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by

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STATEMENT OF THE RESULT

Let A be a unital AF C^*-algebra, inductive limit of the finite dimensional algebras

$$\mathbb{C}\cdot 1 \subset A_1 \subset A_2 \subset A_3 \subset \ldots$$

(1 is the unit of A).

We denote by $m_k = (m_1^k, m_2^k, \ldots, m_{c_k}^k)$ the dimension vector of the algebra A_n and by $R_k = (r_{ij}^k)_{i=1,\ldots,c_k; j=1,\ldots,c_{k+1}}$ the inclusion matrix for $A_k \subset A_{k+1}$ $(k \geq 1)$. In particular, ${}^tR_k m_k = m_{k+1}$.

If w is a real vector, $w \geq 0$ means that its entries are nonnegative.

For $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$, $w \geq 0$, $w \neq 0$, we define

$$\chi(w) := \left(\sum_{k=1}^{n} w_k\right)^{-1} \cdot \min\left\{ \sum_{i\in I} w_i \,\middle|\, I \subset \{1,2,\ldots,n\},\ \mathrm{card}(I) \geq n/2 \right\}.$$

We consider the multiplicative group $G = \bigcup_{k=1}^{\infty} \mathcal{U}(A_k)$ and its action on A by inner automorphisms,

$$g \in G \overset{\theta}{\longmapsto} \mathrm{Ad}\, g \in \mathrm{Int}(A) \subset \mathrm{Aut}\,(A)$$

$$g(x) \equiv (\mathrm{Ad}\, g)(x) = gxg^{-1} \quad (g \in G,\ x \in A).$$

We prove the following:

THEOREM. With the notations introduced above, let

$$\varepsilon_k := \min_{j=1,\ldots,c_{k+1}} \chi\left((m_i^k r_{ij}^k)_{i=1,\ldots,c_k}\right) \quad (k \geq 1).$$

If

$$(*) \quad \sum_{k=1}^{\infty} \varepsilon_k = \infty,$$

then:

(a) There is a unique normalized trace, denoted by τ, on A;

(b) τ is faithful if and only if A is simple;

(c) The action ϑ is mixing with respect to the trace τ, i.e.

$(\forall)\ x,y \in A_h$, $(\exists) g_n \in G$ $(n \in N)$ such that

$$\lim_{n\to\infty} \tau(g_n(x)y) = \tau(x)\tau(y).$$

There are conditions which imply $(*)$ and depend only on the inclusion matrices R_k.

COROLLARY. With the R_k's introduced above, let $\delta_k := \min_{i,j} r_{ij}^k / \max_{i,j} r_{ij}^k$ $(i = 1,\dots,c_k;$ $j = 1,\dots,c_{k+1})$

$$\tilde{\varepsilon}_k := \min_{j=1,\dots,c_{k+1}} \chi\left((r_{ij}^k)_{i=1,\dots,c_k} \right).$$

If

(1) $$\sum_{k=2}^{\infty} \delta_{k-1} \tilde{\varepsilon}_k = \infty,$$

or

(2) $$\sum_{k=2}^{\infty} \delta_{k-1} \delta_k = \infty,$$

then:

(i) The algebra A is simple and has a unique normalized trace τ, which is faithful;

(ii) The action θ is mixing with respect to the trace τ.

Namely, we shall prove that $(2) \Rightarrow (1) \Rightarrow (*)$.

REMARK 1. Condition $(*)$ depends effectively on the particular sequence of algebras A_n defining A. Indeed, let $m_1 = (1,1,1,1)$, and for $k \geq 1$,

$$R_{2k-1} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \qquad R_{2k} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

hence

$$R_{2k-1}R_{2k} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Then the sum in (∗) is zero for the sequence $A_1 \subset A_2 \subset A_3 \subset \dots$, but it is infinite for the sequence $A_1 \subset A_3 \subset A_5 \subset \dots$.

