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Dumitru ADAM*)

ABSTRACT. The mesh independent principle (M.I.P) for Newton's method in the Galerkin type discretizations is studied in the hypothesis of the convergence on the initial space and in the standard approximation property on subspaces. On the line of Algower, Bohmer, Potra and Rheinboldt, [2], and using the framework introduced in [1], we obtain that M.I.P. holds in the "energy-norm" induced by Gram matrix of basis.

Key words: Newton's method, Galerkin discretization, mesh independence.

AMS (MOS): 65F30, 65F35, 65N30

1. Introduction. Let the following nonlinear equation in the separable, real Hilbert space H:

(1.1) $F(u) = 0$

where $F : D \subset H \rightarrow H$, is Lipschitz continuous Fréchet differentiable on the open domain D:

(1.2) $\| F'(u) - F'(v) \| \leq \gamma \|u - v\|, \qquad u,v \in D$

In the assumption that (1.1) has an unique solution $u^* \in D$, which is simple, i.e. there exists $F'(u^*)^{-1}$ and is bounded,

(1.3) $\| F'(u^*)^{-1} \| := \beta^*$

and the ball $B^* := B^*(u^*,r^*) \subset D$, where

(1.4) $r^* = 2/3\beta^*\gamma$

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the following like local convergence result due to Rheinboldt holds: the Newton's sequence, defined by:

(1.5) $u^{j+1} = u^j - F'(u^j)^{-1}F(u^j), \quad j = 0,1,\ldots$

converges for any $u^o \in B^*$ and the iterates verifies:

(1.6) $$\| u^{j+1} - u^* \| \leq \frac{\beta^* \gamma \| u^j - u^* \|^2}{2(1 - \beta^* \gamma \| u^j - u^* \|)}$$

This result and a class of discretizations what are stable, Lipschitz uniform, bounded and consistent of order α (>0), in the sense of [2], was used for proving the M.I.P.: for a prescribed error ε, same number of iterations are necessary for Newton's sequence (1.5) as well as for the corresponding Newton's sequences for the discretizations of (1.1), with starting points u_h^o. The analysis presented in [2] is tipically for finite diference schemas, covering another type of discretizations. In this paper, using same line as in [2], and the "spectral" matrix representation on the real Euclidean space of Galerkin type discretizations, presented in [1], we obtain that M.I.P. holds in the "energy-norm" induced by Gramian. The example on Sobolev space H_o^1, shows that our model is one natural.

Our finite dimension approximation subspaces $S_h \subset H$ are increasing with $h \to 0$ and the following approximation properties hold:

(1.7) $$\inf_{v \in S_h} \| u - v \| := \| u - P_h u \| \leq C_o \| u \|| h^\alpha, \quad \alpha > 0 \quad , h < h_0$$

for any $u \in W \cap H$, where the norm on W majores the norm of H, and P_h is the orthogonal projection operator corresponding of S_h. .We suppose that $W \cap H$ is suficient of rich, for example, is dense in H.

Now, the h-approximation operators defined on $D \cap S_h$, are

(1.8) $F_h := P_h F P_h$, $F'_h(u) := P_h F'(u) P_h$, $u \in D$

and for $u_h := P_h u$, $v_h := P_h v \in D$, we have:

$$\| F_h(u_h + v_h) - F_h(u_h) - F'_h(u_h) v_h \| \leq \| F(u_h + v_h) - F(u_h) - F'_h(u_h) v_h \|$$

what ensures the existence of Fréchet derivative of F_h. Moreover, this is Lipschitz continuous with same constant γ by: if $\xi_h, \eta_h \in S_h \cap D$

$$\|[F_h'(\xi_h) - F_h'(\eta_h)] z_h\| = \| P_h [F'(\xi_h) - F'(\eta_h)] P_h z_h \|$$
$$\leq \gamma \|\xi_h - \eta_h\| \cdot \|z_h\|.$$

for any $z_h \in S_h$. Hence, on $S_h \cap D$:

