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1. INTRODUCTION

Let H_o be the self-adjoint realization in $L^2(R^n)$ of the operator of convolution with a real C^∞ function P defined in R^n:

$$H_o u(x)=P(D)u(x)=(2\pi)^{-n}\int e^{i<x,\xi>}P(\xi)\hat{u}(\xi)d\xi \ , \quad \hat{u}\in C_o^\infty(R^n) \ .$$

Here $\hat{u}$ denotes the Fourier transform of u. We assume that P has the following properties:

(i) there exist two positive constants c and δ such that

$$1+|P(\xi)|+|P'(\xi)|\geq C|\xi|^\delta \ , \quad (\forall) \ \xi\in R^n;$$

(ii) $|P^{(\alpha)}(\xi)|\leq C_\alpha(1+|P(\xi)|+|P'(\xi)|)$, $(\forall)$ $\xi\in R^n$, $(\forall)$ $\alpha\in N^n$, $|\alpha|\geq 2$;

(iii) if $C_v=\{P(\xi);\ P'(\xi)=0\}$ then $\overline{C}_v$ is at most a countable set.

If P is a polynomial we can see using the Tarski-Seidenberg theorem that the assumption (i) may be replaced with

$$\text{(i)'} \quad \lim_{|\xi|\to\infty}(|P(\xi)|+|P'(\xi)|)=\infty \ .$$

A polynomial which satisfies (i)' and (ii) is called a simply characteristic polynomial ([4]).

The spectral properties of H_o which is unitary equivalent (via the Fourier transform) with the operator of multiplication with $P(\xi)$ are wellknown. In this paper we shall describe the spectrum of a perturbed operator $H=H_o+V$. The perturbations V considered here can be decomposed in two parts, $V=V_S+V_L$. V_S is the short-range perturbation and V_L is the long-range perturbation. The conditions imposed on V_S and V_L are the following:

(iv) there exists a number N in $\mathbb{N}$ such that dom $V_S \supset$ dom H_o^N (dom A is the domain of definition of the operator A);

(v) $||(H_o+i)^{-N}V_S\Theta(\frac{\cdot}{r})|| \in L^1(R_+;dr)$ for one (and hence for every) function Θ in $C^\infty(R^n)$, $\Theta(x)=0$ for $|x|\leq 1/2$, $\Theta(x)=1$ for $|x|\geq 1$;

(vi) the operator V_L is an operator of multiplication with a C^∞ function $V_L(x)$ and

$$|\partial_x^\alpha V_L(x)|\leq C_\alpha <x>^{-\delta_o-|\alpha|}, \quad (\forall)\ x\in R^n, \quad (\forall)\ \alpha\in N^n ,$$

where δ_o is a positive constant and $<x>=(1+|x|^2)^{1/2}$;

(vii) the perturbation $V=V_S+V_L$ is a symmetric operator on dom H_o^N;

(viii) the operator H is a selfadjoint extension of $(H_o+V_S+V_L)|\text{dom } H_o^N$;

(ix) $\psi(H)-\psi(H_o)$ is a compact operator for every ψ in $C_o^\infty(R)$;

(x) $\psi(H)(H_o+i)^N$ is a bounded operator for every ψ in $C_o^\infty(R)$.

Now we recall some notations. If A is a self-adjoint operator then $\mathcal{H}_{ac}(A)$ is the subspace of absolute continuity of A, $P_{ac}(A)$ is the orthogonal projection on this subspace, $\mathcal{H}_{sc}(A)$ is the subspace of singularly continuous vectors and Ran A is the range of A.

The main result of this paper is the following.

THEOREM. If the assumptions (i)-(x) are satisfied then there exists a C^∞ function $W: R \times R^n \to R$ such that:

(a) there exist the modified wave operators

$$\tilde{W}_\pm = s\text{-}\lim_{t \to \pm\infty} e^{itH} e^{-iW(t,D)} P_{ac}(H_o) \ ;$$

(b) Ran $\tilde{W}_\pm = \mathcal{H}_{ac}(H)$;

(c) $\mathcal{H}_{sc}(H) = \{0\}$;

(d) the eigenvalues of H which are not in $\overline{C}_V$ are of finite multiplicity and they can accumulate only in the points of $\overline{C}_V$.

REMARK. 1) Instead of (v) and (x) we could assume that $\psi(H)V_S$ is a bounded operator and $||\psi(H)V_S\Theta(\frac{\cdot}{r})|| \in L^1(R_+;dr)$ for every ψ in $C_o^\infty(R)$ and Θ like in (v). Dom H_o^N can be replaced with any set which is included in dom H_o and dense in $L^2(R^n)$.

