choose, for each t, an enumeration $\mu 1, \mu 2, \mu 3, \ldots$ of the multiindices $\mu$ with $|\mu| = d_t - 1$ and consider the RxR matrix whose determinant produces the monomial with the following description:

- for each t, $c^{(k+1)}_{t,\mu 1}$ occurs at the highest possible power (i.e. k+1); next, $c^{(k+1)}_{t,\mu 2}$ occurs at the highest possible power (which now depends on the chosen enumeration); and so on. The coefficient of this monomial will be 1 or $-1$, and the corresponding determinant is not the zero polinomial.

Thus, for generic s, S is smooth along $\pi$ and $H^1(\pi, N_{\pi/S}) = 0$.

Corollary 2.2. The projective k-planes contained in a generic complete intersection $S_n(d)$ of dimension n and multidegree $d=(d_1,\dots,d_r)$ in $P_{n+r}$ define a smooth subvariety $F_k(S_n(d))$ of $G(k+1,n+r+1)$ of codimension $\sum_{t=1}^{r}\binom{d_t+k}{k}$, provided that

$(k+1)(n+r-k) \geqslant \sum_{t=1}^{r}\binom{d_t+k}{k}$ and $S_n(d)$ is not a quadric, in which last case $n \geqslant 2k$ is required.

Remark 2.3. The variety of lines $F_1(S_n(3))$ of a cubic hypersurface $S_n(3) \subset P_{n+1}$ is smooth if the cubic is smooth, but in general, the smoothness of $S_n(d)$ does not imply that of $F_k(S_n(d))$. (cf. [12], [18]).

## §3. Weights

In what follows, we take $\dim G \geqslant \mathrm{rk}\,\mathcal{E}$ (and $n \geqslant 2k$ for quadrics), and assume the complete intersection $S = S_n(d)$ to be such that the codimension of $F = F_k(S)$ in $G = G(k+1,n+r+1)$ be precisely $\mathrm{rk}\,\mathcal{E}$. Generically, this is the case (Corollary 2.2).

Let $J_F$ denote the sheaf of ideals defining F on G.

The Koszul complex of (the section of $\mathcal{E} = q_* p^* E$ defining) $J_F$ gives, for any holomorphic vector bundle M on G, spectral sequences:

$$H^p(G,M \otimes \overset{q}{\wedge}\mathcal{E}^*) \Rightarrow H^{p-q}(F,M|_F)$$
$$H^p(G,M \otimes \overset{q+1}{\wedge}\mathcal{E}^*) \Rightarrow H^{p-q}(G,M \otimes J_F)\ ,\ q \geqslant 0. \qquad (6)$$

If M is a homogeneous vector bundle, we may use the theorem of Bott [5, Th. IV'] for dealing with the groups on the left. To this purpose, we use the following description of the Grassmann manifold $G(k+1,n+r+1)$:

$SL(n+r+1,C)$, which is the universal cover of $Aut(P_{n+r}) = PGL(n+r+1,C)$, has Lie algebra $sl(n+r+1,C) = \{A = (a_{ij}) \mid trA=0\}$. Take as Cartan subalgebra $h = \{A \mid a_{ij}=0 \text{ for } i\neq j\}$. This gives root spaces $L_{ij} = C.E_{ij}$ $(i\neq j)$ where $E_{ij}$ has zeroes everywhere except the $(i,j)$ entry.

The Killing form identifies the corresponding roots $\alpha_{ij}$ with $E_{ii}-E_{jj}$ $(i\neq j)$ so that the root system $A_{n+r}$ may be viewed as embedded in a euclidean space with orthonormal basis $e_i = E_{ii}$, $i=1,\ldots,n+r+1$; the roots being represented by vectors $\alpha$ orthogonal to $e_1+\ldots+e_{n+r+1}$ and of square-norm $(\alpha,\alpha)=2$ (cf. [13, p. 64]).

Put $\alpha_s=\alpha_{s+1,s}=e_{s+1}-e_s$.

$\{\alpha_s \mid s=1,\ldots,n+r\}$ gives a basis of the root system $A_{n+r}$.