(1.9) $\quad \|F_h'(\xi_h) - F_h'(\eta_h)\| \leq \gamma \|\xi_h - \eta_h\|$

Some evaluations are needed for our aim. Firstly, by the approximation subspaces choise, for $h \to 0$, $\|F_h'(u^*) - F'(u^*)\| \to 0$. Then for every $\varepsilon_{h,u^*} > 0$, there exists h_1 such that, for $h < h_1$,

$$\|F_h'(u^*) - F'(u^*)\| < \varepsilon_{h,u^*}$$

choosing ε_{h,u^*} such that $\beta^* \varepsilon_{h,u^*} \leq 1/10$, we obtain by the following trick:

$$F_h'(u^*) = F'(u^*)(I - F'(u^*)^{-1}(F'(u^*) - F_h'(u^*)))$$

that

$$\|F_h'(u^*)^{-1}\| \leq \frac{\beta^*}{1 - \beta^* \varepsilon_{h,u^*}} \leq 1.01 \beta^* := C_1$$

Now, if $u^* \in B^* \cap W$, by (1.7)

(1.10) $\quad \|F_h'(u_h^*) - F_h'(u^*)\| \leq C_0 |||u^*||| \gamma \cdot h^\alpha$

Hence, choosing h_2 such that for $h < h_2$, $C_0 \gamma |||u^*||| \; C_1 h^\alpha \leq 1/10$, we obtain by same trick, that there exists $F_h'(u_h^*)^{-1}$, and,

(1.11) $$\|F_h'(u_h^*)^{-1}\| \leq \frac{\|F_h'(u^*)^{-1}\|}{1 - \|F_h'(u^*)^{-1}\| \cdot \|F_h'(u^*) - F_h'(u_h^*)\|} \leq 1.01^2 \beta^* := C_2$$

REMARK 1. Let $\tilde{h} = \min_{0 \leq i \leq 2} \{h_i\}$ and $u^* \in B^* \cap W$. Then, there exists $F_h'(\xi_h)^{-1}$, for any $\xi_h \in S_h \cap B^*$, and

(1.12) $$\| F_h'(\xi_h)^{-1} \| \leq \sigma := 4\beta^*$$

i.e. the discretization is stable in the sense of [2], locally.

Proof. Observing that,

$$\| F_h'(u_h^*) - F_h'(\xi_h) \| \leq \gamma \| P_h(\xi_h^* - u^*) \| \leq \gamma r^*$$

and $$C_2 \gamma r^* = 1.01^2 \gamma \beta^* 2/3\gamma\beta^* = 2 * 1.01^2/3 < 1,$$

there exists $F_h'(\xi_h)^{-1}$, and (1.12) holds, by same trick. ∎

Second, for $u \in B^* \cap W$, $u_h \in D$, using the following estimation due to Kantorowich ([4])

$$\| F(u) - F(u_h) - F'(u)(u_h - u) \| \leq \gamma \| u - u_h \|^2$$

we obtain $$\| P_h F(u) - F_h(u_h) \| \leq \| F(u) - F(u_h) \|$$

and $$\| F(u) - F(u_h) \| \leq \gamma \| u - u_h \|^2 + M \cdot \| u - u_h \|$$

where M is the constant of boundness of Fréchet derivative. Hence,

(1.13) $$\| P_h F(u) - F_h(u_h) \| \leq C_0 (\gamma |||u||| C_0 h^\alpha + M) |||u||| h^\alpha$$

In same way, for $u, v \in B^* \cap W$, $u_h, v_h \in D$,

(1.14) $$\| P_h (F'(u) v - F_h'(u_h) v_h) \| \leq C_0 (\gamma |||u||| \cdot |||v||| + M |||v|||) h^\alpha$$