2) Slightly modifications in the proof of the theorem permit us to consider functions P which are not smooth in the whole space. More precisely we can assume that P is C^∞ in $R^n \setminus S$, that (i) and (ii) are satisfied in $R^n \setminus S$, that $\overline{C_V \cup P(S)}$ is at most a countable set and that if K is a compact set in $R \setminus \overline{C_V \cup P(S)}$, then dist $(P^{-1}(K), S) > 0$. The last assertion is true in particular if S is a bounded set.

We shall give a time-dependent proof of the theorem stated before. Our proof was inspired by the paper of Muthuramalingam [7]. In that paper he proved a similar result for second degree polynomials P and more restrictive conditions imposed on V_S.

Time-dependent proofs of the asymptotic completeness for the case $|P(\xi)| \to \infty$ when $|\xi| \to \infty$ can be found in a lot of papers. We mention here those of Simon [9], Kitada and Yajima [6] and Iftimie [5]. In fact in [7] Muthuramalingam has adapted the proof of Kitada and Yajima to the case of simply characteristic operators of order two.

2. APPROXIMATE SOLUTIONS OF THE HAMILTON-JACOBI EQUATIONS

In [3] the function $W(t,\xi)$ which defines the modified free evolution is a solution of the Hamilton-Jacobi equation

$$\frac{\partial W(t,\xi)}{\partial t} = P(\xi)+V_L\left(\frac{\partial W(\xi,t)}{\partial \xi}\right) \quad .$$

In this paper we do not use an exact solution of this equation, but an approximate one. However the modified wave operators defined here differ from those defined in [3] only by a constant factor.

We proceed now to the construction of the approximate solutions of the Hamilton-Jacobi equation. Let χ be a function in $C^\infty(R^n)$, $0\leq\chi\leq1$, $\chi(x)=1$ for $|x|\geq2$, $\chi(x)=0$ for $|x|\leq1$ and $\rho\in(0,1]$ to be fixed later.

LEMMA 2.1. If

$$V(\rho,t;x) = \chi(\rho x)\chi\left(\frac{x\log<t>}{<t>}\right)V_L(x), \quad x\in R^n,\ t\in R \ ,$$

then

$$|\partial_x^\alpha V(\rho,t;x)|\leq C_{\alpha,q}\rho^\varepsilon<t>^{-\nu-\varepsilon}<x>^{-|\alpha|+\nu-\varepsilon}$$

for x in R^n, $\varepsilon=\delta_o/4$, $\nu=0,1,\ldots,q$, $\nu\leq|\alpha|$. Here q is a positive integer.

The proof of this lemma consists in a straightforward calculation and we skip over it.

We define by recurrence

$$Y(0,\rho,t,x,\xi) = 0$$

$$Y(m,\rho,t,x,\xi) = \int_o^t V(\rho,\tau,x+\tau P'(\xi)+Y'_\xi(m-1,\rho,\tau,x,\xi))d\tau \ ,$$

$$m=0,1,\ldots,m_o \ ,$$

where $m_o\varepsilon\leq 1<(m_o+1)\varepsilon$.

We also introduce the following notations:

$$Y(m,\rho,t,\xi)=Y(m,\rho,t,0,\xi), \quad Y(m,t,\xi)=Y(m,1,t,\xi),$$

$$Y(\rho,t,x,\xi)=Y(m_o,\rho,t,x,\xi) \ ,$$

$$W(m,\rho,t,x,\xi)=x\cdot\xi+tP(\xi)+Y(m,\rho,t,x,\xi),$$

$$W(m,\rho,t,\xi)=W(m,\rho,t,0,\xi), \quad W(\rho,t,\xi)=W(m_o,\rho,t,\xi),$$

$$W(t,\xi)=W(1,t,\xi), \quad W(\rho,t,x,\xi)=W(m_o,\rho,t,x,\xi).$$

For a fixed number $\sigma\in(0,1)$ and two positive constants a and A we set

$$D_+=D_+(\sigma,a,A)=$$

$$=\{(x,\xi);\ x\cdot P'(\xi)\geq-\sigma|x||P'(\xi)|,|P'(\xi)|\geq a,\ |P(\xi)|\leq A(1+|P'(\xi)|)\},$$

$$D_-=\{(x,\xi);\ x\cdot P'(\xi)\leq\sigma|x||P'(\xi)|,\ |P'(\xi)|\geq a,\ |P(\xi)|\leq A(1+|P'(\xi)|)\} .$$