REMARK 2. Condition (∗) does not imply any of the equivalent conditions in (b): let $m_1 = (1,1,1)$, and

$$R_k = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \qquad \text{for all } k \geq 1$$

Then $\mathcal{E}_k = 1/3$, but the (unique) trace on A has the weights $(1/2\cdot 3^{-k+1}, 0, 1/2\cdot 3^{-k+1})$ on A_k, hence it is not faithful. One can also see from the Bratteli diagram that A is not simple.

REMARK 3. As a special case of the Corollary, (part (i)), we can treat the situation dealt with in a theorem of Elliott (Th. 6.1 in [2]), namely when $R_k = R$ for all k, where R is a primitive matrix, i.e. there is a nonzero p such that R^p has positive entries. Indeed, if we consider the sequence

$$A_1 \subset A_{p+1} \subset A_{2p+1} \subset A_{3p+1} \subset \dots$$

(which also defines A), the inclusion matrices will be constantly R^p, hence the δ_k's will be all equal and nonzero (because R^p has no zero entry), and then clearly (2) holds.

The proof of Elliott follows different ideas.

NOTATIONS AND SKETCH OF THE PROOF

Let $A_n = \bigoplus_{l=1}^{c_n} A_n^l$, $A_n^l \cong \mathrm{Mat}_{m_l^n}(\mathbb{C})$ be the factor decomposition of the A_n's.

For $x \in A_n$, we denote by $[x]_n^1$ its A_n^1-component and by $\alpha_n^1(x)$ the normalized trace of $[x]_n^1 \in A_n^1$:

$$\alpha_n^1(x) = \mathrm{tr}([x]_n^1) = 1/m_\ell^n\ \mathrm{Tr}([x]_n^1)$$

(we denote by Tr the canonical trace on a full matrix algebra -i.e. the sum of all diagonal entries- and by tr the normalized one).

If $v = (v_1, \ldots, v_k) \in \mathbb{C}^k$ is a vector, we write $\omega(v)$ for the "oscillation" of v, i.e.

$$\omega(v) := \max_{i,j=1,\ldots,k} | v_i - v_j |.$$

Now for any $x \in A_n$, we introduce the vector $\alpha_n(x) := (\alpha_n^1(x), \alpha_n^2(x), \ldots, \alpha_n^{c_n}(x))$ and the value $\omega(\alpha_n(x))$.

We shall prove that for any $x \in A_\infty := \bigcup_{n=1}^{\infty} A_n$, $\lim_{n\to\infty} \omega(\alpha_n(x)) = 0$, i.e. the entries of $\alpha_n(x)$ tend to become equal. It is here that we use condition (*). This implies that the entries of $\alpha_n(x)$ tend to be all equal to some complex number $\tau(x)$ (Lemma 3). Then the map

$$x \in A_\infty \longmapsto \tau(x) \in \mathbb{C}$$

defines a tracial state on A_∞ which is the unique tracial state on A_∞ (Lemma 4). Assertion (b) of the Theorem is proved in Lemma 5, assertion (c) in Lemma 6 after some preparation, and the Corollary in Lemma 7.

We emphasize that the whole proof depends on the fact that $\lim_{n\to\infty} \omega(\alpha_n(x)) = 0$. This is deduced from condition (*) by the estimate given in Lemma 2. One can look for other estimates in order to obtain the same fact from other conditions. Our estimate is insensitive to the equality of all rows of Q, when $\| Q \|_\omega = 0$, regardless of ε (see the notations in Lemma 2). We have chosen it because of its relative simplicity.

THE PROOF

First of all we clarify how the inclusion matrices R_k and the dimension vectors m_k allow the computation of $\alpha_{n+1}(x)$ from $\alpha_n(x)$. Let $Q_n = (q_{ij}^n)_{i=1,\ldots,c_{n+1};\, j=1,\ldots,c_n}$ be the matrix given by $q_{ij}^{n} = m_j^n r_{ji}^n / m_i^{n+1}$, i.e.

$$Q_n = \begin{pmatrix} m_1^{n+1} & & 0 \\ & \ddots & \\ 0 & & m_{c_{n+1}}^{n+1} \end{pmatrix}^{-1} \cdot {}^tR_n \cdot \begin{pmatrix} m_1^n & & 0 \\ & \ddots & \\ 0 & & m_{c_n}^n \end{pmatrix}$$

and $1_m = (1,1,\ldots,1) \in \mathbb{C}^m$. Note that $Q_n(1_{c_n}) = 1_{c_{n+1}}$ because $m_l^{n+1} = \sum_{k=1}^{c_n} m_k^n \; r_{kl}^n$ $(l = 1,\ldots,c_{n+1})$.