REMARK 2. If $|||u||| < C$ for any $u \in B^* \cap W$, then ([2]) the discretization is consistent of order α, with (1.13) and (1.14). This isn't possible always, and we remark that in the proving of M.I.P. in [2] it is sufficed that the consistence property to hold only on the Newton sequence. So, we can define the special form of consistence in the following manner:

(1.15) $\quad \| P_h F(u^j) - F_h(u_h^j) \| \leq C_4 h^{\alpha}$

(1.16) $\quad \| P_h (F'(u^j) u^k - F'_h(u_h^j) u_h^k \| \leq C_5 h^{\alpha}$

where C_4 and C_5 can depend only $u^o \in B^* \cap W$, the starting point in the Newton's sequence $\{ u^j,\ j = 0,1,2,\ldots \}$ defined by (1.5).

2. M.I.P. for h-approximations. In consens with [1], we separe the analysis of the approximations on subspaces, of their matrix representation on Euclidean real spaces. This permits to work for approximations in B^* and to transport the obtained estimations on real Euclidean spaces for Galerkin discretizations. On S_h we are encountered with the existence of the solution for the approximation of (1.1):

(2.1) $\quad F_h(u_h) = 0$

and with the convergence of the Newton's sequence. The hypothesis of the theorem of Rheinboldt enounced in section 1, and the approximation property (1.7) ensures this. We will name this as standard hypothesis (S.H.) in the following sections.

LEMMA 1. If (S.H.) hold, and $u^* \in B^* \cap W$ is the solution of (1.1) then there exists $\tilde{h}$ such that for $h < \tilde{h}$, (2.1) has an unique solution $\xi_h^* \in B_h(u_h^*, r_h)$ and the Newton's sequence with the starting point $u_h^o := P_h u^o$ is convergent to it, quadratically, i.e. a similar relation as (1.5) holds.

Proof. Because $u^* \in B^* \cap W$, $\| u^* - u_h^* \| \leq C_o \, |\!|\!| u^* |\!|\!| \, h^{\alpha}$. Let h_4 such that for $h < h_4$, $C_o \, |\!|\!| u^* |\!|\!| \, h^{\alpha} < r^*$, i.e. $u_h^* \in B^*$ and we redefine $\tilde{h}$ of the remark 1., as $\tilde{h} = \min_{0 \leq i \leq 4} \{h_i\}$. Then, there exists $F'_h(u_h^*)^{-1}$, and

(2.2) $\quad \| F'_h(u_h^*)^{-1} \| \leq \sigma$

Now, we can evaluate the cantity α_h,

$$\alpha_h := 2\gamma \| F_h'(u_h^*)^{-1} \| \cdot \| F_h'(u_h^*)^{-1} F_h(u_h^*) \|$$

$$\leq 2\sigma^2 \gamma \| F_h(u_h^*) \| = 2\sigma^2 \gamma \| F_h(u_h^*) - P_h F(u^*) \|$$

By (1.13) we have that,

$$\alpha_h \leq 2\sigma^2 \gamma C_6 h^\alpha := C_7 h^\alpha \tag{2.3}$$

Hence, there exists h_5 such that $\alpha_h < 1$ for any $h < \tilde{h}$, where redefined $\tilde{h}$ is: $\tilde{h} = \min_{1 \leq i \leq 5} \{h_i\}$. By a classical theorem of Kantorowich, because hold (2.2) and (2.3) there exists $\xi_h^* \in B_h(u_h^*, r_h)$, that is the unique solution of (2.1) in this ball where

$$r_h := \frac{1 - \sqrt{1 - \alpha_h}}{\alpha_h} \| F_h'(u_h^*)^{-1} F_h(u_h^*) \| \leq$$

$$\leq \sigma C_6 h^\alpha := C_8 h^\alpha$$

We wish to have $\xi_h^* \in B^*$; by

$$\| \xi_h^* - u^* \| \leq r_h + \| u^* - u_h^* \| \leq C_8 h^\alpha + C_6 |||u^*||| h^\alpha := C_9 h^\alpha$$