LEMMA 2.2. For every q in N there exists a number $\rho_o\in(0,1]$ such that

(i) $|\partial_\xi^\alpha Y(m,\rho,t,x,\xi)| \leq C_m \rho^\varepsilon |t|^{1-\varepsilon} \langle x\rangle^{-\varepsilon}$, $|\alpha| \leq q+m_o-m$, $m \leq m_o$, $\rho \leq \rho_o$, $t \gtrless 0$, $(x,\xi) \in D_\pm$;

(ii) $|\partial_\xi^\alpha V(\rho,t,W'(m,\rho,t,x,\xi)| \leq C_m \rho^\varepsilon \langle t\rangle^{-\varepsilon} \langle x\rangle^{-\varepsilon}$, $|\alpha| \leq q+m_o-m-1$, $m \leq m_o$, $\rho \leq \rho_o$, $t \gtrless 0$, $(x,\xi) \in D_\pm$;

(iii) $|\partial_\xi^\alpha \partial_x^\beta Y(m,\rho,t,x,\xi)| \leq C_m \rho^\varepsilon \langle x\rangle^{-\varepsilon}$, $|\alpha| \leq q+m_o-m$, $0<|\beta| \leq q$, $m \leq m_o$, $\rho \leq \rho_o$, $t \gtrless 0$, $(x,\xi) \in D_\pm$;

(iv) $|\partial_\xi^\alpha \partial_x^\beta V(\rho,t,W'_\xi(m,\rho,t,x,\xi)| \leq C_m \rho^\varepsilon \langle t\rangle^{-1-\varepsilon} \langle x\rangle^{-\varepsilon}$, $|\alpha| \leq q+m_o-m-1$, $0<|\beta| \leq q$, $m \leq m_o$, $\rho \leq \rho_o$, $t \gtrless 0$, $(x,\xi) \in D_\pm$;

(v) $|\partial_\xi^\alpha \partial_x^\beta [Y(m,\rho,t,x,\xi) - Y(m-1,\rho,t,x,\xi)]| \leq C_m \rho^\varepsilon \langle t\rangle^{1-m\varepsilon} \langle x\rangle^{-\varepsilon}$, $|\alpha| \leq q+m_o-m$, $|\beta| \leq q$, $m \leq m_o$, $\rho \leq \rho_o$, $t \gtrless 0$, $(x,\xi) \in D_\pm$.

<u>Proof</u>. We shall prove only (i) and (ii); the proof of the other statements of the lemma is similar. We consider only the case of the sign "+" (most of the assertions in this paper will be proved only for this case since the case of the sign "−" is in general analogous). The estimations (i) and (ii) can be verified simultaneously by induction with respect to m. If (i) is true for m, then (ii) is also true for m. Indeed, $\partial_\xi^\alpha V(\rho,t,W'_\xi(m,\rho,t,x,\xi))$ is a sum of terms of the form

$$C\,(\partial_x^\beta V)(\rho,t,W'(m,\rho,t,x,\xi)) \prod_{i=1}^{|\beta|} \partial_\xi^{\gamma_i+(j_i)} (tP(\xi)+Y(m,\rho,t,x,\xi)),$$

where (j_i) is the multiindex of length one whose j_i-th component is nonzero. Here $0<|\beta| \leq |\alpha|$, $\sum_{i=1}^{|\beta|} \gamma_i = \alpha$, $|\gamma_i| \neq 0$, $i=1,2,\ldots|\beta|$, $\sum_{i=1}^{|\beta|} (j_i) = \beta$.

Now $Y(m,\rho,t,x,\xi)$ satisfies (i), the function P-the assumption (ii) from the introduction and $V(\rho,t,\cdot)$-the conclusion of Lemma 2.1. Hence the absolute value of such a term can be estimated on D_+

by

$$C\rho^{\varepsilon}<t>^{-\varepsilon}<x+tP'(\xi)+Y'_{\xi}(m,\rho,t,x,\xi)>^{-|\beta|-\varepsilon}(t|P'(\xi)|+C_m\rho^{\varepsilon}t^{1-\varepsilon})^{|\beta|} .$$

But on D_+ we have

$$|x+tP'(\xi)|\geq\sqrt{\frac{1-\sigma}{2}}(|x|+t|P'(\xi)|)=:c_o(|x|+t|P'(\xi)|).$$

If $t^{\varepsilon}|P'(\xi)|\geq 1$, then

$$|Y'_{\xi}(m,\rho,t,x,\xi)|\leq C_m\rho^{\varepsilon}t^{1-\varepsilon}\leq C_m\rho^{\varepsilon}t|P'(\xi)|\leq\frac{c_o}{2}t|P'(\xi)| ,$$

if ρ is smaller than $(\frac{c_o}{2C_m})^{1/\varepsilon}$. Hence

$$<x+tP'(\xi)+Y'_{\xi}(m,\rho,t,x,\xi)>^{-1}\leq C\min\{<x>^{-1},<t|P'(\xi)>^{-1}\} .$$

So we can further estimate our expresion by $C\rho^{\varepsilon}<t>^{-\varepsilon}<x>^{-\varepsilon}$.