LEMMA 1. For any $x \in A_n$ we have

(a) $\alpha_{n+1}(x) = Q_n \alpha_n(x)$;

(b) $\min\limits_{1 \le k \le c_n} \operatorname{Re} \alpha_n^k(x) \le \operatorname{Re} \alpha_{n+1}^l(x) \le \max\limits_{1 \le k \le c_n} \operatorname{Re} \alpha_n^k(x)$;

$\min\limits_{1 \le k \le c_n} \operatorname{Im} \alpha_n^k(x) \le \operatorname{Im} \alpha_{n+1}^l(x) \le \max\limits_{1 \le k \le c_n} \operatorname{Im} \alpha_n^k(x)$ for all $l = 1,\ldots,c_{n+1}$.

Proof. (a) Using the information given by the inclusion matrix, it follows that

$$\alpha_{n+1}^l(x) = \mathrm{Tr}([x]_{n+1}^l)/m_l^{n+1} = \Big(\sum_{k=1}^{c_n} r_{kl}^n \, \mathrm{Tr}([x]_n^k)/m_l^{n+1} = \Big(\sum_{k=1}^{c_n} m_k^n r_{kl}^n \alpha_n^k(x)\Big)/m_l^{n+1} .$$

(b) This is a consequence of the relation $Q_n(1_{c_n}) = 1_{c_{n+1}}$ and of the fact that Q_n has real nonnegative entries (hence $\alpha_{n+1}^l(x)$ is a weighted average of the entries of $\alpha_n(x)$).

Let us study the matrices $Q = (q_{ij})_{i=1,\ldots,n;j=1,\ldots,m}$ with real nonnegative entries which satisfy

$$Q(1_m) = 1_n.$$

Note that if $v \in R^m$ and $\omega(v) = 0$, then $\omega(Q(v)) = 0$ ($\omega(w) = 0 \Leftrightarrow w$ is proportional to the vector 1_m). Since ω defines a seminorm on any R^p, from the above remark we see that Q induces a liniar map $Q : R^m/\omega \longrightarrow R^n/\omega$, where R^p/ω denotes the quotient space $R^p/\{v \in R^p / \omega(v) = 0\}$. Hence

$$\| Q \|_{\omega} := \sup \{ \omega(Q(v)) \mid v \in \mathbb{R}^n, \omega(v) \leq 1 \}$$

is finite. Clearly

$$\omega(Q(v)) \leq \| Q \|_{\omega} \cdot \omega(v),$$

and

$$\| Q_1 Q_2 \|_{\omega} \leq \| Q_1 \|_{\omega} \cdot \| Q_2 \|_{\omega}$$

whenever $Q_1 Q_2$ is defined.

LEMMA 2. Let $Q = (q_{ij})_{i=1,\dots,n;j=1,\dots,m}$ be a matrix with real nonnegative entries which satisfies $Q(1_m) = 1_n$. Then

$$\| Q \|_{\omega} \leq 1 - \mathcal{E},$$

where

$$\mathcal{E} := \min_{i=1,\dots,n} \chi((q_{ij})_{j=1,\dots,m}).$$

Proof. It is enough to show that if $v = (v_1, \dots, v_m) \in \mathbb{R}^m$, $w = (w_1, \dots, w_m) \in \mathbb{R}^m$ are such that $v \geq 0$, $w \geq 0$, $\sum_{k=1}^{m} v_k = 1$, $\sum_{k=1}^{m} w_k = 1$, $\chi(v) \geq \mathcal{E}$, $\chi(w) \geq \mathcal{E}$, then

$$| \langle \alpha, v \rangle - \langle \alpha, w \rangle | \leq (1 - \mathcal{E}) \cdot \omega(\alpha)$$

for any $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$, where $\langle \cdot, \cdot \rangle$ stands for the canonical scalar product of $\mathbb{R}^m$. The desired result will then follow by considering $v = (q_{ik})_{k=1,\dots,m}$, $w = (q_{jk})_{k=1,\dots,m}$ for all $1 \leq i,j \leq n$.