Hence, there exists h_6 such that for $h < \tilde{h} = \min_{1 \leq i \leq 6} \{h_i\}$, $\xi_h^* \in B^*$, and there exists $F_h'(\xi_h^*)^{-1}$, with

$$\| F_h'(\xi_h^*)^{-1} \| := \beta_h^* \leq \sigma \tag{2.4}$$

Now, by Rheinboldt's theorem, the Newton's sequence, for (2.1) :

$$\xi_h^{j+1} = \xi_h^j - F_h'(\xi_h^j)^{-1} F(\xi_h^j) \qquad , \ j = 0, 1, 2, \ldots \tag{2.5}$$

converges for any $\xi_h^0 \in B_h^*(\xi_h^*, r_h^*)$, where

$$r_h^* = 2/3\gamma \beta_h^* \tag{2.6}$$

and the iterates verifies:

$$\| \xi_h^{j+1} - \xi_h^* \| \leq \frac{\beta_h^* \gamma \| \xi_h^j - \xi_h^* \|^2}{2(1 - \beta_h^* \gamma \| \xi_h^j - \xi_h^* \|)} \tag{2.7}$$

■

Our interest for M.I.P. is that the starting point in (2.5) be $u_h^o := P_h u^o$, i.e. we need that $u_h^o \in B_h^*$. We can reevaluate r_h^*,

$$r_h^* \geq \frac{2(1 - \gamma \|F_h'(u_h^*)^{-1}\| \cdot \|u_h^* - \xi_h^*\|)}{3\gamma \|F_h'(u_h^*)^{-1}\|} \geq$$

$$\geq \frac{1 - 4\beta^* \gamma r_h}{6\gamma\beta^*}$$

by same trick for β_h^* and (1.12). By

$$\|u_h^o - \xi_h^*\| \leq \|u_h^o - u_h^*\| + r_h$$

with $\|u^o - u^*\| \leq qr^*$, $q < \frac{1}{4}$, if

(2.8) $$q r^* + r_h < \frac{1 - 4\beta^* \gamma r_h}{6\gamma\beta^*}$$

then, there exist h_7, such that for $h < \tilde{h} := \min_{1 \leq i \leq 7} \{h_i\}$, $u_h^o \in B_h^*(\xi_h^*, r_h^*) := B_h^*$.

THEOREM 1. (M.I.P. for h-approximation). In the following assumptions:

(S.H.) hold, with $u^* \in W \cap B^*$;

Starting point $u^o \in W \cap B_q^* := B(u^*, qr^*)$, $q < \frac{1}{4}$;

Newton's sequence $\{u^j, j = 1,2,\dots\} \subset W \cap B^*$, and is bounded in the norm of $W : |||u^j||| \leq C_9$, $C_9 := C(u^o)$, then there exists h such that for $h < \tilde{h}$, the M.I.P. in the sense of [2], holds, when the starting point of Newton's sequence of h-approximation (2.1) is $P_h u^o := u_h^o$; i.e.:

(2.9) $$\xi_h^j = P_h u^j + O(h^\alpha)$$

(2.10) $$F_h(\xi_h^j) = P_h F(u^j) + O(h^\alpha)$$

(2.11) $$\xi_h^j - \xi_h^* = P_h(u^j - u^*) + O(h^\alpha, \quad j = 0,1,2,\dots$$

and in the stronger version: for any $\varepsilon > 0$,

(2.12) $$\left| \min\{j \geq 0;\ \|u^j - u^*\| < \varepsilon\} - \min\{j \geq 0;\ \|\xi_h^j - \xi_h^*\| < \varepsilon\} \right| \leq 1$$