If $t\leq|P'(\xi)|^{-1/\varepsilon}\leq a^{-1/2}$, the derivatives of $V(\rho,t,W'_{\xi}(m,\rho,t,x,\xi))$ admit the majorant $c(a)\rho^{\varepsilon}<t>^{-\varepsilon}<x>^{-\varepsilon}$ since $t|P'(\xi)|\leq a^{-(1-\varepsilon)/\varepsilon}$. Summing up one obtains (ii).

In order to conclude the proof we remark that (i) is trivial for $m=0$ and for $m\neq 0$ it is an immediate consequence of (ii) for $m-1$.

Q.E.D.

LEMMA 2.3. For every q in N and for every compact set $K\subset R^n$ we have that

(i) $|\partial_{\xi}^{\alpha}Y(m,\rho,t,\xi)|\leq C_m|t|^{1-\varepsilon}$, $|\alpha|\leq q+m_o-m$, $\rho\in(0,1]$, $m\leq m_o$, $\xi\in K$, $t\in R$;

(ii) $|\partial_{\xi}^{\alpha}(\partial_x^{\beta}V)(\rho,t,W'_{\xi}(m,\rho,t,\xi))|\leq C_m<t>^{-\varepsilon-|\beta|}$, $|\alpha|\leq q+m_o-m-1$, $|\beta|\leq q$, $\rho\in(0,1]$, $m\leq m_o$, $\xi\in K$, $t\in R$;

(iii) $|\partial_{\xi}^{\alpha}(Y(m,\rho,t,\xi)-Y(m-1,\rho,t,\xi))|\leq C_m<t>^{1-m\varepsilon}$, $|\alpha|\leq q+m_o-m$, $\rho\in(0,1]$, $m\leq m_o$, $\xi\in K$, $t\in R$.

The proof of this lemma is similar to that of Lemma 2.2 if we make use of the inequalities

$$|\partial_x^\alpha V(\rho,t,x)|\leq C_\alpha\langle t\rangle^{-|\alpha|-\varepsilon},\quad x\in R^n,\ \rho\in(0,1],\quad t\in R\ .$$

LEMMA 2.4. For every ρ in $(0,1]$ there exist

$$\lim_{t\to\pm\infty}(W(1,t,\xi)-W(\rho,t,\xi))$$

uniformly with respect to ξ in a compact set $K\subset R^n$.

<u>Proof</u>. Since $W(1,t,\xi)-W(\rho,t,\xi)=Y(m_o,t,\xi)-Y(m_o,\rho,t,\xi)$, it is sufficient to prove by induction that there exist

$$\lim_{t\to+\infty}\partial_\xi^\alpha(Y(m,t,\xi)-Y(m,\rho,t,\xi)),\quad |\alpha|\leq m_o-m,\quad m=0,1,\dots,m_o\ .$$

We suppose that this assertion is true for m-1 and we prove it for m. We have

$$Y(m,t,\xi)-Y(m,\rho,t,\xi)=\int_o^t\Big[V(1,\tau,\tau P'(\xi)+Y_\xi'(m-1,\tau,\xi))-$$

$$-V(\rho,\tau,\tau P'(\xi)+Y_\xi'(m-1,\rho,\tau,\xi))\Big]d\tau\ .$$

But

$$V(1,t,x)-V(\rho,t,y)=(\chi(x)-\chi(\rho x))\chi\Big(\frac{x\log\langle t\rangle}{\langle t\rangle}\Big)V_L(x)+$$

$$+(x-y)\cdot\int_o^1(\partial_xV)(\rho,t,\theta x+(1-\theta)y)d\theta\ .$$

Hence

$$Y(m,t,\xi)-Y(m,\rho,t,\xi)=\int_o^t\Big[V(1,\tau,W_\xi'(m-1,\tau,\xi))-$$

$$-V(\rho,\tau,W_\xi'(m-1,\tau,\xi))\Big]d\tau+\int_o^t(Y_\xi'(m-1,\tau,\xi)-Y_\xi'(m-1,\rho,\tau,\xi))\cdot$$

$$\cdot\int_o^1(\partial_xV)(\rho,\tau,\tau P'(\xi)+\theta Y_\xi'(m-1,\tau,\xi)+(1-\theta)Y_\xi'(m-1,\rho,\tau,\xi))\,d\theta d\tau\ .$$

In the first integral we have to integrate only from 0 to t_o , where t_o is the smallest number which has the properties that $\langle t_o\rangle \geq e$ and $\frac{\langle t_o\rangle}{\log\langle t_o\rangle} \geq 2\rho^{-1}$; if t is greater than t_o, then the function to integrate is equal to zero. The existence of the limit of the second term is established using Lemma 2.3, (ii) and the hypothesis of induction.

