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by

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NEGATIVE DEFINITE FUNCTIONS AND PSEUDO-DIFFERENTIAL OPERATORS

By Emil Popescu

In this paper we propose a connection between the negative definite functions and the pseudo-differential operators. The section 1 presents the notations and terminology which we shall use in this paper. In section 2 we shall prove that a continuous negative definite function can not be a symbol of order > 2. We shall give conditions in which a family of pseudo-differential operators is a Feller semigroup on $\mathbb{R}^n$ and we shall characterise the semigroups which commute with the translations. The result from the end of section (Proposition 2.4) contains conditions in which a pseudo-differential operator whose symbol depends not on x is the infinitesimal generator for a Feller semigroup on $\mathbb{R}^n$. This fact corresponds, for a differential operator, to the case of "constant coefficients". In the section 3 we shall use Proposition 2.4 for the deduction of the main result of this paper, the Theorem 3.1 (the case of "variable coefficients"). In the Theorem 3.1 we shall give conditions in which a pseudo-differential operator is the infinitesimal generator for a Feller semigroup on $\mathbb{R}^n$ and we shall present an explicit formula for the construction of a such semigroup. We shall prove that this result can be considered as an extension of Roth's paper [4], while the methods of proof are different. They are depending on a series of theorems due to Chernoff[2].

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1. Definitions and Preliminaries

In this section we present the definitions and some basic results concerning the negative definite functions, the Feller semigroups and the pseudo-differential operators. For details see [1], [3], [5].

Definition 1.1. A function $\psi : \mathbb{R}^n \longrightarrow \mathbb{C}$ is called negative definite if for all natural numbers p and all n-tuples $(u_1, \ldots, u_p)$ of elements from $\mathbb{R}^n$

$$\sum_{i=1}^{p} \sum_{j=1}^{p} \left[\psi(u_i) + \overline{\psi(u_j)} - \psi(u_i - u_j) \right] c_i \bar{c}_j \geqslant 0$$

for any n-tuple $(c_1, c_2, \ldots, c_p) \in \mathbb{C}^p$.

A function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{C}$ is called positive definite if for all natural numbers p and all n-tuples $(u_1, u_2, \ldots, u_p)$ of elements from $\mathbb{R}^n$

$$\sum_{i=1}^{p} \sum_{j=1}^{p} \varphi(u_i - u_j) c_i \bar{c}_j \geqslant 0$$

for any n-tuple $(c_1, c_2, \ldots, c_p) \in \mathbb{C}^p$.

Proposition 1.1. A function $\psi : \mathbb{R}^n \longrightarrow \mathbb{C}$ is negative definite if and only if the following two conditions are satisfied:

(i) $\psi(0) \geqslant 0$

(ii) The function $e^{-t\psi}$ is positive definite for all $t > 0$.

Let μ be a bounded positive measure on $\mathbb{R}^n$. We define the Fourier transformation of μ, denoted $\hat{\mu}$, by the formula

(1) $\hat{\mu}(\xi) = \int e^{-i \langle y, \xi \rangle} \mu(dy), \qquad \xi \in \mathbb{R}^n$

where $\langle y, \xi \rangle$ is the scalar product from $\mathbb{R}^n$. $\hat{\mu}$ is a contiuous positive definite function and take place the following result:

Proposition 1.2. Let μ be a bounded positive measure on $\mathbb{R}^n$. If the Fourier transformation $\hat{\mu}$ belongs to $L^1(\mathbb{R}^n)$, then the measure μ has a continuous density g given by

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(2) $$g(x) = (2\pi)^{-n} \int e^{i \langle x, \xi \rangle} \hat{\mu}(\xi)\, d\xi \quad , x \in \mathbb{R}^n$$

Definition 1.2. A convolution semigroup on $\mathbb{R}^n$ is a family $(\mu_t)_{t>0}$ of positive bounded measures on $\mathbb{R}^n$ with the properties:

(i) $\mu_t(\mathbb{R}^n) = 1$ for $t > 0$

(ii) $\mu_t * \mu_s = \mu_{t+s}$ for $t, s > 0$

(iii) $\lim_{t \to 0} \mu_t = \varepsilon_0$ vaguely.

There is a one-to-one correspondence between convolution semigroups $(\mu_t)_{t>0}$ on $\mathbb{R}^n$ and continuous negative definite functions on $\mathbb{R}^n$:

Theorem 1.1.

a) Let $(\mu_t)_{t>0}$ be a convolution semigroup on $\mathbb{R}^n$.

Then there exists a uniquely determined continuous negative definite function ψ on $\mathbb{R}^n$ such that

$$\hat{\mu}_t = e^{-t\psi} \quad \text{for } t > 0 \text{ and } \psi(0) = 0.$$

b) Conversely, if ψ is a continuous, negative definite function ψ on $\mathbb{R}^n$ such that $\psi(0) = 0$, then there exists a uniquely determined convolution semigroup $(\mu_t)_{t>0}$ such that

$$\hat{\mu}_t = e^{-t\psi} \quad \text{for } t > 0.$$

A continuous, negative definite function on $\mathbb{R}^n$ is described by the Lévy-Khinchin formula.

Theorem 1.2. Let $\psi : \mathbb{R}^n \longrightarrow \mathbb{C}$ be a continuous negative definite function. There exist:

(i) a constant $c \geq 0$

(ii) a continuous linear form $l : \mathbb{R}^n \longrightarrow \mathbb{R}$

$$l(\xi) = \sum_{i=1}^{n} b_i \xi_i , \qquad b_i \in \mathbb{R}$$

(iii) a continuous, positive quadratic form $q : \mathbb{R}^n \longrightarrow \mathbb{R}$,

$$q(\xi) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \xi_i \xi_j, \quad a_{ij} \in \mathbb{R} \text{ and } a_{ij} = a_{ji}$$

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(iv) a positive, bounded measure ϑ on $\mathbb{R}^n \setminus \{0\}$, such that for $\xi \in \mathbb{R}^n$

$$(3)\quad \psi(\xi) = c + il(\xi) + q(\xi) + \int_{\mathbb{R}^n \setminus \{0\}} \left[1 - e^{-i\langle \xi, y\rangle} - i\frac{\langle \xi, y\rangle}{1+|y|^2}\right] \frac{1+|y|^2}{|y|^2}\, d\vartheta(y)$$

where $|y| := \langle y, y\rangle$

Conversely, if (c, l, q, ϑ) is a quadruple as specified above, then (3) defines a continuous negative definite function.

Definition 1.3. Let $(E, \|\ \|)$ be a complex Banach space. A strongly continuous contraction semigroup on E is a family $(P_t)_{t>0}$ of linear and bounded operators on E satisfying:

(i) $\|P_t\| \leqslant 1$ for $t > 0$

(ii) $P_t P_s = P_{t+s}$ for $t, s > 0$

(iii) $P_o = I$

(iv) $\lim\limits_{t \to o} P_t f = f$ for all $f \in E$.

The infinitesimal generator $(A, \mathcal{D}(A))$ for $(P_t)_{t>o}$ is the operator on E with domain

$$\mathcal{D}(A) = \left\{ f \in E \;\middle|\; \lim_{t \to o} \frac{1}{t}(P_t f - f) \text{ exists in } E \right\}$$

and given by

$$Af = \lim_{t \to o} \frac{1}{t}(P_t f - f) \text{ for } f \in \mathcal{D}(A).$$

The semigroup generated by A will be denoted by $(e^{tA})_{t>o}$, and the smallest closed extension of A, the closure of A, will be noted by $\overline{A}$.