Proof. The last $\tilde{h}$ ensures that the Newton's sequence for h-discretizations with $h < \tilde{h}$ converges to the unique solution ξ_h^* of (2.1), with starting point $P_h u^o$ what lies in B_h^*. Now, the estimations for

$$\| \xi_h^j - P_h u^j \| \, , \quad \| F_h(\xi_h^j) - P_h F(u^j) \| \, , \quad \| (\xi_h^j - \xi_h^*) - P_h(u^j - u^*) \|$$

and (2.12) follows by the Theorem 2 and Corollary 1 of [2], via Lemma 2, see below, and Remark 2; in fact, this estimations make the object of [2]. ∎

LEMMA 2. In the hypotheses of the theorem 1, there exists $\tilde{h}$ such that, for $h < \tilde{h}$, for every $u^o \in W \cap B^*(u^*, qr^*)$, the h-approximations are Lipschitz uniform

$$(2.13) \qquad \| F_h'(u_h) - F_h'(v_h) \| \leq \gamma \| u_h - v_h \|$$

stable

$$(2.14) \qquad \| F_h'(u_h)^{-1} \| \leq \sigma$$

bounded

$$(2.15) \qquad \| P_h u \| \leq \| u \|$$

and consistent of order α

$$(2.16) \qquad \| P_h F(u) - F_h(u_h) \| \leq C_{10} h^\alpha$$

$$(2.17) \qquad \| P_h (F'(u)v) - F_h'(u_h) v_h \| \leq C_{11} h^\alpha$$

on the set of Newton iterates defined by u^o and (1.5), where C_{10}, C_{11} are constants what depend only u^o. Moreover, (2.15) holds for every $u \in D$, (2.13) holds for every $u_h, v_h \in D \cap S_h$, (2.14) holds for every $u_h \in S_h \cap B^*$.

Proof. Because P_h is orthogonal projection, (2.15) holds. (2.13) is (1.9) and (2.16), (2.17) are obtained by (1.13), (1.14) using the boundness of the Newton sequence in the norm of W. Now, by

$$\| \xi_h^j - u^* \| \leq \| \xi_h^j - \xi_h^* \| + \| \xi_h^* - u^* \|$$

$$\leq \| u_h^0 - \xi_h^* \| + \| \xi_h^* - u^* \| \leq q r^* + 2 r_h + C_0 ||| u^* ||| h^\alpha$$

$$\leq q r^* + C_{12} h^\alpha$$

there exists h_8 such that for $h < \tilde{h} := \min_{1 \leq i \leq 8} \{h_i\}$ the Newton sequence $\{\xi_h^j\}$ lies in B^*. Hence (2.14) hold on this. Moreover, by

$$\| u_h^j - u^* \| \leq \| u_h^j - u_h^* \| + \| u_h^* - u^* \| \leq \| u^j - u^* \| + \| u_h^* - u^* \|$$

$$\leq \| u^0 - u^* \| + \| u_h^* - u^* \|$$

$$\leq q r^* + C_0 ||| u^* ||| h^\alpha$$

the projection of the Newton's sequence (1.5) lies in B^*. Then, (2.14) holds, for $h < \tilde{h}$. ∎

3. M.I.P. for matrix representations. Let R^{n_h} be the Euclidean real space with same dimension as S_h, where S_h is spanned by the linear independent family $\{\phi_h^i ;\ i = 1,n_h\}$. There exists an unique representation for every $\xi_h \in S_h$ in R^{n_h}, that is the vector $\tilde{\xi}_h$ with the entries the components of ξ_h for the basis $\{\phi_h^i\}$. Let $\{e_h^i,\ i = 1,n_h\}$ be the canonical basis of R^{n_h}; we define the linear operator $J^h \in \mathcal{L}(R^{n_h}, S_h)$ by $J^h e_h^i = \phi_h^i$, $i = 1,n_h$, and let J_h be the adjoint of this

(3.1) $\quad \langle J_h \xi_h, \tilde{\eta}_h \rangle_{R^{n_h}} := \langle J_h \xi_h, \tilde{\eta}_h \rangle_h = \langle \xi_h, J^h \tilde{\eta}_h \rangle$