Q.E.D.

The following result is an immediate consequence of Lemma 2.2, (iii).

LEMMA 2.5. If $|\alpha|, |\beta| \leq q$, then

$$\sup_{(t,x,\xi)\in D_\pm} \langle x\rangle^{-1} |\partial_\xi^\alpha \partial_x^\beta [Y(m_o-1,\rho,t,\xi)-Y(m_o-1,\rho,t,x,\xi)]| < \infty .$$

The last lemma from this section compares the approximate solution $W(t,\xi)$ of the Hamilton-Jacobi equation with the exact one $W_1(t,\xi)$

$$\partial_t W_1(t,\xi) = P(\xi) + V_L(\partial_\xi W_1(t,\xi))$$

constructed in [3] . (In fact for every compact set $K \subseteq \{\xi;\ |P'(\xi)| \neq 0\}$ there exists a number $t_K > 0$ such that this equation is satisfied for ξ in K and $|t| \geq t_K$.) Moreover

$$|\partial_\xi^\alpha (W_1(t,\xi) - tP(\xi))| \leq C_\alpha t^{1-\delta_o} \quad \text{for } \xi \text{ in K.}$$

The constants C_α may depend on the compact set K.

LEMMA 2.6. Let K be a compact set in $\{\xi;\ |P'(\xi)| \neq 0\}$. Then the limits

$$\lim_{t\to\pm\infty} (W(t,\xi)-W_1(t,\xi))$$

exist uniformly with respect to ξ in K.

<u>Sketch of proof</u>. We set $Y_1(t,\xi)=W_1(t,\xi)-tP(\xi)$. It can be proved by induction that

$$|\partial_\xi^\alpha(Y_1(t,\xi)-Y(m,t,\xi))|\leq Ct^{1-(m+1)\varepsilon},\quad |\alpha|\leq m_o-m,\ \xi\in K,\ m=0,1,\ldots m_o-1.$$

Then

$$Y_1(t,\xi)-Y(m_o,t,\xi)=\int_o^t (1-\chi(W'_{1,\xi}(m_o-1,\tau,\xi))\cdot$$

$$\cdot\chi(\frac{W'_{1,\xi}(m_o-1,\tau,\xi)\log<\tau>}{<\tau>})V_L(W'_{1,\xi}(m_o-1,\tau,\xi))+$$

$$+\int_o^t (Y'_{1,\xi}(\tau,\xi)-Y'_\xi(m_o-1,\tau,\xi))\cdot\int_o^1 (\partial_x V)(1,\tau,\tau P'(\xi)+$$

$$+\theta Y'_{1,\xi}(\tau,\xi)+(1-\theta)Y'_\xi(m-1,\tau,\xi))d\theta d\tau\ .$$

Now we can proceed like in the proof of Lemma 2.4.

Q.E.D.

3. THE EXISTENCE OF THE MODIFIED WAVE OPERATORS

In this section we shall outline the proof of the part a) of the theorem. Let $W(t,\xi)$ be the approximate solution of the Hamilton-Jacobi equation constructed in the previous section. Then the limits

$$\lim_{t\to\pm\infty} e^{itH}e^{-iW(t,D)}u \qquad (1)$$

exist for u in $\hat{\mathcal{D}}(\Omega)=\{u\in\mathcal{S}(R^n);\ \text{supp}\ \hat{u}\subset\Omega\}$, where $\Omega=\{\xi;P'(\xi)\neq 0\}$. The set $\hat{\mathcal{D}}(\Omega)$ is dense in $\mathcal{H}_{ac}(H_o)$ and

$$e^{-iW(t,D)}u(x)=(2\pi)^{-n}\int e^{i<x,\xi>}e^{-iW(t,\xi)}\hat{u}(\xi)d\xi \quad .$$

LEMMA 3.1. If u is in $\hat{\mathcal{D}}(\Omega)$, then $e^{-iW(t,D)}u$ converges weakly to zero when $t\to\pm\infty$.