Let $a = \min_k \alpha_k$, $b = \max_k \alpha_k$, $I = [a,b]^m$. Then $\alpha \in I$, $\omega(\alpha) = b - a$. Since the map $f : I \to \mathbb{R}$, $f(u) := \langle u,v \rangle - \langle u,w \rangle$ is an affine map, $f(I) = \mathrm{co}\, f(\mathrm{ext}(I))$, where ext I denotes the set of extreme points of I and co stands for the convex hull.

Let $\beta \in \mathrm{ext}\, I$, $\beta = (\beta_1, \dots, \beta_m)$. Then $\beta_k \in \{a,b\}$ for any $k = 1, \dots, m$. Denote

$$K_a = \{ k \mid 1 \leq k \leq m, \beta_k = a \}, \quad K_b = \{ k \mid 1 \leq k \leq m, \beta_k = b \}.$$

One of the sets K_a and K_b has at least n/2 elements. Suppose card $K_a \geq n/2$. Since $\langle \beta, w \rangle \geq a$, we have

$$f(\beta) = (a \sum_{k \in K_a} v_k + b \sum_{k \in K_b} v_k) - \langle \beta, w \rangle =$$

$$= [b - (b - a)\sum_{k\in K_a} v_k] - \langle\beta, w\rangle \leq b - (b - a)\chi(v) - a \leq (1 - \varepsilon)(b - a).$$

For v instead of w we also obtain

$$f(\beta) \geq -(1 - \varepsilon)(b - a)$$

The case card $K_b \geq n/2$ can be treated similarly and we obtain the same results.

Thus for any $\beta \in \mathrm{ext}\, I$ we have

$$-(1 - \varepsilon)(b - a) \leq f(\beta) \leq (1 - \varepsilon)(b - a)$$

hence $\qquad f(I) \subset [-(1 - \varepsilon)(b - a), (1 - \varepsilon)(b - a)]$

and so

$$|f(\alpha)| \leq (b - a)(1 - \varepsilon) = \omega(\alpha)(1 - \varepsilon)$$

Recall that $\prod_{n=1}^{\infty}(1 - \eta_n) = 0$ whenever $0 \leq \eta_n \leq 1$ and $\sum_{n=1}^{\infty}\eta_n = \infty$. Therefore, by condition $(*)$ we have

$$(**) \qquad \prod_{n=n_o}^{\infty}(1 - \varepsilon_n) = 0 \quad (\forall)\, n_o \geq 1.$$

Note that due to Lemma 1(a), the ε_n's defined in the Theorem have the same meaning for the matrices Q_n as ε for the matrix Q in Lemma 2.

For $v = (v_1, \dots, v_m) \in \mathbb{C}^m$, define $\|v\|_\infty = \max_{k=1,\dots,m} |v_k|$

Now we can prove the following:

LEMMA 3. For any $x \in A_{n_o}$ we have

(a) $\lim_{n\to\infty} \omega(\alpha_n(x)) = 0$;

(b) $\lim_{n\to\infty} \|\alpha_n(x) - \tau(x)\cdot 1_{c_n}\|_\infty = 0$ for some $\tau(x) \in \mathbb{C}$.

Proof. Let $n \geq n_o$. Since both ω and $\|\cdot\|_\infty$ are seminorms, we can deal separately with the real and imaginary parts of $\alpha_n(x)$. Denote by $\mathrm{Re}\,\alpha_n(x)$ and $\mathrm{Im}\,\alpha_n(x)$ the vectors whose entries are the real and respectively, the imaginary parts of the entries of $\alpha_n(x)$.