Then, the h-discretization of F is

(3.2) $\quad \tilde{F}_h := J_h F_h J^h$

and the h-discretization of the Fréchet derivative is

(3.3) $\quad \tilde{F}_h'(\tilde{\xi}_h) := J_h F_h'(\xi_h) J^h ,$

where $\xi_h = J^h \tilde{\xi}_h$. We note that if $A = [H]$, then the h-discretization $\tilde{A}_h$ has the matrix representation in canonical basis the Galerkin matrix ([1]) because his entries are

$$\tilde{A}_h|_{ij} = \langle \tilde{A}_h e_i , e_j \rangle_h = \langle J_h A_h J^h e_i , e_j \rangle_h = \langle A_h \phi_h^i , \phi_h^j \rangle$$

We identify the operators on R^{n_h} with their matrix representation in canonical basis.

By this observations, for $\xi_h \in S_h$, $\tilde{F}'_h(\tilde{\xi}_h)$ represents the Galerkin matrix representation of the Fréchet derivative $F'(\xi_h) \in [H]$.

Now, let $G_h \in [R^{n_h}]$ be defined by

$$G_h := J_h J^h \tag{3.4}$$

what has the Gram matrix of $\{\phi_h^i\}$ as matrix representation, and let the Choleski factorization

$$G_h = L_h L_h^* \tag{3.5}$$

where L_h is a low-triangular matrix.

Defining $\hat{J}^h \in \mathcal{L}(R^{n_h}, S_h)$ being the adjoint of $\hat{J}_h \in \mathcal{L}(S_h, R^{n_h})$,

$$\hat{J}_h := L_h^{-1} J_h \tag{3.6}$$

and the mapping $\Lambda_h \in \mathcal{L}([S_h], [R^{n_h}])$ by

$$\Lambda_h (A_h) := \hat{J}_h A_h \hat{J}^h = L_h^{-1} \tilde{A}_h L_h^{-*} \tag{3.7}$$

we have the following theorem of spectral matrix representation ([1]):

THEOREM 2. Λ_h is an isomorfism of operator algebras which preserves the spectrum, norm and condition number, i.e. for $A_h \in [S_h]$,

$$\sigma(\Lambda_h(A_h)) = \sigma(A_h)$$

$$\| \Lambda_h(A_h) \|_h = \| A_h \|$$

$$\kappa(\Lambda_h(A_h)) = \kappa(A_h) := \| A_h \| \cdot \| A_h^{-1} \| ,$$

if A_h is invertible.

Proof. Schetching, we observe that $\hat{J}_h^{-1} = \hat{J}^h$. Then Λ_h is a similarity application. Because we have

(3.8) $$\| \hat{J}_h \xi_h \|_h = \| \xi_h \|$$

the our affirmations can easy proved. ∎

We use in the following this representation theorem to transfer our results of h-approximations on R^{n_h}.

Let $\xi_h := J^h \tilde{\xi}_h := \hat{J}^h \hat{\xi}_h \in S_h$, where $\tilde{\xi}_h = L_h^{-*} \hat{\xi}_h \in R^{n_h}$. By Λ_h, the matrix representation of the derivative Fréchet is

(3.9) $$\hat{F}_h'(\hat{\xi}_h) := \Lambda_h(F_h'(\xi_h)) = L_h^{-1} \tilde{F}_h'(\tilde{\xi}_h) L_h^{-*}$$

Then, the Newton's sequence corresponding of the h-approximation by Λ_h is

(3.10) $$\hat{\xi}_h^{j+1} = \hat{\xi}_h^{j} - \hat{F}_h'(\hat{\xi}_h)^{-1} \hat{F}_h(\hat{\xi}_h) \quad , \hat{\xi}_h^0 = \hat{J}_h \xi_h^0 \quad , j = 0, 1, 2, \ldots$$

and the Newton's sequence for Galerkin discretization is

(3.11) $$\tilde{\xi}_h^{j+1} = \tilde{\xi}_h^{j} - \tilde{F}_h'(\tilde{\xi}_h)^{-1} \tilde{F}_h(\tilde{\xi}_h) \quad , \tilde{\xi}_h^0 = J_h^{-1} \xi_h^0 \quad , j = 0, 1, 2, \ldots$$

which is that of the practical interest.