Proposition 1.3. Let $(A, \mathcal{D}(A))$ be the infinitesimal generator for the strongly continuous contraction semigroup on E, $(e^{tA})_{t>o}$. Let $D \subset \mathcal{D}(A)$ satisfy:

(i) $\overline{D} \subset E$

(ii) $(\forall)\ t > 0,\ e^{tA}(D) \subset D$

Then $A = \overline{A\big|_D}$

We remember two results due to Chernoff [2], which we use in this paper.

Theorem 1.3. Let $\{F_t\}_{t \geq 0}$ be a family of linear contractions on a Banach space E such that the function $t \longrightarrow F_t f$ is continuous for each $f \in E$. Suppose that $F_o = I$ and for each $t > 0$

$$\lim_{n \to \infty} (F_{\frac{t}{n}})^n f = G_t f, \text{ for } f \in E.$$

Then $\{G_t\}_{t > o}$ is a strongly continuous contraction semigroup on E.

Theorem 1.4. Let $\{F_t\}_{t \geq o}$ be a family of linear contractions on a Banach space E with $F_o = I$. Assume that there is a strongly continuous contraction semigroup on E, $(e^{tA})_{t \geq 0}$ such that

$$\lim_{n \to \infty} (F_{\frac{t}{n}})^n f = e^{tA} f ,$$

uniformly on compact t intervals.

Then $A \supset F'(o)$, i.e. A is an extension of the strong derivative $F'(0)$.

In the sequel, we denote by $C(\mathbb{R}^n)$ the set of continuous complex functions on $\mathbb{R}^n$. By $C_c(\mathbb{R}^n)$, resp. $C_o(\mathbb{R}^n)$ we denote the set of functions from $C(\mathbb{R}^n)$ which have compact support, resp. which tend to zero at infinity. $C_o(\mathbb{R}^n)$ is a Banach space with respect to uniform norm:

$$\|f\| = \sup_{x \in \mathbb{R}^n} |f(x)|$$

$C_c^{\infty}(\mathbb{R}^n)$ denote the set of C^{∞} functions on $\mathbb{R}^n$ which have compact support. $C_c^{\infty}(\mathbb{R}^n)$ is dense in $C_o(\mathbb{R}^n)$.

Definition 1.4. A strongly continuous contraction semigroup $(P_t)_{t > o}$ on $C_o(\mathbb{R}^n)$, for which all the operators P_t are positive, i.e. such that for all $t > 0$

$$f \in C_o^+(\mathbb{R}^n) \text{ implies } P_t f \in C_o^+(\mathbb{R}^n),$$

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is called a Feller semigroup on $\mathbb{R}^n$. A Feller semigroup $(P_t)_{t>0}$ on $\mathbb{R}^n$ commutes with the translations of $\mathbb{R}^n$ if

(4) $(\forall)\ a \in \mathbb{R}^n,\ t>0$ and $f \in C_0(\mathbb{R}^n)$, $P_t(\tau_a f) = \tau_a(P_t f)$,

where $\tau_a f(x) = f(x-a)$ for $x \in \mathbb{R}^n$.

A convolution semigroup $(\mu_t)_{t>0}$ on $\mathbb{R}^n$ induces a Feller semigroup on $\mathbb{R}^n$, $(P_t)_{t>0}$, which commutes with translations, by the definition:

(5) $P_t f = \mu_t * f$ for $f \in C_0(\mathbb{R}^n)$ and $t > 0$.

Then, from the Theorem 1.1.b) results that a continuous negative definite function ψ such that $\psi(0) = 0$ induces a Feller semigroup which commute with the translations of $\mathbb{R}^n$.

For the Feller semigroups on $\mathbb{R}^n$, the condition (i) of Definition 1.3 is equivalent with the next condition:

(6) $(\forall) t>0,\ (\forall)\ f \in C_0(\mathbb{R}^n)$ and $0 \leq f \leq 1 \Rightarrow 0 \leq P_t f \leq 1$.

The infinitesimal generator $(A, \mathcal{D}(A))$ for the Feller semigroup on $\mathbb{R}^n$, $(P_t)_{t>0}$, satisfies the positive maximum principle:

(7) $(\forall)\ u \in \mathcal{D}(A)$ and $x \in \mathbb{R}^n$, $u(x) = \sup u \geq 0 \Rightarrow Au(x) \leq 0$

In the sequel the basic notation and terminology will be established for the pseudo-differential operators on $\mathbb{R}^n$.

For the derivatives operators we use the notations:

$$\partial_j = \frac{\partial}{\partial x_j}, \qquad D_j = -i\,\partial_j$$

where $i = \sqrt{-1}$.

Let Ω be an open $\subset \mathbb{R}^n$, $m, \rho, \delta \in \mathbb{R}$ such that $0 < \rho \leq 1$, $0 \leq \delta < 1$. We denote with $S^m_{\rho,\delta}(\Omega)$ the set of functions $a \in C^\infty(\Omega \times \mathbb{R}^n)$ with the property that for any compact $K \subset \Omega$ and for any multi-indices α and β there exists $C = C(K, \alpha, \beta) > 0$ such that

$$\left| \partial_x^\beta \partial_\theta^\alpha a(x, \theta) \right| \leq C(1 + |\theta|)^{m - \rho|\alpha| + \delta|\beta|},$$

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for all $x \in K$, $\theta \in \mathbb{R}^n$. The elements of $S^m_{\rho,\delta}(\Omega)$ are called symbols on Ω of order m and type (ρ, δ). $S^m_{\rho,\delta}(\Omega)$ is a complex linear space. $S^{-\infty}_{\rho,\delta} := \bigcap_m S^m_{\rho,\delta}(\Omega)$. The most important case is that in which $\rho = 1$, $\delta = 0$. Instead of $S^m_{1,0}(\Omega)$ we shall write $S^m(\Omega)$. The set of symbols of order $-\infty$ and type (1, 0) we shall denote with $S^{-\infty}(\Omega)$. If $m_1 \leqslant m_2$ then $S^{m_1}(\Omega) \subset S^{m_2}(\Omega)$.

By $\mathcal{S}(\mathbb{R}^n)$ or $\mathcal{S}$ we shall denote the set of all functions $\varphi \in C^\infty(\mathbb{R}^n)$ such that

$$\sup_{x \in \mathbb{R}^n} \left| x^\beta \partial^\alpha \varphi(x) \right| < +\infty ,$$

for all multi-indices α and β .

Therefore the functions of $\mathcal{S}$ are indefinite derivable functions, which for $x \longrightarrow \infty$, tend to zero together with all their derivatives of all orders, faster than any power of $|x|^{-1}$.

$\mathcal{S}$ is a linear space and the relation

$$C_c^\infty(\mathbb{R}^n) \subset \mathcal{S} \subset C_o(\mathbb{R}^n)$$

is satisfied.

The space $\mathcal{S}$ is invariant with respect to the differentiation and multiplication by the temperate functions, i.e. with the functions which grow not at infinity faster than any polinomial. Thus, $x^\beta \partial^\alpha \varphi \in \mathcal{S}$ is bounded and integrable on $\mathbb{R}^n$, for all $\varphi \in S$ and multi-indices α and β . Therefore, between the spaces $\mathcal{S}$ and L^p the relation $\mathcal{S} \subset L^p$, $p \geqslant 1$ is satisfied.

The Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ is a complex function $\hat{f} : \mathbb{R}^n \longrightarrow C$ defined by

$$\hat{f}(\xi) = \int e^{-i\langle x, \xi \rangle} f(x)\, dx$$

$\mathcal{S}$ is invariant with respect to the Fourier transform and

$$f(x) = (2\pi)^{-n} \int e^{i\langle x, \xi \rangle} \hat{f}(\xi)\, d\xi$$

The form of a pseudo-differential operator of order m and

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type (ϱ, δ) is:

$$Af(x) = (2\pi)^{-n} \int e^{i\langle x, \xi\rangle} a(x, \xi) \hat{f}(\xi)\, d\xi =$$
$$= (2\pi)^{-n} \iint e^{i\langle x - y, \xi\rangle} a(x, \xi) f(y)\, dy\, d\xi$$

where $f \in C_c^{\infty}(\mathbb{R}^n)$ and $a \in S^m_{\varrho,\delta}(\Omega)$.

The set of pseudo-differential operators of order m and type (ϱ, δ) on Ω is denoted with $PS(m, \varrho, \delta)$. If $\varrho = 1, \delta = 0$ then $PS(m) := PS(m, 1, 0)$. The operators from $PS(-\infty) := PS(-\infty, 1, 0)$ will be named in the sequel, regularizant pseudo-differential operators. The composition of two operators from $PS(-\infty)$ is an operator from $PS(-\infty)$.

<u>Definition 1.5</u>. $A \in PS(m, \varrho, \delta)$ is called elliptic of order m if for each compact $K \subset \Omega$ there are positive constants C_K, R_K such that

(8) $$|a(x, \xi)| \geq C_K |\xi|^m, \ (\forall)\ x \in K, \quad |\xi| \geq R_K.$$

2. Negative Definite Functions and Symbols. Semigroups of Pseudo-differential Operators

In this section we shall prove that a continuous negative definite function can not be a symbol of order > 2. The effect of this result will be that of consideration in the rest of the paper only of the pseudo-differential operators of order $\leqslant 2$. We shall distinguish the class of symbols which caracterise the Feller semigroups of regularizant pseudo-differential operators which commute with translations. In Proposition 2.4 we shall give conditions in which a pseudo-differential operator, whose symbol depends not on x, is the infinitesimal generator for a Feller semigroup on $\mathbb{R}^n$ as before.

Proposition 2.1. Let $\psi: \mathbb{R}^n \longrightarrow \mathbb{C}$ be a continuous negative definite function. Then there exists $C_0 > 0$ such that

(9) $\quad (\forall)\ \xi \in \mathbb{R}^n, \quad |\psi(\xi)| \leqslant C_0(1 + |\xi|)^2.$

Proof. Because ψ is continuous negative definite function, the relation (3) from Theorem 1.2 takes place. Obvious, for c, l, q we can indicate a positive constant such that their sum is satisfying an inequality of type (9). For the integral from (3) we shall use the inequalities:

$$\left|1 - e^{-i\langle y, \xi\rangle} - i\langle y, \xi\rangle\right| \leqslant \langle y, \xi\rangle^2 \quad \text{and}$$

$|\langle y, \xi\rangle| \leqslant |y||\xi|$. We have:

$$\left|1 - e^{-i\langle y, \xi\rangle} - i\frac{\langle y, \xi\rangle}{1+|y|^2}\right| \leqslant \left|1 - e^{-i\langle y, \xi\rangle} - i\frac{\langle y, \xi\rangle}{1+|y|^2} - i\frac{\langle y, \xi\rangle |y|^2}{1+|y|^2}\right| + \left|\frac{i\langle y, \xi\rangle |y|^2}{1+|y|^2}\right| = \left|1 - e^{-i\langle y, \xi\rangle} - i\langle y, \xi\rangle\right| +$$

$$+ \frac{|\langle y, \xi\rangle| \cdot |y|^2}{1+|y|^2} \leqslant \langle y, \xi\rangle^2 + \frac{|\langle y, \xi\rangle|\,|y|^2}{1+|y|^2}$$

Hence:

$$\left|1 - e^{-i\langle y, \xi\rangle} - i\frac{\langle y, \xi\rangle}{1+|y|^2}\right| \leqslant |y|^2|\xi|^2 + \frac{|\xi|\,|y|^3}{1+|y|^2}$$

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Let $0 < \varepsilon < 1$ be .

$$\int_{0\neq|y|\leqslant\varepsilon} \left| 1 - e^{-i<y,\xi>} - i\frac{<y,\xi>}{1+|y|^2} \right| \frac{1+|y|^2}{|y|^2}\, d\vartheta(y) \leqslant$$

$$\leqslant |\xi|^2 \int_{0\neq|y|\leqslant\varepsilon} (1+|y|^2)\, d\vartheta(y) + |\xi| \int_{0\neq|y|\leqslant\varepsilon} |y|\, d\vartheta(y) \leqslant (2|\xi|^2 + \varepsilon|\xi|) \int_{0\neq|y|\leqslant\varepsilon} d\vartheta(y)$$

$$\int_{|y|>\varepsilon} \left| 1 - e^{-i<y,\xi>} \right| \frac{1+|y|^2}{|y|^2}\, d\vartheta(y) \leqslant 2 \int_{|y|>\varepsilon} \frac{1}{|y|^2}\, d\vartheta(y) +$$

$$+ 2\int_{|y|>\varepsilon} d\vartheta(y) < 2\frac{1}{\varepsilon^2} \int_{|y|>\varepsilon} d\vartheta(y) + 2\int_{|y|>\varepsilon} d\vartheta(y) = \left(\frac{2}{\varepsilon^2} + 2\right) \int_{|y|>\varepsilon} d\vartheta(y)$$

$$< \frac{4}{\varepsilon^2} \int_{|y|>\varepsilon} d\vartheta(y) .$$

$$\int_{|y|>\varepsilon} \left| \frac{i<y,\xi>}{1+|y|^2} \right| \frac{1+|y|^2}{|y|^2}\, d\vartheta(y) \leqslant \int_{|y|>\varepsilon} \frac{|\xi||y|}{1+|y|^2} \cdot \frac{1+|y|^2}{|y|^2}\, d\vartheta(y)$$

$$= |\xi| \int_{|y|>\varepsilon} \frac{1}{|y|}\, d\vartheta(y) < \frac{|\xi|}{\varepsilon} \int_{|y|>\varepsilon} d\vartheta(y).$$

It follows that:

$$\int_{R^n\setminus\{0\}} \left| \left[1 - e^{-i<y,\xi>} - i\frac{<y,\xi>}{1+|y|^2} \right] \right| \frac{1+|y|^2}{|y|^2}\, d\vartheta(y) \leqslant (2|\xi|^2 + \varepsilon|\xi|) \int_{0\neq|y|\leqslant\varepsilon} d\vartheta(y) +$$

$$+ \left(\frac{4}{\varepsilon^2} + \frac{|\xi|}{\varepsilon}\right) \int_{|y|>\varepsilon} d\vartheta(y) \leqslant b\,(1 + |\xi|)^2,$$ where b is a constant > 0.

Remark 2.1. There exist many exemples of continuous negative definite functions which are symbols (for example, $\psi : \mathbb{R}^n \longrightarrow \mathbb{C}$, $\psi(\xi) = |\xi|^{\alpha}$, $0 < \alpha \leq 2$).

The importante of Proposition 2.1. consists in the fact that a continuous negative definite function can not be a symbol of order > 2.