If $U_{k+1}$ denotes the subgroup of $SL(n+r+1,C)$ consisting of transformations which preserve the linear space $\{x_{k+2}=\ldots=x_{n+r+1}=0\}\subset C^{n+r+1}$ with coordinates $(x_1,\ldots,x_{n+r+1})$, the Lie algebra $u_{k+1}$ of $U_{k+1}$ will contain $h$, all the negative roots $(\alpha_{ij},\ i<j)$ and all positive roots not involving $\alpha_{k+1}$ when expressed in terms of the given basis.

We have: $G(k+1,n+r+1) = SL(n+r+1,C)/U_{k+1}$, which is the description we shall use.

Let us now investigate the weights associated to various homogeneous vector bundles over $G = G(k+1,n+r+1)$.

Such a bundle is defined by a holomorphic representation $\rho : U_{k+1} \to GL(N,C)$ and the weights are taken with respect to h.

(a) Consider first the tautological bundle $\tau$ over G. It corresponds to the natural representation of $U_{k+1}$ on the invariant subspace $\{x_{k+2}=\ldots=x_{n+r+1}=0\}$.

Let $\beta_s$ denote the weight characterized by:

$$(\beta_s,\alpha_t) = 0 \text{ for } t\neq s \quad \text{and} \quad (\beta_s,\alpha_s) = \tfrac{1}{2}(\alpha_s,\alpha_s) = 1 .$$

An elementary computation then gives the weights of $\tau_{k+1}$:

$$t_1=-\beta_1, \quad t_2=\beta_1-\beta_2, \quad \ldots , \quad t_{k+1}=\beta_k-\beta_{k+1} .$$

(b) The line bundle $\det(\tau^*_{k+1})$, which gives the Plücker embedding of $G(k+1,n+r+1)$, has therefore associated weight: $\beta_{k+1}$.

(c) The tangent bundle of G: $\Theta_G$ is given by the adjoint representation of $U_{k+1}$ on $sl(n+r+1,C)/u_{k+1}$. Consequently, its weights are precisely the positive roots involving $\alpha_{k+1}$ in their expression, namely: $\alpha_{ij}$ , $i > k+1 \geqslant j$ .

(d) $\mathcal{E}^* = \bigoplus_{m=1}^{r} S^{d_m}(\tau^*_{k+1})^*$ and (a) immediately gives that its weights are of the form:

$$\sum_{i=1}^{k+1} a_i t_i = (a_2-a_1)\beta_1 + (a_3-a_2)\beta_2 + \ldots + (a_{k+1}-a_k)\beta_k - a_{k+1}\beta_{k+1}$$

with $a_i \in \mathbb{N}$ , $\sum_{i=1}^{k+1} a_i = d_m$ for some $m \leq r$.

We now draw up a table of scalar products of positive roots and various weights, which will be relevant in estimating indices of weights.

$\delta$ is half the sum of all positive roots.

$\omega = \sum_{i=1}^{k+1} a_i t_i$ , $a_i \in Z$ (motivated by (d) above and the spectral sequences (6)).

$1 \leqslant m \leqslant k$ .

| | Conditions | $\delta$ | $\omega$ | $\alpha_{n+r+1,m}$ | | $\alpha_{n+r+1,k+1}$ | |
|---|---|---|---|---|---|---|---|
| $\alpha_p$ | $p \neq m-1,\ m$ <br> $p \leqslant k$ | 1 | $a_{p+1}-a_p$ | | 0 | $p<k$ | 0 |
| | | | | | | $p=k$ | -1 |
| $\alpha_{m-1}$ | | 1 | $a_m-a_{m-1}$ | | -1 | | 0 |
| $\alpha_m$ | | 1 | $a_{m+1}-a_m$ | | 1 | $m<k$ | 0 |
| | | | | | | $m=k$ | -1 |
| $\alpha_q$ | $k+2 \leqslant q \leqslant n+r$ | 1 | 0 | $q<n+r$ | 0 | | 0 |
| | | | | $q=n+r$ | 1 | | 1 |
| $\alpha_{t,k+1}$ | $t>k+1$ | $t-k-1$ | $-a_{k+1}$ | $t<n+r+1$ | 0 | | 1 |
| | | | | $t=n+r+1$ | 1 | | 2 |
| $\alpha_{t,m}$ | $t>k+1$ | $t-m$ | $-a_m$ | $t<n+r+1$ | 1 | | 0 |
| | | | | $t=n+r+1$ | 2 | | 1 |
| $\alpha_{t,p}$ | $t>k+1>p$ <br> $p \neq m$ | $t-p$ | $-a_p$ | $t<n+r+1$ | 0 | | 0 |
| | | | | $t=n+r+1$ | 1 | | 1 |

Table 1.