Now, because

$$\| L_h^{-1} (\tilde{F}_h(\tilde{\xi}_h + \tilde{\eta}_h) - \tilde{F}_h(\tilde{\xi}_h) - \tilde{F}_h'(\tilde{\xi}_h)\tilde{\eta}_h) \|_h$$

$$= \| \hat{F}_h(\hat{\xi}_h + \hat{\eta}_h) - \hat{F}_h(\hat{\xi}_h) - \hat{F}_h'(\hat{\xi}_h)\hat{\eta}_h \|_h$$

$$= \| F_h(\xi_h + \eta_h) - F_h(\xi_h) - F_h'(\xi_h)\eta_h \|$$

$\tilde{F}_h'(\tilde{\xi}_h)$ is derivative Fréchet iff $\hat{F}_h'(\hat{\xi}_h)$ is derivative Fréchet, iff $F_h'(\xi_h)$ is derivative Fréchet, the our model is consistent and because by an above observation, the matrix representation of $\tilde{F}_h'(\tilde{\xi}_h)$ is the Galerkin matrix of $F'(\xi_h)$, this model is natural.

Defining the following "energy"-inner product induced by the Gram matrix

(3.11) $$\langle \tilde{\xi}_h, \tilde{\eta}_h \rangle_{G_h} := \langle G_h \tilde{\xi}_h, \tilde{\eta}_h \rangle_h$$

we give the following result:

THEOREM 3. (M.I.P. for Galerkin discretizations). In the hypotheses of theorem 1, with same h, M.I.P. holds for matrix representation of h-approximation, h < h, in the norm induced by Gramian. In the stronger formulation, for any $\varepsilon > 0$,

$$|\min\{j \geq 0; \|u^j - u^*\| < \varepsilon\} - \min\{j \geq 0; \|\tilde{\xi}_h^j - \tilde{\xi}_h^*\|_{G_h} < \varepsilon\}| \leq 1 \tag{3.12}$$

Proof. By theorem 2, the Newton's method for h-approximation of (1.1) with starting point $P_h u^o := {}_h^o$ converges for $h < h_o$ and M.I.P. holds. Now by theorem 2, all estimations on S_h are automatically transfered with same constants on R^{n_h} for (3.10); hence

$$\|\hat{\xi}_h^j - \hat{\xi}_h^*\|_h = \|\xi_h^j - \xi_h^*\| \tag{3.13}$$

and (3.12) holds in Euclidean norm for $\{\xi_h^j\}$.

Now, because

$$\|\hat{\xi}_h^j - \hat{\xi}_h^*\|_h = \|L_h^*(\tilde{\xi}_h^j - \tilde{\xi}_h^*)\|_h = \|\tilde{\xi}_h^j - \tilde{\xi}_h^*\|_{G_h} \tag{3.14}$$

the Newton's method for Galerkin discretization converge to $\tilde{\xi}_h^*$ and M.I.P. holds in the G_h-norm. ∎

4. An example on H_o^1. We wish to apply our model for P.D.E. equations. Let the following problem

$$\tilde{f}(u) := \frac{\partial^2 u}{\partial x^2} - f\left(x, u, \frac{\partial u}{\partial x}\right) = 0, \quad x \in \Omega := (0,1) \tag{4.1}$$

$$u(0) = u(1) = 0$$

whose variational formulation is in $H := H_o^1(\Omega)$, the Sobolev space equipped with the inner product involving only first derivative.