<u>Proof.</u> It is sufficient to show that

$$(e^{-iW(t,D)}u,v)\to 0, \quad t\to\pm\infty \quad , \quad (\forall)\ u\in\hat{\mathcal{D}}(\Omega), \quad (\forall)\ v\in\mathcal{S}(R^n) \ .$$

But

$$(e^{-iW(t,D)}u,v)=(2\pi)^{-n}\int e^{-iW(t,\xi)}\hat{u}(\xi)\bar{\hat{v}}(\xi)d\xi \quad .$$

Let us remark that $\hat{u}\bar{\hat{v}}\in C_o(\Omega)$. Hence the conclusion of the lemma is a consequence of the following result.

LEMMA 3.2. Let a be equal to $\inf\{|P'(\xi)|;\xi \text{ supp } \hat{u}\}$ and b to $\sup\{|P'(\xi)|;\ \xi\in\text{supp } \hat{u}\}$. Then for every m in N there exists a constant C_m and $t_o>0$ such that

$$|e^{-iW(t,D)}u(x)|\leq C_m(1+|x|+|t|)^{-m} \quad \text{for } |\tfrac{x}{t}|\notin(\tfrac{a}{2},2b),\ |t|>t_o \ .$$

<u>Proof.</u> Accordingly to the definition

$$(e^{-iW(t,D)}u)(x)=(2\pi)^{-n}\int e^{i<x,\xi>-itP(\xi)-iY(m_o,t,\xi)}\hat{u}(\xi)d\xi \ .$$

Since

$$|Y'_\xi(m_o,t,\xi)|\leq c<t>^{1-\varepsilon}$$

for ξ in supp $\hat{u}$, there exists a number $t_o>0$ such that

$$|x-tP'(\xi)-Y'_\xi(m_o,t,\xi)|\geq\frac{a}{8}(t+|x|)$$

for $|\frac{x}{t}|\leq\frac{a}{2}$, $\xi\in$supp $\hat{u}$ and $t>t_o$. Therefore we can introduce the operator

$$L=-i\frac{x-W'_\xi(t,\xi)}{|x-W'_\xi(t,\xi)|^2}\cdot\partial_\xi$$

which has the property that

$$Le^{i<x,\xi>-iW(t,\xi)}=e^{i<x,\xi>-iW(t,\xi)}.$$

Thus

$$e^{-iW(t,D)}u(x)=(2\pi)^{-n}\int e^{i<x,\xi>-iW(t,\xi)}({}^tL)^m\hat{u}(\xi)d\xi \quad .$$

As

$$|({}^tL)^m\hat{u}(\xi)|\leqq C_m(1+|x|+t)^{-m}$$

for x and t like before, we obtain the desired estimation for the case $|\frac{x}{t}|\leqq\frac{a}{2}$, $t>t_o$. The other cases can be treated in a similar way.

Q.E.D.

Now, since u is in $\hat{\mathcal{D}}(\Omega)$, there exists a function ϕ in $C_o^\infty(R)$ with $\phi(H_o)u=u$. Applying Lemma 3.1 and the assumption (ix) from the introduction we deduce that

$$s-\lim_{t\to\pm\infty} e^{itH}(\phi(H)-\phi(H_o))e^{-iW(t,D)}u=0.$$

Therefore the limits (1) exist if there exist

$$\lim_{t\to\pm\infty} \phi(H)e^{itH}e^{-iW(t,D)}u \quad ,$$

for ϕ in $C_o^\infty(R)$ and u in $\hat{\mathcal{D}}(\Omega)$. The existence of these limits is proved by Cook's method. We consider only the case $t\to+\infty$.

LEMMA 3.3. Let u be in $\hat{\mathcal{D}}(\Omega)$ and ϕ in $C_o^\infty(R)$. Then

$$\int_o^\infty||\frac{d}{dt}\phi(H)e^{itH}e^{-iW(t,D)}u||dt<\infty \quad .$$

Proof. We have

$$\frac{d}{dt}\phi(H)e^{itH}e^{-iW(t,D)}u=ie^{itH}\phi(H)(H-\partial_t W(t,D))e^{-iW(t,D)}u=$$

$$=ie^{itH}\phi(H)\Big[V_S+(V_L(x)-V(1,t,x))+(V(1,t,W_\xi'(t,s))-\partial_t Y(m_o,t,D))+$$

$$+(V(1,t,x)-V(1,t,W_\xi'(t,D)))\Big]e^{-iW(t,D)}u\ .$$

We must estimate four integrals. In what follows t_o is a fixed number greater than zero and sufficiently large.

a) $\int_{t_o}^\infty ||\psi(H)V_S e^{-iW(t,D)}u||dt<\infty$. This estimation results in a standard way from Lemma 3.2 and the assumptions (v) and (x) from the introduction (see [8]).

b) $\int_{t_o}^\infty ||(V_L(x)-V(1,t,x))e^{-iW(t,D)}u||dt<\infty$. First we remark that $V_L(x)-V(1,t,x)=0$ for $|x|>2\max\ (1,\ \langle t\rangle/\log\langle t\rangle)$. Then we take t_o such that $2\langle t\rangle/\log\langle t\rangle\ <at/2$ for $t\geq t_o$ and we apply again Lemma 3.2.

c) $\int_{t_o}^\infty ||(V(1,t,W_\xi'(t,D))-\partial_t Y(m_o,t,D))e^{-iW(t,D)}u||dt<\infty$.