We anticipate here on the type of reasoning to be used in the sequel. Given a homogeneous vector bundle over G, defined by a representation $U_{k+1} \to GL(N,C)$, we first produce a filtration with consecutive quotients corresponding to irreducible representations of $U_{k+1}$. Such an irreducible representation determines a highest weight, say $\rho$. This $\rho$ has to be one of the weights of the original representation and further satisfy $(\rho,\alpha_s) \geqslant 0$ for all $s \neq k+1$.

In our computations $\rho$ will be either of type $\omega$ or $\omega+\alpha_{n+r+1,m}$ $(m \leq k+1)$.

In order to obtain the vanishing of $H^s(G,\rho)$, it will suffice either to ascertain the singularity of the weight $\rho+\delta$ or to prove: $s < \mathrm{index}(\rho+\delta)$.

In this context, the main feature of our table of products is that $(\alpha_{t,m}, \rho+\delta)$ increases by 1 when t increases by 1, except the last step for $\rho = \omega+\alpha_{n+r+1,m}$ $(m \leq k+1)$.

Note also that for $1 \leq p \leq k+1$, $(\alpha_{k+2,p}, \rho+\delta) < (\alpha_{k+2,p-1}, \rho+\delta)$ since $(\alpha_{p-1}, \rho) \geq 0$.

## §4. Connectedness

Suppose: $\dim F = \dim G - \mathrm{rk}\mathcal{E} \geq 1$. $(A_1)$

F is connected if and only if $H^o(\mathcal{O}_F) = C$.

We have: $H^s(G, \overset{s}{\wedge}\mathcal{E}^*) \Rightarrow H^o(\mathcal{O}_F)$, therefore the vanishing of $H^s(G, \overset{s}{\wedge}\mathcal{E}^*)$ for $s>0$ will imply the connectedness of F.

According to our method, described at the end of §3, we examine $H^s(G,\rho)$, with $\rho$ an irreducible representation of $U_{k+1}$ with highest weight (again denoted $\rho$) among the weights of $\overset{s}{\wedge}\mathcal{E}^*$. Thus $\rho = \omega = \sum_{i=1}^{k+1} a_i t_i$ and we know (see Table 1):

1) $a_{k+1} \geq a_k \geq \ldots \geq a_1 \geq 0$ ;

2) $\rho+\delta$ is either singular or of index $u(n+r-k)$, $1 \leq u \leq k$ ; ( $u=k+1$ is excluded because $\mathrm{rk}\mathcal{E} < \dim G$).

Suppose therefore $s = u(n+r-k)$.

For $\rho+\delta$ to have index s, we must have $(\alpha_{t,p}, \rho+\delta) > 0$ for $p = 1,\ldots,k+1-u$; in particular: $a_{k+1-u} \leq u$ .

Now remember that $\rho$ is a weight of $\overset{s}{\wedge}\mathcal{E}^*$ , thus a sum of s weights of $\mathcal{E}^*$ , each weight counted at most as many times as the dimension of its eigenspace. There are (multiplicities included) $\sum_{m=1}^{r}\binom{d_m+u-1}{u-1}$ weights involving only $t_i$ , $i > k+1-u$.

Adding any other weight increase some $a_j$ , $j \leqslant k+1-u$, thus we must not add more than $u(k+1-u)$ such weights. This will be clearly impossible if n satisfies the following conditions:

$$\sum_{m=1}^{r}\binom{d_m+u-1}{u-1} + u(k+1-u) < u(n+r-k) = s \qquad (C_u)$$

with u running from 1 to k.