By Lemma 1(a), we see that

$$\mathrm{Re}\,\alpha_{n+1}(x) = Q_n(\mathrm{Re}\,\alpha_n(x)),\ \mathrm{Im}\,\alpha_{n+1}(x) = Q_n(\mathrm{Im}\,\alpha_n(x)).$$

Lemma 2 implies that

$$\omega\,(\mathrm{Re}\,\alpha_{n+1}(x)) \le \| Q_n \|_\omega \cdot \omega(\mathrm{Re}\,\alpha_n(x)) \le (1 - \varepsilon_n)\,\omega(\mathrm{Re}\,\alpha_n(x)).$$

Iterating, we get

$$(\mathrm{Re}\,\alpha_{n+1}(x)) \le \prod_{k=n_o}^{n} (1 - \varepsilon_k)\ \ \omega(\mathrm{Re}\,\alpha_{n_o}(x)) \text{ and then, by } (**),$$

$$\lim_{n\to\infty} \omega(\mathrm{Re}\,\alpha_n(x)) = 0.$$

Since

$$\omega(\mathrm{Re}\,\alpha_n(x)) = \max_{1\le l\le c_n} \mathrm{Re}\,\alpha_n^l(x) - \min_{1\le l\le c_n} \mathrm{Re}\,\alpha_n^l(x),$$

Lemma 1(b) implies that

$$\lim_{n\to\infty} \| \mathrm{Re}\,\alpha_n(x) - a\cdot 1_{c_n} \|_\infty = 0$$

for some $a \in \mathbb{R}$.

The vectors $\mathrm{Im}\,\alpha_n(x)$ can be treated similarily.

LEMMA 4. (a) The mapping

$$x \in A_\infty \xrightarrow{\tau} \tau(x) \in \mathbb{C}$$

is a continuous, normalized trace on A_∞ which can be extended by continuity to the whole A.

(b) Any normalized trace on A_∞ equals τ.

Proof. (a) The linearity follows from the fact that

$$\alpha_n(ax + by) = a\alpha_n(x) + b\alpha_n(y) \text{ for any } x,y \in A_n \text{ and } a,b \in \mathbb{C}.$$

It is easy to see that $\alpha_n(1) = 1_{c_n}$, hence $\tau(1) = 1$.

Since $|\ \mathrm{tr}([x]_n^l)| \le \| [x]_n^l \| \le \|x\|$, we see that $\| \alpha_n(x) \|_\infty \le \|x\|$, and hence $|\tau(x)| \le \|x\|$.

Similarily, $\alpha_n(x^*x) \ge 0$, hence $\tau(x^*x) \ge 0$.

That τ is a trace follows from the relation

$$\alpha_n(xy) = \alpha_n(yx) \text{ for any } x,y \in A_n,$$

which is a consequence of the definition of α_n.

(b) Let μ be any normalized trace on A. Since the factors A_n^l have unique normalized traces, the restriction of μ to the algebra A_n is described by a nonnegative vector $t_n = (t_n^1, t_n^2, \ldots, t_n^{c_n})$ with $\sum_{k=1}^{c_n} t_n^k = 1$: if $x \in A_n$, then

$$\mu(x) = \sum_{k=1}^{c_n} t_n^k \alpha_n^k(x)$$

Then for all $x \in A_n$,

$$|\mu(x) - \tau(x)| = \Big|\sum_{k=1}^{c_n} t_n^k [\alpha_n^k(x) - \tau(x)]\Big| \leq \sum_{k=1}^{c_n} t_n^k \|\alpha_n(x) - \tau(x)\cdot 1_{c_n}\|_\infty =$$

$$= \|\alpha_n(x) - \tau(x) 1_{c_n}\|_\infty .$$

Hence Lemma 3 (a) implies that $\mu(x) = \tau(x)$ for all $x \in A_\infty$.

LEMMA 5. Suppose (*) holds and τ is the above defined trace. Then τ is faithful if and only if the algebra A is simple.

Proof. Denote by e_n^l the minimal central projection of A_n corresponding to A_n^l.