Proposition 2.2. Let $(W_t)_{t\geqslant 0}$ be a family of regularizant

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pseudo-differential operators on $C_o(\mathbb{R}^n)$ defined by the relation:

(10) $(\forall)\ t > 0,\ (\forall)\ f \in C_c^{\infty}(\mathbb{R}^n),\ W_t f(x) = (2\pi)^{-n} \iint e^{i<x-y,\xi>} .$

$. \ a_t(x, \xi)\ f(y)\ dy d\xi$

If the following conditions are satisfied:

(i) $a_o = 1,\ \lim_{t \to o} a_t = 1$

(ii) $(\forall)\ t > 0,\ (\forall)\ x, y \in \mathbb{R}^n,\ \widehat{a_t(x, .)}\ (y) \geqslant 0$

(iii) $(\forall)\ t, s > 0,\ (\forall)\ x, y, z \in \mathbb{R}^n,\ \widehat{a_{t+s}(z, .)}\ (z - x) =$

$= (2\pi)^{-n} \int \widehat{a_t(x, .)}\ (y - x)\ .\ \widehat{a_s(y, .)}\ (z - y)\ dy =$

$= (2\pi)^{-n} \int \widehat{a_s(x, .)}\ (y - x)\ .\ \widehat{a_t(y, .)}\ (z - y)\ dy$

(iv) $(\forall)\ t > 0,\ (\forall)\ x \in \mathbb{R}^n,\ (2\pi)^{-n} \int \widehat{a_t(x, .)}\ (u)\ du \leqslant 1,$

then $(W_t)_{t \geqslant 0}$ is a Feller semigroup on R^n.

Proof. The proof is immediately since $a_t \in S^{-\infty}(R^n x R^n)$ and, therefore, we can use the Fubini theorem. Changing the succession of integration in (10) it follows that

$(\forall)\ t > 0,\ (\forall)\ f \in C_c^{\infty}(\mathbb{R}^n),\ W_t f(x) = \int K_t(x, y)\ f(y)\ dy$

where $K_t(x, y) = (2\pi)^{-n} \int e^{i<x - y, \xi>}\ a_t(x, \xi)\ d\xi.$

The conditions (ii) and (iv) provide the fact that $(W_t)_{t \geqslant 0}$ are sub-Markovian positive linear functions. The conditions (i) and (iii) imply that $(W_t)_{t \geqslant o}$ is a strongly continuous contraction semigroup. For to finish the proof we use the fact that $C_c^{\infty}(\mathbb{R}^n)$ is dense in $C_o(\mathbb{R}^n)$ and by a change of variable

$$W_t f(x) = (2\pi)^{-n} \iint e^{i<z, \xi>} a_t(x, \xi)\ f(x - z)\ dz\ d\xi$$

It follows that $(W_t)_{t \geqslant o}$ is a Feller semigroup on $\mathbb{R}^n$.

Proposition 2.3. Let $(W_t)_{t \geqslant o}$ be such as in Proposition 2.2. Then $(W_t)_{t \geqslant o}$ commute with the translations if and only if for any $t > 0$, $a_t(x, \xi)$ is a function depending only on ξ.

Proof. Let $t > 0$, $a \in \mathbb{R}^n$ and $f \in C_c^{\infty}(\mathbb{R}^n)$.

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$W_t(\tau_a f) = \tau_a(W_t f)$, where $\tau_a f(x) = f(x - a)$, $x \in \mathbb{R}^n$ is equivalent with the relation

$$(\forall)\ x \in \mathbb{R}^n,\ \iint e^{i < x-y-a,\ \xi >} \left[a_t(x, \xi) - a_t(x-a, \xi) \right] f(y) dy d\xi = 0$$

or with:

$$(\forall)\ x \in \mathbb{R}^n,\ \int e^{i < x-a,\ \xi >} \left[a_t(x, \xi) - a_t(x-a, \xi) \right] \hat{f}(\xi)\, d\xi = 0$$

From the injectivety of Fourier transform it follows

$$(\forall)\ x,\ \xi \in R^n,\ a_t(x, \xi) = a_t(x-a, \xi).$$

Since a was arbitrarly chosen it follows that $a_t(x, \xi)$ is a function depending only on ξ.

Example 2.1. The brownian semigroup is a Feller semigroup on $\mathbb{R}^n$, which commute with the translations, the operators of semigroup being regularizant pseudo-differential operators on $C_c^\infty(\mathbb{R}^n)$.

Indeed, for any $t \geq 0$, let $a_t(\xi) = e^{-\frac{t|\xi|^2}{2}}$, $\xi \in R^n$

For $f \in C_c^\infty(\mathbb{R}^n)$ and $t \geq 0$ we define

$$W_t f(x) = (2\pi)^{-n} \iint e^{i<x-y,\ \xi>}\ a_t(\xi)\, f(y)\, dy\, d\xi \quad .$$

Since $a_t \in S^{-\infty}(\mathbb{R}^n)$ it follows that $(W_t)_{t \geq 0}$ are regularizant pseudo-differential operators. Using the Proposition 2.2 and the Proposition 2.3. we obtain that $(W_t)_{t \geq 0}$ is a Feller semigroup on $\mathbb{R}^n$ which commute with the translations. We remark now that

$$(\forall)\ t > 0,\quad W_t f(x) = (2\pi t)^{-\frac{n}{2}} \int e^{-\frac{|y-x|^2}{2t}} f(y)\, dy,$$

i.e. the formula of brownian semigroup.

Proposition 2.4. Let $\psi : \mathbb{R}^n \longrightarrow \mathbb{C}$, $\psi(0) = 0$, $\psi \in S^m(\mathbb{R}^n)$ with $0 < m \leq 2$, ψ being negative definite such that there are the constants $K > 0$, $r > 0$ with the property

$$(11)\quad (\forall)\ \xi \in \mathbb{R}^n,\ \ \mathrm{Re}\, \psi(\xi) \geq K |\xi|^r .$$

Let the pseudo-differential operator of order m defined by:

$$\mathcal{D}(A) = \mathcal{S} = \left\{ f \in C^\infty(\mathbb{R}^n)\ \middle|\ (\forall)\ \alpha, \beta \text{ multi-indices } \sup_{x \in \mathbb{R}^n} \left| x^\beta \partial^\alpha f(x) \right| < \infty \right\}$$

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$$(\forall)\ f \in \mathcal{D}(A),\ Af(x) = (2\pi)^{-n} \iint e^{i\langle x-y,\xi\rangle}(-\psi(\xi))\, f(y)\, dy\, d\xi$$

Then there exists a Feller semigroup on $\mathbb{R}^n$ such that his infinitesimal generator is the closure of operator A, the semigroup being given, on $C_c^{\infty}(\mathbb{R}^n)$, by the following formula:

$$(\forall) t \geqslant 0,\ (\forall) f \in C_c^{\infty}(\mathbb{R}^n),\ W_t f(x) = (2\pi)^{-n} \iint e^{i\langle x-y,\xi\rangle - t\psi(\xi)} f(y)\,dy\,d\xi$$

where $(W_t)_{t \geqslant 0}$ are regularizant pseudo-differential operators which commute with the translations.

Remark 2.2. From the Proposition 2.1 it follows, necessarily, that $m \leqslant 2$. On the other hand, the existence of $r > 0$ which verify (11), imply $m > 0$. Also, the relation (11) imply $r \leqslant m$.

The proof of Proposition 2.4. We denote by $a_t(\xi) := e^{-t\psi(\xi)}$, $\xi \in \mathbb{R}^n$. We show that $a_t \in S^{-\infty}(\mathbb{R}^n)$. Indeed, let $p \in \mathbb{N}$ be. Then:

$$\left| e^{-t\psi(\xi)} \right| = e^{-t \operatorname{Re} \psi(\xi)} \leqslant \frac{1}{e^{tK|\xi|^r}} \leqslant \frac{1}{\left(\frac{tK}{l}|\xi|^r + 1\right)^l}$$

with $l \in \mathbb{N}$ such that $rl \geqslant p$. It follows that there exists a positive constant C_0 such that

$$\left| e^{-t\psi(\xi)} \right| \leqslant C_0 (1 + |\xi|)^{-p},\ (\forall)\ \xi \in \mathbb{R}^n.$$

We easily deduce that the last inequality takes place for any $p \in \mathbb{R}$. Since $\psi \in C^{\infty}(\mathbb{R}^n)$, it follows $a_t \in C^{\infty}(\mathbb{R}^n)$.