Now, use (repeatedly) the formula:

$$\frac{1}{q+1}\binom{d_m+q}{q} - \frac{1}{q}\binom{d_m+q-1}{q-1} = \frac{d_m-1}{q(q+1)}\binom{d_m+q-1}{q-1} \qquad (7)$$

to show that if some $d_m \geqslant 3$, or at least two degrees in d are $\geqslant 2$, then $(C_u)$, $1 \leqslant u \leqslant k$ , is a consequence of our assumption $(A_1)$. Note that $(C_1)$ reads: $n > 2k$.

We have therefore:

Theorem 4.1. Let $S = S_n(d_1,\ldots,d_r)$ be a complete intersection in $P_{n+r}$ and $F = F_k(S)$ its variety of projective k-planes.

Suppose $\dim F = (k+1)(n+r-k) - \sum_{m=1}^{r}\binom{d_m+k}{k} \geqslant 1$ , or, in case S is a quadric, suppose $n > 2k$.

Then F is connected.

Remark 4.2. For a smooth quadric $S = S_{2k}(2)$, $F_k(S)$ consists of two isomorphic (hermitian symmetric) connected components.

This should rather be viewed as the exception which confirms the rule: $S_{2k}(2)$ is a homogeneous (hermitian symmetric) space (of rank one) in its own right, and the generating k-planes of the two families in $F_k(S)$ correspond to Schubert cycles which are not homologically equivalent.

Remark 4.3. There is a simple formula for the canonical bundle of $F = F_k(S_n(d))$, when smooth.

Let $\mathcal{O}_G(1)$ denote the positive generator of Pic(G), restricting to $\mathcal{O}_F(1)$ on F.

Set: $K = \sum_{m=1}^{r} \binom{d_m+k}{k+1} - (n+r+1)$ .

Then: $K_F = \mathcal{O}_F(K)$.

§5. Deformations

In this section we assume that $F = F_k(S_n(d))$ has the 'right' codimension and dimension at least two:

$$\dim F = \dim G - \operatorname{rk}\mathcal{E} \geqslant 2 \,. \qquad (A_2)$$

Our purpose is to produce conditions on (n,d,k) which ensure the completeness of the natural deformation of F, parametrized by a neighbourhood of the section $\Phi(s) \in H^0(G,\mathcal{E})$ defining F. Notice that the family of complete intersections to which $S_n(d)$ belongs (parametrized by a neighbourhood of $s \in H^0(P,E) \cong H^0(G,\mathcal{E})$, i.e. the 'same' base) is itself complete (see [4], [17], [21]).

A sufficient condition for completeness is the vanishing of $H^1(G, \mathcal{E}\otimes J_F)$ and $H^1(F, \Theta_G|_F)$. This is a general result for varieties defined by sections in a vector bundle (see [21]).

We look therefore at the spectral sequences (6) abutting to the above two groups.

(5.1) Take first $H^s(G, \mathcal{E}\otimes \overset{s}{\wedge}\mathcal{E}^*)$, $s \geq 1$.

We obtain vanishing conditions for these groups as we did for $H^s(G, \overset{s}{\wedge}\mathcal{E}^*)$ in §4.

Let $D = \max_{1\leq m\leq r}(d_m)$. Filtering and taking highest weights will produce as above weights $\rho = \omega = \sum_{i=1}^{k+1} a_i t_i$, with $(\alpha_p, \rho) \geq 0$ for $p \leq k$.

Since $\rho$ is the sum of a weight $\omega'$ of $\mathcal{E}$ and a weight $\omega''$ of $\overset{s}{\wedge}\mathcal{E}^*$, adding $\omega'$ to $\omega'' = \sum_{i=1}^{k+1} a_i'' t_i$ decreases some of its coefficients $a_i''$, diminishing their sum by at most D.

This means that our sufficient conditions $(C_u)$, $1 \leq u \leq k$, for the vanishing of $H^s(G, \overset{s}{\wedge}\mathcal{E}^*)$, $s \geq 1$, become, by the same type of reasoning, sufficient conditions $(C_u^D)$, $1 \leq u \leq k$, for the vanishing of $H^s(G, \mathcal{E}\otimes\overset{s}{\wedge}\mathcal{E}^*)$, once we add D to the left hand side of each inequality:

$$\sum_{m=1}^{r}\binom{d_m+u-1}{u-1} + u(k+1-u) + D < u(n+r-k) \qquad (C_u^D)$$

(5.2) Consider now $H^{s+1}(G, \Theta_G \otimes \overset{s}{\wedge}\mathcal{E}^*)$, $s \geq 0$.