Assume that f is such that (4.1) verifies the Rheimboldt's hypotheses and u^* is the unique solution for it. Then the variational formulation as well as the operator equation defined by this on $H_o^1(\Omega)$, has the same solution $u^* \in H^2 \cap H_o^1$. Moreover, we assume that f is such that this operator equation, (1.1), verifies the Rheimboldt's hypotheses, eventually with modified constants. The following remark, that isn't complicated to prove, is the support of our suppositions, and ensures that the Newton's sequence for (4.1) is same for variational formulation as well as for operator equation on H^1_o.

REMARK 3. In the stated hypothesis of initial problem, holds in $H^1_o(\Omega)$ the "commutativiy" between the Newton's process and the variational formulation: both ordering of their conducts at same equations in $H^1_o(\Omega)$.

For $u^o \in H^2(\Omega)$, $u^o(0) = u^o(1) = 0$, the Newton's iterates of (4.1) are the unique solutions of the linear problems:

(4.2)
$$\mathcal{F}'(u^j)\, u^{j+1} = g(u^j) := \mathcal{F}'(u^j)\, u^j - \mathcal{F}(u^j)$$
$$u^{j+1}(0) = u^{j+1}(1) = 0 \qquad ; \qquad j = 0, 1, 2, \ldots$$

Then, they are the unique solutions of the variational formulations associated to (4.2), and $u^{j+1} \in H^2(\Omega) \cap H^1_o(\Omega)$.

Let $W = H^2(\Omega) \cap H^1_0(\Omega)$. We have,

$$|||u^j|||^2 := \|u^j\|^2_{H^2(\Omega)} = \|u^j\|^2_{H^1(\Omega)} + \left\|\frac{\partial^2 u^j}{\partial x^2}\right\|^2_{L^2(\Omega)}$$
$$\leq C\|u^j\|^2_{H^1_0(\Omega)} + \left(\|f\|_{L^2(\Omega)} + \|\mathcal{F}(u^j)\|_{L^2(\Omega)}\right)^2$$

Hence, if $\|f(u)\|_{L^2(\Omega)} < C$ in the ball of convergence of Newton's method for $\mathcal{F}$, and because $\mathcal{F}$ is so, then the suplimentary hypothesis of theorem 1 are satisfied, for every $u^o \in W \cap B^*(u^*, qr^*)$ with the constant C_9 independent of u^o.

Now, let S_h be sppanned by the linear piecewise family of functions corresponding to the uniform discretization of the domain Ω, of the mesh $h(n+1) = 1$, $\phi_j(ih) = \delta_{ij}$, $i,j = 1,\ldots,n$. For this polinomial basis of functions, the approximation property verifies ([3]):

$$\inf_{v \in S_h} \|u - v\|_{H^1(\Omega)} \leq C\|u\|_{H^2(\Omega)} \cdot h \quad , \qquad u \in H^2 \cap H^1_o$$

i.e. $\alpha = 1$ in (1.7); and the Gramian coincides with the discretization of the Laplace operator:

$$G_h = \frac{1}{h}\begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}$$

with $\lambda_{min}(G_h) = 2(1 - \cos\pi h)/h = h\pi^2\theta_h^2$, where $\theta_h \to 1$, $h \to 0$. Notting that the norm on R^n is weighted by $h^{\frac{1}{2}}$ for the equivalence with L^2-norm, we have:

$$\|\tilde{\xi}_h\|_{\mathbb{R}^n} := h^{\frac{1}{2}} \|\tilde{\xi}_h\|_h$$
$$\leq (h/\lambda_{min}(G_h))^{1/2} \|\tilde{\xi}_h\|_{G_h}$$
$$\leq \frac{1}{\pi\theta_h}\|\xi_h\|_{H_0^1(\Omega)} \leq \|\xi_h\|_{H_0^1(\Omega)}$$

for h sufficiently small. Then, the M.I.P holds in the R^n-norm, for Galerkin discretization, because:

We remark that this model can easily be extended to nonlinear equations with the linear part an elliptic P.D.E operator on multidimensional domain.

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