By derivation with respect to $\xi = (\xi_1, \ldots, \xi_n)$ we have the relation:

$$(12)\quad \left| \partial^{\alpha} a_t(\xi) \right| \leqslant \left| e^{-t\psi(\xi)} \right| \cdot \left| P(\psi(\xi), \partial^1 \psi(\xi), \ldots, \partial^{\alpha} \psi(\xi) \right|$$

where $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$ and $P(\psi(\xi), \partial^1\psi(\xi), \ldots, \partial^{\alpha}\psi(\xi))$ is a polynomial in $\psi(\xi), \partial^1\psi(\xi), \ldots, \partial^{\alpha}\psi(\xi)$.

Since $\psi \in S^m(\mathbb{R}^n)$, $0 < m \leqslant 2$, it follows that there exists $M_{\alpha} > 0$ such that

$$(13)\qquad (\forall)\ \xi \in \mathbb{R}^n,\ \left| \partial^{\alpha}\psi(\xi) \right| \leqslant M_{\alpha} (1 + |\xi|)^{m - |\alpha|}$$

On the other hand we have seen that for any $p \in \mathbb{R}$ there exists

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$C_o > 0$ such that

(14) $(\forall)\quad \xi \in \mathbb{R}^n, \quad \left| e^{-t\,\psi(\xi)} \right| \leqslant C_o(1 + |\xi|)^{-p}$

From the relations (12), (13), (14) it follows that for any $s \in \mathbb{R}$ there exists $C_\alpha > 0$ such that

$$(\forall)\quad \xi \in \mathbb{R}^n, \quad \left| \partial^\alpha a_t(\xi) \right| \leqslant C_\alpha (1 + |\xi|)^{s - |\alpha|}$$

Hence $a_t \in \bigcap_{s \in \mathbb{R}} S^s(\mathbb{R}^n) = S^{-\infty}(\mathbb{R}^n)$.

Since ψ is a continuous negative definite function with $\psi(0) = 0$, from the Theorem 1.1.b) there exists an uniquely determined convolution semigroup $(\mu_t)_{t>0}$ on $\mathbb{R}^n$ such that

$$\hat{\mu}_t = e^{-t\psi} \quad \text{for } t > 0.$$

From the relation (5) it follows that for any $f \in C_o(\mathbb{R}^n)$, $(\mu_t * f)_{t \geqslant 0}$ determines a Feller semigroup on $\mathbb{R}^n$ which commute with the translations. For any $t \geqslant 0$ and any $f \in C_c^\infty(\mathbb{R}^n)$, we define

$$W_t f := \mu_t * f, \quad \text{i.e.} \quad W_t f(x) = \int f(x-y)\, d\mu_t(y)$$

Since $\hat{\mu}_t = a_t \in S^{-\infty}$ we can apply Proposition 1.2. For $t \geqslant 0$ we denote by g_t a comtinuous density of measure μ_t from the Proposition 1.2:

$$g_t(y) = (2\pi)^{-n} \int e^{i\langle y, \xi \rangle} a_t(\xi)\, d\xi$$

We deduce that

$$W_t f(x) = (2\pi)^{-n} \int f(x-y) \left(\int e^{i\langle y, \xi \rangle} a_t(\xi)\, d\xi \right) dy$$

Since $a_t \in S^{-\infty}$ we can apply the Fubini Theorem: we can integrate in any order in the formula of W_t. It follows that, for any $f \in C_c^\infty(\mathbb{R}^n)$ and any $t \geqslant 0$

$$W_t f(x) = (2\pi)^{-n} \iint e^{i\langle x-y, \xi \rangle - t\psi(\xi)} f(y)\, dy\, d\xi$$

and that $(W_t)_{t \geqslant 0}$ are regularizant pseudo-differential operators which commute with the translations.

Let $f \in \mathcal{D}(A)$ be. Then, we deduce easily that,

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$$\lim_{t \to 0} \frac{W_t f(x) - f(x)}{t} = Af(x) \text{ in } C_0(\mathbb{R}^n)$$

It follows that the infinitesimal generator for $(W_t)_{t \geq 0}$ extends A. On the other hand, we notice that for any $f \in \mathcal{S}$, $W_t f \in \mathcal{S}$, $(\forall)\ t \geq 0$, i.e. $(\forall)\ t \geq 0$, $W_t(\mathcal{D}(A)) \subset \mathcal{D}(A)$. Also, $\mathcal{D}(A) = \mathcal{S}$ is dense in $C_0(\mathbb{R}^n)$ since $\mathcal{S} \supset C_c^\infty(\mathbb{R}^n)$.

From the Proposition 1.3 it follows that the infinitesimal generator for $(W_t)_{t \geq 0}$ is equal with $\overline{A|_{\mathcal{D}(A)}}$. Hence the infinitesimal generator for $(W_t)_{t \geq 0}$ is the closure of operator A.

<u>Remark 2.3.</u>

For the result from the Proposition 2.4 see [3] also.

The novelty taken by our proposition consists in the selection of the conditions in which the semigroup $(W_t)_{t \geq 0}$ have a canonical form and the choosing of the subspace $\mathcal{D}(A)$ such that it is stable at $(W_t)_{t \geq 0}$.

On the other hand this Proposition will be used in the next section.

3. Pseudo-differential Operators as Infinitesimal Generators of Feller Semigroups

This section contains the main result of this paper: the Theorem 3.1. In this theorem we give conditions in which the results of the Proposition 2.4 can be generalized to the symbols that depende also on x. This corresponds, for a differential operator, to the case of variable coefficients.

To the end of section we obtain, from our theorem as a corollary, one of the results contained in [4].

Theorem 3.1.

Let $\psi : \mathbb{R}^n \longrightarrow \mathbb{C}$ be with the properties:

(i) $\psi \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$ with $0 < m \leqslant 2$

(ii) $(\forall)\ x \in \mathbb{R}^n, \quad \psi(x, o) = 0$

(iii) $(\forall)\ x \in \mathbb{R}^n$, the function $\xi \longrightarrow \psi_x(\xi) := \psi(x, \xi)$ is negative definite.

(iv) There exist the constants $K > 0$ and $r > 0$ such that

$$(\forall)\ x \in \mathbb{R}^n,\ (\forall)\ \xi \in \mathbb{R}^n,\ \mathrm{Re}\ \psi(x, \xi) \geqslant K|\xi|^r$$

(v) $(\forall)\ \xi \in \mathbb{R}^n$, the function $x \longrightarrow \psi(x, \xi)$ is bounded.

Let the pseudo-differential operator of order m defined by:

$$\mathcal{D}(B) = \mathcal{S}$$

$$(\forall)\ f \in \mathcal{D}(B),\ Bf(x) = (2\pi)^{-n} \iint e^{i<x-y,\,\xi>}(-\psi(x, \xi))f(y)dyd\xi$$

Then there exists a Feller semigroup on $\mathbb{R}^n$ such that his infinitesimal generator is the closure of operator A, the semigroup being given, on $C_c^\infty(\mathbb{R}^n)$, by the following formula:

$$(\forall)\ t \geqslant 0,\ (\forall) f \in C_c^\infty(\mathbb{R}^n),\ P_t f = \lim_{m \to \infty} (W_{\frac{t}{m}})^m f,$$

$$W_t f(x) = (2\pi)^{-n} \iint e^{i<x-y,\,\xi> - t\psi(x, \xi)} f(y)\, dy\, d\xi$$

where $(W_t)_{t \geqslant 0}$ are regularizant pseudo-differential operators.