For $s = 0$, we have $H^1(G, \Theta_G) = 0$, because G is rigid [5]. Suppose $s \geq 1$.

Again, using a filtration (actually, the representations we are dealing with are all completely reducible) and successive quotients corresponding to irreducible representations of $U_{k+1}$, we find that the highest weight $\rho$ associated to such a representation is necessarily of the form $\rho = \omega + \alpha_{t,m}$, with

$$\omega = \sum_{i=1}^{k+1} a_i t_i \quad \text{a weight of } \overset{s}{\wedge}\mathcal{E}^*, \quad t > k+1 \geqslant m \qquad \text{(cf.§3(c)), and}$$

further conditions: $(\rho, \alpha_q) \geqslant 0$ for all $q \neq k+1$, which imply in particular $t = n+r+1$.

Take therefore $\rho = \omega + \alpha_{n+r+1,m}$ $(m \leqslant k+1)$ and consider the series of integers: $(\rho + \delta, \alpha_{t,p})$ with $p \leqslant k+1$ fixed and t increasing from k+2 to n+r+1. If $\rho + \delta$ is non-singular, this series of non-zero integers will keep the same sign, except possibly at the last step $t = n+r+1$, when it might 'jump' precisely over zero (see Table 1).

Now let p decrease from k+1 to 1 and notice the relations of the starting values in each series:

$$(\rho + \delta, \alpha_{k+2,k+1}) < (\rho + \delta, \alpha_{k+2,k}) < \ldots < (\rho + \delta, \alpha_{k+2,1}) .$$

This means that we might encounter non-vanishing cohomology $H^{s+1}(G, \rho)$ at most for s+1 or s a multiple of n+r-k, say u(n+r-k) $(u < k+1$ by our assumption $rk\mathcal{E} \leqslant \dim G - 2)$.

For the coefficients $a_i$ in $\omega = \sum_{i=1}^{k+1} a_i t_i$, we have either:

1) $a_{k+1} > a_k \geqslant \ldots \geqslant a_1$ for $m = k+1$, or

2) $a_{k+1} \geqslant \ldots \geqslant a_{m+1}$ ; $a_{m+1} + 1 \geqslant a_m > a_{m-1} \geqslant \ldots \geqslant a_1$ for $m \leqslant k$ .

Since $\omega$ is a weight of $\overset{s}{\wedge}\mathcal{E}^*$, it appears that $(C_u^2)$ above is a sufficient condition for the vanishing of $H^{s+1}(G, \rho)$.

Summing-up, we obtain:

Theorem 5.3. Let $S = S_n(d_1,\dots,d_r)$ be a complete intersection in $P_{n+r}$ and suppose that its variety of k-planes $F = F_k(S)$ satisfies:

$$\dim F = (k+1)(n+r-k) - \sum_{m=1}^{r} \binom{d_m+k}{k} \geqslant 2 \qquad (A_2).$$

The following conditions $(C_u^D)$, $1 \leqslant u \leqslant k$, ensure that any small deformation of F is induced by a deformation of S:

$$\sum_{m=1}^{r} \binom{d_m+u-1}{u-1} + u(k+1-u) + D < u(n+r-k) \qquad (C_u^D)$$

where $D = \max_{1\leqslant m\leqslant r}(d_m)$.

If no $d_m$ equals 1, these conditions are already implied by $(A_2)$, as soon as d avoids the following list:

(2), (2,2), (2,2,2), (3), (2,3), (4), and $n > 2k+D$.

The last statement is easily derived, using e.g. the identity (7) in §4. Actually, if the given list is excepted and $k \geqslant 2$, $(A_2) \Rightarrow (C_k^D) \Rightarrow \dots \Rightarrow (C_2^D)$, while $n > 2k+D$ is $(C_1^D)$.

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