Indeed, Lemma 2.3 (ii) and (iii) imply that

$$|V(1,t,W_\xi'(t,\xi))-\partial_t Y(m_o,t,\xi)|\leq C\langle t\rangle^{-(m_o+1)\varepsilon}$$

for ξ in supp $\hat{u}$. But $(m_o+1)\varepsilon$ is greater than 1.

d) $\int_{t_o}^\infty ||(V(1,t,x)-V(1,t,W_\xi'(t,D)))e^{-iW(t,D)}u||dt<\infty$. We have

$$(V(1,t,x)-V(1,t,W_\xi'(t,D))e^{-iW(t,D)}u(x)=$$

$$=(2\pi)^{-n}\int e^{i<x,\xi>-iW(t,\xi)}\left[V(1,t,x)-V(1,t,W'_{\xi}(t,\xi))\right]\hat{u}(\xi)d\xi=$$

$$=(2\pi)^{-n}\int e^{i<x,\xi>-iW(t,\xi)}\left[(x-W'_{\xi}(t,\xi))\cdot\int_0^1(\partial_x V)(1,t,\theta x+(1-\theta)W'_{\xi}(t,\xi))d\theta\right]\hat{u}(\xi)\cdot d\xi=$$

$$=-i(2\pi)^{-n}\int\partial_{\xi}e^{i<x,\xi>-iW(t,\xi)}\cdot\int_0^1(\partial_x V)(1,t,\theta x+(1-\theta)W'_{\xi}(t,\xi))d\theta\hat{u}(\xi)d\xi=$$

$$=i\left(\sum_{j=1}^{n}a_j(t,x,D)e^{-iW(t,D)}u+\sum_{j=1}^{n}b_j(t,x,D)e^{-iW(t,D)}(-ix_ju)\right),$$

where

$$a_j(t,x,\xi)=\int_0^1(\partial_{x_j}\partial_x V)(1,t,\theta x+(1-\theta)W'_{\xi}(t,\xi))(1-\theta)d\theta\cdot\partial_{\xi}\partial_{\xi_j}W(t,\xi),$$

$$b_j(t,x,\xi)=\int_0^1(\partial_{x_j}V)(1,t,\theta x+(1-\theta)W'_{\xi}(t,\xi))d\theta .$$

The amplitudes a_j and b_j satisfy the following inequalities:

$$|\partial_x^{\alpha}\partial_{\xi}^{\beta}a_j(t,x,\xi)|+|\partial_x^{\alpha}\partial_{\xi}^{\beta}b_j(t,x,\xi)|\leq c<t>^{-1-\varepsilon},\quad |\alpha|,|\beta|\leq\left[\frac{n}{2}\right]+1 ,$$

for ξ in a neighbourhood of supp $\hat{u}$. We can now apply the theorem of continuity of pseudodifferential operators due to Calderon and Vaillancourt.

Q.E.D.

The next proposition is an immediate consequence of this lemma.

PROPOSITION 3.4. If the assumptions (i)-(x) are satisfied and if $W(t,\xi)$ is the approximate solution of the Hamilton-Jacobi equation constructed in the second section then there exist the modified wave operators

$$\tilde{W}_{\pm}=\text{s-}\lim_{t\to\pm\infty} e^{itH}e^{-iW(t,D)}P_{ac}(H_o) .$$

REMARK 3.5. Let $W_1(t,\xi)$ be the exact solution of the Hamilton-Jacobi equation constructed in [3] and let $F_\pm(\xi)$ be equal to $\lim_{t\to\pm\infty} (W(t,\xi)-W_1(t,\xi))$, $\xi\in\Omega$. Then

$$\lim_{t\to\pm\infty} e^{itH} e^{-iW_1(t,D)} u = \tilde{W}_\pm e^{iF_\pm(D)} u \quad ,$$

for u in $\hat{\mathcal{D}}(\Omega)$. Thus the modified wave operators defined as in [3] exist and, moreover, their ranges are equal to Ran $\tilde{W}_\pm$.

The following proposition is proved in [4].