Remark 3.1. The observations from the Remark 2.2 are available for Theorem 3.1, also. If $r = m$, by the Definition 1.5, the condition

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(iv) implies that B is elliptic of order m.

The proof of Theorem 3.1.

Let $z \in \mathbb{R}^n$ be fixed. We define the operator A_z as follows:

$$\mathcal{D}(A_z) = \mathcal{D}(B)$$

$$(\forall) f \in \mathcal{D}(A_z),\ A_z f(x) = (2\pi)^{-n} \iint e^{i\langle x-y,\xi\rangle} a(z,\xi) f(y) dy d\xi$$

where $a(z,\xi) = -\psi_z(\xi)$.

From the assumptions (i) - (iv) it follows that $\psi_z \in S^m(\mathbb{R}^n)$, ψ_z is negative definite with $\psi_z(o) = 0$ and there are the constants $K > 0$ and $r > 0$ such that

$$(\forall)\ \xi \in \mathbb{R}^n,\ \operatorname{Re}\psi_z(\xi) \geqslant K|\xi|^r$$

Then, from the Proposition 2.4, it follows that the closure of operator A_z is the infinitesimal generator for the Feller semigroup on $\mathbb{R}^n$, which, for $f \in C_c^\infty(\mathbb{R}^n)$, is given by the formula

$$(\forall)\ t \geqslant 0,\ V_t^z f(x) = (2\pi)^{-n} \iint e^{i\langle x-y,\xi\rangle - t\psi_z(\xi)} f(y)\, dy\, d\xi$$

For $t \geqslant 0$, let W_t be the operator defined for all $x \in \mathbb{R}^n$ and all $f \in C_c^\infty(\mathbb{R}^n)$, by

$$W_t f(x) := V_t^x f(x),$$

i.e.

$$W_t f(x) = (2\pi)^{-n} \iint e^{i\langle x-y,\xi\rangle - t\cdot\psi(x,\xi)} f(y)\, dy\, d\xi$$

We notice that W_t have the properties:

(a) W_t is linear, positive and contraction

(b) $W_t(\mathcal{D}(B)) \subset \mathcal{D}(B)$

(c) $(\forall)\ f \in C_c^\infty(\mathbb{R}^n)$, the function $t \longrightarrow W_t f$ is continuous in $C_o(\mathbb{R}^n)$.

For to continue the proof we need of some supplementary results.

Lemma 3.1. Let $(S, \mathcal{D}(S))$ and $(T, \mathcal{D}(T))$ be the infinitesimal generators for the strongly continuous contraction semigroups $(e^{tS})_{t\geqslant 0}$ and $(e^{tT})_{t\geqslant 0}$ in the Banach space $(X, \|\ \|)$ such that

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$\mathfrak{D}(S) = \mathfrak{D}(T) = X$. S and T are linear and continuous on X.

Then we have the inequality:

$$\left\| e^{tS}x - e^{tT}x \right\| \leqslant t\left(\sup_{\|y\|\leqslant 1} \|Sy - Ty\| \right) \|x\|$$

for all $t \geqslant 0$ and $x \in X$.

Proof. We have the following relation for any $x \in X$:

(15) $$\left\| e^{tS}x - e^{tT}x \right\| \leqslant \left\| e^{tS} - e^{tT} \right\|_{L(X)} \cdot \|x\|.$$

$$\left\| e^{tS} - e^{tT} \right\|_{L(X)} = \left\| \left(e^{\frac{t}{n}S}\right)^n - \left(e^{\frac{t}{n}T}\right)^n \right\|_{L(X)} =$$

$$= \Big\| \left(e^{\frac{t}{n}S}\right)^{n} - \left(e^{\frac{t}{n}S}\right)^{n-1} \left(e^{\frac{t}{n}T}\right) + \left(e^{\frac{t}{n}S}\right)^{n-1} \left(e^{\frac{t}{n}T}\right) - \left(e^{\frac{t}{n}S}\right)^{n-2} \left(e^{\frac{t}{n}T}\right)^2 +$$
$$+ \left(e^{\frac{t}{n}S}\right)^{n-2} \left(e^{\frac{t}{n}T}\right)^2 - \dots - \left(e^{\frac{t}{n}S}\right) \left(e^{\frac{t}{n}T}\right)^{n-1} + \left(e^{\frac{t}{n}S}\right)\left(e^{\frac{t}{n}T}\right)^{n-1} - \left(e^{\frac{t}{n}T}\right)^{n} \Big\|_{L(X)}$$

$$= \Big\| \left(e^{\frac{t}{n}S}\right)^{n-1} \left(e^{\frac{t}{n}S} - e^{\frac{t}{n}T}\right) + \left(e^{\frac{t}{n}S}\right)^{n-2} \left(e^{\frac{t}{n}S} - e^{\frac{t}{n}T}\right)\left(e^{\frac{t}{n}T}\right) + \dots +$$
$$+ \left(e^{\frac{t}{n}S} - e^{\frac{t}{n}T}\right) . \left(e^{\frac{t}{n}T}\right)^{n-1} \Big\|_{L(X)} \leqslant n \left\| e^{\frac{t}{n}S} - e^{\frac{t}{n}T} \right\|_{L(X)}$$

$$\left\| e^{\frac{t}{n}S} - e^{\frac{t}{n}T} \right\|_{L(X)} = \sup_{\|y\|\leqslant 1} \left\| e^{\frac{t}{n}S}y - e^{\frac{t}{n}T}y \right\|$$

Hence $$\left\| e^{tS} - e^{tT} \right\|_{L(X)} \leqslant n \sup_{\|y\|\leqslant 1} \left\| e^{\frac{t}{n}S}y - e^{\frac{t}{n}T}y \right\| = t \sup_{\|y\|\leqslant 1} \left\| \frac{e^{\frac{t}{n}S}y - y}{\frac{t}{n}} - \frac{e^{\frac{t}{n}T}y - y}{\frac{t}{n}} \right\|$$

If n tends to infinity, it follows that:

$$\left\| e^{tS} - e^{tT} \right\|_{L(X)} \leqslant t \sup_{\|y\|\leqslant 1} \|Sy - Ty\|$$

From (15) it follows the desired relation.

Lemma 3.2. (Chernoff)

Let T be a linear and continuous operator on the Banach space $(X, \|\ \|)$ such that $\|T\| \leqslant 1$.

Then

$$\left\| e^{n(T-I)}x - T^n x \right\| \leqslant \sqrt{n}\, \|Tx - x\|$$

for any $n \geqslant 1$ and for any $x \in X$.