It is known that A is simple if and only if for any $n \geq 1$ and any $1 \leq l \leq c_n$, there is a $p > n$ such that the inclusion matrix $R_{n,p} = (r_{ij}^{n,p})_{i=1,\ldots,c_n; j=1,\ldots,c_p}$ for $A_n \subset A_p$ has only nonzero entries on the l-th row (i.e. A_n^l "enters" in all factor summands of A_p)-just look at the description of the ideals in the Bratteli diagram of A_p.

Since

$$\alpha_p^i(e_n^l) = (r_{li}^{n,p} m_l^n)/m_i^p, \quad i = 1, \ldots, c_p,$$

we see that the above condition on the inclusion matrix is equivalent to the fact that $\alpha_p(e_n^l)$ has only nonzero entries.

Suppose first that τ is faithful. Choose $n \geq 1$ and $1 \leq l \leq c_n$. Since $\tau(e_n^l) \neq 0$, and

$$\lim_{p\to\infty} \|\alpha_p(e_n^l) - \tau(e_n^l)\cdot 1_{c_p}\|_\infty = 0,$$

we infer that for p large enough, all the entries of $\alpha_p(e_n^l)$ are nonzero. Thus by the above remark, A must be simple.

Suppose now that A is simple. It suffices to prove the faithfulness of τ on A_∞. Indeed, since $J = \{ x \in A \mid \tau(x^*x) = 0 \}$ is a bilateral ideal, it is known that ([1])

$$J = \operatorname{clos} \bigcup_{n=1}^{\infty} (J \cap A_n).$$

Since the support projection of any trace on A_n is central, it suffices to show that $\tau(e_n^l) > 0$ for all $l = 1, \ldots, c_n$.

Since A is simple, we know that there is a $p > n$ such that $\alpha_p(e_n^l)$ has no zero entries, hence

$$a_o = \min\{ \alpha_p^k(e_n^l) \mid k = 1, \ldots, c_p \} > 0.$$

From Lemma 1(b) we infer that $\tau(e_n^l) \geq a_o > 0$.

For proving the mixing property of τ we need two elementary and possible well known results which we record below.

For a finite dimensional C^*-algebra, with a fixed system of matrix units and $x \in N$, we denote by Diag(x) the set of values which are on the diagonal of x.

REMARK 4. Let $x \in \mathrm{Mat}_n(\mathbb{C}) \cong \mathcal{B}(\mathbb{C}^n)$, $x = x^*$.

Then there is a unitary $u \in \mathrm{Mat}_n(\mathbb{C})$ such that Diag((Ad u)(x)) has only one element (namely tr(x)). (This statement also holds for $x \neq x^*$ but its proof would be more intricate.)

To see this, notice first that since $x = x^*$, there is an orthogonal basis of $\mathbb{C}^n$ with respect to which x has diagonal form, hence the corresponding matrix has real entries. If we consider (Ad u)(x) instead of x, where u is the unitary that describes the change of coordinates, we may assume that $x \in \mathrm{Mat}_n(\mathbb{R})$.

We shall obtain the assertion by induction.

Let $n = 2$, $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{R})$. We define

$$u_t = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \in \mathcal{U}(\mathrm{Mat}_2(\mathbb{R})),\ t \in [0, \pi/2].$$

Since $(\mathrm{Ad}\, u_o)(x) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $(\mathrm{Ad}\, u_{\pi/2})(x) = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$, and $t \longmapsto (\mathrm{Ad}\, u_t)(x)$ is a continuous function with values in $\mathrm{Mat}_2(\mathbb{R})$, the Darboux property of it implies that there is a $t \in [0, \pi/2]$ such that $(\mathrm{Ad}\, u_t)(x)$ has equal diagonal entries. Moreover,

$$(3)\quad (\forall)\lambda \in \mathbb{R},\ \min\{a,d\} \le \lambda \le \max\{a,d\} \Rightarrow (\exists)t \in [0, \pi/2], \text{ such that } (\mathrm{Ad}\, u_t)(x) = \begin{pmatrix} \lambda & * \\ * & * \end{pmatrix}$$

The statement is proved for $n = 2$. Assume we have proved it for $n-1$, $n \ge 3$. Let $x = (a_{ij}) \in \mathrm{Mat}_n(\mathbb{R})$. If x has different diagonal entries, one of them differs from $\mathrm{tr}(x)$; let's say it is a_{11}. We may assume that $a_{11} < \mathrm{tr}(x)$. There must be an $i_o \neq 1$ such that $a_{i_o i_o} > \mathrm{tr}(x)$. We may consider $i_o = 2$.