Proof. First we observe that

$$\left\| e^{t(T-I)} \right\| = e^{-t} \left\| \sum_{n=0}^{\infty} \frac{t^n}{n!} T^n \right\| \leqslant e^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} \|T\|^n \leqslant 1$$

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Let $x \in X$ be.

$$\| e^{n(T-I)}x - T^n x \| = \| e^{-nI} e^{nT} x - T^n x \| = \| e^{-n} . I \sum_{k=0}^{\infty} \frac{n^k T^k}{k!} x - T^n x \| \leqslant$$

$$\leqslant e^{-n} \sum_{k=0}^{\infty} \frac{n^k}{k!} \quad \| T^k x - T^n x \|$$

Let $k \geqslant n$ be

$$\| T^k x - T^n x \| = \| \sum_{j=n}^{k-1} (T^{j+1} x - T^i x) \| \leqslant \sum_{j=n}^{k-1} \| T^j (T-I) x \| \leqslant |k-n| \cdot \| Tx - x \| .$$

Hence $\| e^{n(T-I)}x - T^n x \| \leqslant e^{-n} \sum_{k=0}^{\infty} \frac{n^k}{k!} | k - n | . \| Tx - x \|$

From the Cauchy-Schwartz inequality we deduce:

$$\sum_{k=0}^{\infty} \frac{n^k}{k!} |k - n| \leqslant \left(\sum_{k=0}^{\infty} \frac{n^k}{k!} \right)^{\frac{1}{2}} \quad \left(\sum_{k=0}^{\infty} \frac{n^k}{k!} (n-k)^2 \right)^{\frac{1}{2}}$$

Since $\sum_{k=0}^{\infty} \frac{n^k}{k!} (n - k)^2 = ne^n$ it follows:

$$\| e^{n(T-I)}x - T^n x \| \leqslant \sqrt{n} \cdot \| Tx - x \| .$$

Lemma 3.3.

For any $t \geqslant 0$ and any $f \in C_c^{\infty} (\mathbb{R}^n)$,

$$(16) \quad \lim_{\substack{m \to \infty \\ p \to \infty}} \| (W_{\frac{t}{m}})^m f - (W_{\frac{t}{p}})^p f \| = 0$$

in $C_o(\mathbb{R}^n)$, uniformly on compact t intervals.

Proof. Let $f \in C_c^{\infty} (\mathbb{R}^n)$ and $t > 0$. From Lemma 3.2 it follows:

$$(17) \quad \| (W_{\frac{t}{m}})^m f - e^{m(W_{\frac{t}{m}} - I)} f \| \leqslant \sqrt{m} \| (W_{\frac{t}{m}} - I) f \|$$

$$(18) \quad \| (W_{\frac{t}{p}})^p f - e^{p(W_{\frac{t}{p}}) - I)} f \| \leqslant \sqrt{p} \| (W_{\frac{t}{p}} - I) f \| ; \| (W_{\frac{t}{m}} - I) f \| =$$

$$= \sup_{x \in \mathbb{R}^n} \left| (2\pi)^{-n} \int e^{i\langle x, \xi \rangle} \left[e^{-\frac{t}{m} \psi(x, \xi)} - 1 \right] \hat{f}(\xi) \, d\xi \right| \leqslant$$

$$\leqslant \sup_{x \in \mathbb{R}^n} \int \frac{t}{m} | \psi(x, \xi) | . | \hat{f}(\xi) | \, d\xi \leqslant \frac{t}{m} C \int (1 + |\xi|)^2 | \hat{f}(\xi) | \, d\xi$$

$= \frac{t}{m} C \| \delta \hat{f} \|_{L^1}$, where $\delta(\xi) := (1 + |\xi|)^2$, $\xi \in \mathbb{R}^n$ and C is

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a positive constant (we have used the assumption (v), also). Analogous,

$$\left\| (W_{\frac{t}{p}} - I) f \right\| \leqslant \frac{t}{p} C \; \left\| \delta \hat{f} \right\|_{L^1}$$

It follows that each left side of the inequalities (17) and (18) tends to zero, uniformly on compact t intervals, when $m \to \infty$ and $p \to \infty$.

From Lemma 3.1 it follows that

$$(19) \quad \left\| e^{t \frac{W_{\frac{t}{m}} - I}{\frac{t}{m}}} f - e^{t \frac{W_{\frac{t}{p}} - I}{\frac{t}{p}}} f \right\| \leqslant t \left(\sup_{\|g\| \leqslant 1} \left\| \frac{W_{\frac{t}{m}} - I}{\frac{t}{m}} g - \frac{W_{\frac{t}{m}} - I}{\frac{t}{m}} g \right\| \right) \|f\|$$

For any $g \in C_c^\infty(\mathbb{R}^n)$ with $\|g\| \leqslant 1$ we have

$$\left\| \frac{W_{\frac{t}{m}} - I}{\frac{t}{m}} g - \frac{W_{\frac{t}{p}} - I}{\frac{t}{p}} g \right\| = \sup_{x \in \mathbb{R}^n} \left| \frac{W_{\frac{t}{m}} - I}{\frac{t}{m}} g(x) - \frac{W_{\frac{t}{p}} - I}{\frac{t}{p}} g(x) \right| \leqslant$$

$$\sup_{x \in \mathbb{R}^n} \left| \int e^{i\langle x, \xi \rangle} \left[\frac{e^{-\frac{t}{m}\psi(x,\xi)} - 1}{\frac{t}{m}} - \frac{e^{-\frac{t}{p}\psi(x,\xi)} - 1}{\frac{t}{p}} \right] \hat{g}(\xi) \, d\xi \right| \leqslant$$

$$\sup_{x \in \mathbb{R}^n} \int \left| \frac{e^{-\frac{t}{m}\psi(x,\xi)} - 1}{\frac{t}{m}} - \frac{e^{-\frac{t}{p}\psi(x,\xi)} - 1}{\frac{t}{p}} \right| \left| \hat{g}(\xi) \right| d\xi$$

Passing to limit when $m \to \infty$ and $p \to \infty$ it follows that:

$$\left\| \frac{W_{\frac{t}{m}} - I}{\frac{t}{m}} g - \frac{W_{\frac{t}{p}} - I}{\frac{t}{p}} g \right\| \longrightarrow 0,$$

and we obtain

$$(20) \quad \left\| e^{t \frac{W_{\frac{t}{m}} - I}{\frac{t}{m}}} f - e^{t \frac{W_{\frac{t}{p}} - I}{\frac{t}{p}}} f \right\| \longrightarrow 0$$

and from (19) it follows that the convergence is uniform on compact t intervals.

From the relations (17), (18) and (20) we deduce the desired

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relation (16) and the uniform convergence on compact t intervals.

<u>The proof of Theorem 3.1. - sequel</u>

Using Lemma 3.3 it follows that for any $t > 0$ and for any $f \in C_c^{\infty}(\mathbb{R}^n)$, $(W_{\frac{t}{m}})^m f$ converges uniform on compact t intervals to an element from $C_o(\mathbb{R}^n)$, denoted $P_t f$. Since $(W_{\frac{t}{m}})^m$ are bounded by 1 and $C_c^{\infty}(\mathbb{R}^n)$ is dense in $C_o(\mathbb{R}^n)$, the above convergence takes place for all $f \in C_o(\mathbb{R}^n)$.

Moreover considering the relation (c) and the fact that $P_o = I$ we can apply Theorem 1.3. It follows that $(P_t)_{t \geqslant 0}$ is a strongly continuous contraction semigroup on $\mathbb{R}^n$.

From (a) we deduce that $(P_t)_{t \geqslant 0}$ is a Feller semigroup.

In the sequel we shall prove that the infinitesimal generator for $(P_t)_{t \geqslant o}$ is $(B, \mathcal{D}(B))$. We denote with $(C, \mathcal{D}(C))$ the infinitesimal generator for $(P_t)_{t \geqslant o}$. From Theorem 1.4 it follows that $C \supset \frac{dW_t}{dt}\Big|_{t=o}$. Since $\frac{dW_t}{dt}\Big|_{t=o} \supset B$ it follows that $C \supset B$, whence $\overline{B} \subset C$. Now, we shall prove that $\overline{B} \supset C$. Since the composition of two operators from $PS(-\infty)$ is an operator from $PS(-\infty)$, there exists $b_m^t \in S^{-\infty}$ such that

$$(W_{\frac{t}{m}})^m f(x) = (2\pi)^{-n} \int e^{i\langle x, \xi\rangle} \, b_m^t(x, \xi)\, \hat{f}(\xi)\, d\xi ,$$

for any $f \in \mathcal{S}$ and any $t \geqslant 0$. Since $\lim_{m \to \infty} (W_{\frac{t}{m}})^m f$ exists and it is equal with $P_t f$, it follows that:

$$P_t f(x) = (2\pi)^{-n} \int e^{i\langle x, \xi\rangle} \Big[\lim_{m \to \infty} b_m^t(x, \xi)\Big] \hat{f}(\xi)\, d\xi$$

We deduce that $P_t f \in \mathcal{S}$, for any $f \in \mathcal{S}$, i.e. $P_t(\mathcal{D}(B)) \subset \mathcal{D}(B)$. $\mathcal{D}(B)$ is dense in $C_o(\mathbb{R}^n)$ since $\mathcal{S} \supset C_c^{\infty}(\mathbb{R}^n)$. From the Proposition 1.3 it follows that C is equal with $\overline{B\big|_{\mathcal{D}(B)}}$. Hence the infinitesimal generator for the semigroup $(P_t)_{t \geqslant o}$ is the closure of operator B.

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As a consequence of the Theorem 3.1 we obtain a result due to Roth in [4]: the characterization of elliptic operators as infinitesimal generators of Feller semigroups.

Corollary 3.1. Let $Q(x) = [q_{ij}(x)]$ be a symmetric square matrix of order n, with the elements real bounded functions of class C^{∞} on R^n, such that there exists $a > 0$ with the property

$$(\forall)\ x \in \mathbb{R}^n,\ (\forall)\ \xi \in \mathbb{R}^n,\ Q(x)\xi \geqslant a|\xi|^2$$

where we have identified the matrix $Q(x)$ with the bi-linear form associated on $\mathbb{R}^n$.

Let $l(x) = [l_i(x)]$ be a linear form on $\mathbb{R}^n$ with the elements real bounded functions of class C^{∞}, where we have identified the vector $[l_i(x)]$ with the linear form associated on $\mathbb{R}^n$.

Let B be the operator defined by:

$$\mathcal{D}(B) = \mathcal{S}$$

$$Bf = \sum_{i=1}^{n} \sum_{j=1}^{n} q_{ij}(x)\, \partial_i \partial_j f + \sum_{i=1}^{n} l_i(x)\, \partial_i f$$

Then there exists a Feller semigroup on $\mathbb{R}^n$ such that his infinitesimal generator is the closure of operator B, the semigroup being by the following formula:

$$(\forall)\ t > 0,\ (\forall)\ f \in C_0(\mathbb{R}^n),\ P_t f = \lim_{m \to \infty} (W_{\frac{t}{m}})^m f,$$

$$W_t f(x) = \frac{1}{\pi^{\frac{n}{2}}} \int e^{-|u|^2} f(x + tl(x) - 2\sqrt{t}\, P(x)u)\, du$$

where $P^t(x)P(x) = Q(x)$, $(\forall)\ x \in \mathbb{R}^n$.

Proof. We apply the Theorem 3.1. for the function $\psi : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{C}$ given by the relation

$$\psi(x, \xi) = \sum_{i=1}^{n} \sum_{j=1}^{n} q_{ij}(x)\, \xi_i \xi_j - i \sum_{i=1}^{n} l_i(x)\, \xi_i$$

The conditions (i) - (v) from the Theorem 3.1 are satisfied. We see that:

$$Bf(x) = -\sum_{i=1}^{n} \sum_{j=1}^{n} q_{ij}(x)\, D_i D_j f(x) + i \sum_{i=1}^{n} l_i(x)\, D_i f(x) =$$

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$$= (2\pi)^{-n} \iint e^{i\langle x-y,\xi\rangle}\Big(-\sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij}(x)\,\xi_i\,\xi_j + i\sum_{i=1}^{n} l_i(x)\,\xi_i\Big).$$

$$.f(y)\,dy\,d\xi = (2\pi)^{-n}\iint e^{i\langle x-y,\xi\rangle}\,(-\psi(x,\xi))\,f(y)\,dy\,d\xi$$

Then from the Theorem 3.1, there exists a Feller semigroup on $\mathbb{R}^n$ such that his infinitesimal generator is the closure of operator B, the semigroup being given, on $C_c^{\infty}(\mathbb{R}^n)$, by the formula (21). ψ can be written: $\psi(x,\xi) = \langle Q(x)\xi,\xi\rangle - i\langle l(x),\xi\rangle$.

It proves ([4]) that for any matrix Q which verifies the above properties, there exists a square matrix P, of order n, such that:

$$(\forall)\ x \in \mathbb{R}^n,\ Q(x) = P^t(x)\,P(x)$$

Then $W_t f(x) = (2\pi)^{-n}\iint e^{i\langle x-y+tl(x),\xi\rangle - t\langle P(x)\xi,\,P(x)\xi\rangle} f(y)\,dy\,d\xi$.

By the change of variable $\sqrt{t}\,P(x)\xi = v$, we obtain

$$W_t f(x) = (2\pi)^{-n}\int\Big(\int e^{i\langle x-y+tl(x),\frac{1}{\sqrt{t}}P^{-1}(x)v\rangle - |v|^2}dv\Big)\,\frac{1}{(\sqrt{t})^n}\,|P^{-1}(x)|.$$

$$.\ f(y)\,dy =$$

$$= (2\pi)^{-n}\int \pi^{\frac{n}{2}}\,e^{-\frac{\left|\frac{1}{\sqrt{t}}P^{-1}(x)(x-y+tl(x))\right|^2}{4}}\ .\ \frac{1}{(\sqrt{t}\,)^n}|P^{-1}(x)|\,f(y)\,dy$$

If $\frac{1}{\sqrt{t}}\ P^{-1}(x)\ (x-y+tl(x)) = u$ then

$$\frac{1}{(2\sqrt{t})^n}\ |P^{-1}(x)|\ dy = du.\ \text{Hence}$$

$$W_t f(x) = (2\pi)^{-n}\ \pi^{\frac{n}{2}}\ .\ 2^n\int e^{-|u|^2} f(x + tl(x) - 2\sqrt{t}\,P(x)u)\,du$$

$$= \frac{1}{\pi^{\frac{n}{2}}}\int e^{-|u|^2}\ f(x + tl(x) - 2\sqrt{t}\,P(x)\,u)\,du.$$

Certainly, this expression of W_t can be extended for all $f \in C_o(\mathbb{R}^n)$.

Corollary 3.2. In the conditions of Theorem 3.1, the operator $\bar{B}$ satisfies the positive maximum principle: (in $C_0(\mathbb{R}^n,\mathbb{R})$)

$$(\forall)\ u \in \mathcal{D}(\bar{B})\ \text{and}\ x \in \mathbb{R}^n,\ u(x) = \sup u \geq 0 \Rightarrow \bar{B}u(x) \leq 0$$

Proof. Since $\bar{B}$ is the infinitesimal generator of a Feller semigroup on $\mathbb{R}^n$, we write (7).

/...

Remark 3.2. Instead of the Banach space $C_o(\mathbb{R}^n)$ we can consider any Banach space E such that

(i) $C_c^{\infty}(\mathbb{R}^n)$ is dense in E

(ii) $\mathcal{S} \subset E$

(for example, L^p with $p \geqslant 1$).

Certainly, then we shall obtain strongly continuous contraction semigroup of positive operators.

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