Due to (3), there is a unitary

$$\tilde{u}_t = \begin{pmatrix} u_t & 0 \\ 0 & I_{n-2} \end{pmatrix} \in \mathrm{Mat}_n(\mathbb{R})$$

such that

$$x' := (\mathrm{Ad}\, \tilde{u}_t)(x) = \begin{pmatrix} \mathrm{tr}(x) & * \\ * & x'' \end{pmatrix}.$$

where $x'' \in \mathrm{Mat}_{n-1}(\mathbb{R})$. By the inductive assumption there is a $u'' \in \mathrm{Mat}_{n-1}(\mathbb{C})$ such that $\mathrm{Diag}((\mathrm{Ad}\, u'')(x''))$ has only one value, namely $\mathrm{tr}(x'')$. But $\mathrm{tr}(x'') = \mathrm{tr}(x)$, hence if

$$u' = \begin{pmatrix} 1 & 0 \\ 0 & u'' \end{pmatrix},$$

then $\mathrm{Diag}\,((\mathrm{Ad}\, u'\tilde{u}_t)(x))$ has only one value.

REMARK 5. Let N be a finite dimensional C^*-algebra with a fixed system of matrix units, and let μ be a normalized trace on N.

If $x,y \in N$ and y has a diagonal form, then

$$|\mu(xy) - \mu(x)\mu(y)| \leq \|y\| \Delta_N(x),$$

where $\Delta_N(x) = \max\{|a - a'| \mid a, a' \in \mathrm{Diag}(x)\}$.

This follows by an easy computation. Suppose that $N = \bigoplus_{i=1}^{m} \mathrm{Mat}_{n_i}(\mathbb{C})$, and let $t = (t_1, \ldots, t_m)$ be the vector of the weights of the minimal projections of the factor summands of N in the trace μ (so that $\sum_{i=1}^{m} n_i \cdot t_i = 1$). Let the diagonal entries of x and y be

$$a_1^1, a_2^1, \ldots, a_{n_1}^1, a_1^2, \ldots, a_{n_2}^2, \ldots, a_1^m, \ldots, a_{n_m}^m$$

and respectively

$$b_1^1, b_2^1, \ldots, b_{n_1}^1, b_1^2, \ldots, b_{n_2}^2, \ldots, b_1^m, \ldots, b_{n_m}^m$$

(the upper index indicates the factor summand of N).

Then

$$\mu(x) = \sum_{l=1}^{m} t_l \sum_{i=1}^{n_l} a_i^l,$$

$$\mu(y) = \sum_{k=1}^{m} t_k \sum_{j=1}^{n_k} b_j^k,$$

$$\mu(xy) = \sum_{k=1}^{m} t_k \sum_{j=1}^{n_k} a_j^k b_j^k$$

(because y has a diagonal form). Since $1 = \sum_{l=1}^{m} \sum_{i=1}^{n_l} t_l$,

$$|\mu(xy) - \mu(x)\mu(y)| = \left|\sum_{l=1}^{m} \sum_{k=1}^{m} \sum_{i=1}^{n_l} \sum_{j=1}^{n_k} t_l t_k (a_j^k b_j^k - a_i^l b_j^k)\right| \leq$$

$$\leq \left(\sum_{l=1}^{m} \sum_{k=1}^{m} \sum_{i=1}^{n_l} \sum_{j=1}^{n_k} t_l t_k\right) \cdot \max_{k,l,i,j} |a_j^k - a_i^l| \cdot \max_{k,j} |b_j^k| = \Delta_N(x) \cdot \|y\|.$$

LEMMA 6. Suppose (*) holds and τ is the trace in A given in Lemma 4. Then the action ϑ is mixing with respect to τ.

Proof. Choose the systems of matrix units in the A_n's such that the matrix units of A_n are sums of matrix units of A_{n+1} for all n.

Let $x,y \in A_\infty$, $x = x^*$, $y = y^*$. We may assume $x,y \in A_{n_0}$. Since y is selfadjoint, there is a $u_0 \in \mathcal{U}(A_{n_0})$ such that $(\mathrm{Ad}\, u_0)(y)$ is diagonal in the matrix units system of A_{n_0}; moreover, this will hold in all A_n, $n \geq n_0$.

From the Remark 4. we infer that for $n \geq n_0$, there is a $u_n \in \mathcal{U}(A_n)$ such that

$$\mathrm{Diag}\,([(\mathrm{Ad}\, u_n)(x)]^1_n) = \{\mathrm{tr}([x])^1_n\} = \{\alpha^1_n(x)\} \text{ for all } 1 = 1, \ldots, c_n.$$

Hence $\Delta_{A_n}((\mathrm{Ad}\, u_n)(x)) = \omega(\alpha_n(x))$. Since $\lim_{n\to\infty} \omega(\alpha_n(x)) = 0$ and $\tau((\mathrm{Ad}\, u)(x)) = \tau(x)$, by Remark 5. we see that

$$|\tau((\mathrm{Ad}\, u_0^* u_n)(x)y) - \tau(x)\tau(y)| = |\tau((\mathrm{Ad}\, u_n)(x)(\mathrm{Ad}\, u_0)(y)) - \tau((\mathrm{Ad}\, u_n)(x))\tau((\mathrm{Ad}\, u_0)(y))| \leq$$

$$\leq \|(\mathrm{Ad}\, u_0)(y)\| \Delta_{A_n}((\mathrm{Ad}\, u_n)(x)) = \|y\|\, \omega(\alpha_n(x)) \longrightarrow 0$$

as $n \to \infty$.

So we proved the mixing property for $x,y \in (A_\infty)_h$. That it also holds for any $x,y \in A_h$ can be proved using an obvious approximation argument.

LEMMA 7. (a) With the notations of the Corollary,

$$1/2\, \delta_{k-1} \cdot \delta_k \leq \delta_{k-1} \cdot \tilde{\varepsilon}_k \leq \varepsilon_k \qquad (k \geq 2),$$

hence $(2) \Rightarrow (1) \Rightarrow (*)$.

(b) If any of (1) or (2) holds, then the algebra A is simple, hence, by the Theorem, part (b), the unique normalized trace on A is faithful.

Proof. (a) Since $m_k = {}^t R_{k-1} m_{k-1}$, we see that

$$(\max_{i,j} r_{ij}^{k-1}) \cdot \sum_{l=1}^{c_{k-1}} m_l^{k-1} \geq \sum_{l=1}^{c_{k-1}} r_{lj}^{k-1} \cdot m_l^{k-1} = m_j^k \geq$$

$$\geq (\min_{i,j} r_{ij}^{k-1}) \cdot \sum_{l=1}^{c_{k-1}} m_l^{k-1}$$

for any $j = 1, \ldots, c_k$.

Hence

$$(4) \qquad \min_j m_k^j / \max_j m_k^j \geq \min_{i,j} r_{ij}^{k-1} / \max_{i,j} r_{ij}^{k-1} = \delta_{k-1}.$$

The result can now be obtained using the following straightforward inequalities: for any nonnegative nonzero vectors $w = (w_1, \dots, w_n)$, $a = (a_1, \dots, a_n)$

$$1/2 \cdot \min_i w_i / \max_i w_i \leq \chi(w);$$

$$(\min_i a_i / \max_i a_i) \cdot \chi(w) \leq \chi((a_1 w_1, a_2 w_2, \dots, a_n w_n)).$$

The first one of these inequalities gives $1/2\,\delta_k \leq \tilde{\varepsilon}_k$, while the second one and (4) gives $\delta_{k-1} \tilde{\varepsilon}_k \leq \varepsilon_k$.

(b) Both (1) and (2) imply that there are an infinity of R_k's with no zero entries. This implies that A is simple, by the same argument as that used in the proof of Lemma 5.

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