PROPOSITION 3.6. The modified wave operators $\tilde{W}_\pm$ are partial isometries which intertwine H and H_o and Ran $\tilde{W}_\pm \subset \mathcal{H}_{ac}(H)$.

4. THE ASYMPTOTIC COMPLETENESS OF THE MODIFIED WAVE OPERATORS

DEFINITION. The modified wave operators are asymptotic complete if Ran $\tilde{W}_\pm = \mathcal{H}_{ac}(H)$ and $\mathcal{H}_{sc}(H)=\{0\}$.

In order to prove the asymptotic completeness of the modified wave operators, we shall construct for every ψ in $C_o^\infty(R\setminus\overline{C}_V)$ two families of bounded operators $\{P_{r,\pm}\}_{r\geq 1}$ such that $P_{r,+}+P_{r,-}$ approximates the identity and

$$\lim_{r\to\infty} \sup_{t\gtrless 0} ||(1-\tilde{W}_\pm\tilde{W}_\pm^*)\psi(H)P_{r,+}e^{-itH}u||=0 \quad , \tag{2}$$

$$\lim_{r\to\infty} \sup_{t\gtrless 0} ||(1-\tilde{W}_\pm\tilde{W}_\pm^*)\psi(H)P_{r,-}e^{-itH}u||=0 \quad , \tag{2'}$$

for every u in L^2.

The operators $P_{r,\pm}$ will be pseudodifferential operators.

The amplitudes of these operators are defined with the aid of the following functions:

1) $g \in C^\infty(R)$, $g(\lambda)=0$ for $\lambda \leqq a$, $g(\lambda)=1$ for $\lambda \geqq 2a$, where a is a positive constant;

2) $f \in C_o^\infty(R)$, $f(\lambda)=1$ for $|\lambda| \leqq 1$, $f(\lambda)=0$ for $|\lambda| \geqq 2$;

3) $\theta \in C^\infty(R^n)$ is like in the assumption (v) from the introduction;

4) $\psi_\pm \; C^\infty(R)$, $\psi_+(\lambda)+\psi_-(\lambda)=1$ for $|\lambda| \leqq 1$, $0 \leqq \psi_\pm(\lambda) \leqq 1$, $\psi_+(\lambda)=0$ for $\lambda \leqq -\sigma_o$, $\psi_-(\lambda)=0$ for $\lambda \geqq \sigma_o$, where $\sigma_o \in (0,1)$ is a fixed constant.

We fix another constant $C>0$ and set

$$h(\xi)=f\left(\frac{P(\xi)}{C(1+|P'(\xi)|)}\right)g(|P'(\xi)|),$$

$$a_{r,\pm}(y,\xi)=h(\xi)\psi_\pm\left(\frac{y \cdot P'(\xi)}{|y||P'(\xi)|}\right)\theta\left(\frac{y}{r}\right), \quad r \geqq 1.$$

Then

$$|\partial_\xi^\alpha a_{r,\pm}(y,\xi)| \leqq C_\alpha, \quad (y,\xi) \in R^{2n}, \quad r \geqq 1 \tag{3}$$

and

$$|\partial_\xi^\alpha \partial_y^\beta a_{r,\pm}(y,\xi)| \leqq C_{\alpha\beta} r^{-1}, \quad (y,\xi) \in R^{2n}, \quad r \geqq 1, \quad \beta \neq 0 . \tag{4}$$

Next we define

$$P_{r,\pm}u(x)=(2\pi)^{-n}\iint e^{i<x-y,\xi>}a_{r,\pm}(y,\xi)u(y)dyd\xi, \quad u \in \mathcal{S}(R^n) .$$

LEMMA 4.1. (i) The operators $P_{r,\pm}$ can be extended by continuity to operators defined on $L^2(R^n)$ and

$$\sup_{r \geqq 1} ||P_{r,\pm}|| < \infty .$$

(ii) $P_{r,+}+P_{r,-}=h(D)\theta(\frac{\cdot}{r})$.

(iii) $||P_{r,\pm}-P^*_{r,\pm}||\leq Cr^{-1}$, $r\geq 1$.

Proof. The assertion (i) is an immediate consequence of the Calderon-Vaillancourt theorem of continuity of pseudodifferential operators, (ii) results by a straightforward calculation and (iii) from (4) and the same theorem of Calderon and Vaillancourt (for details, see [5], [8]).

Q.E.D.

Now we consider some approximations of the modified free evolution of $P_{r,+}u$ and $P_{r,-}u$:

$$E_{r,\pm}(t)u(x)=(2\pi)^{-n}\int(\int e^{i\langle x,\xi\rangle-iW(\rho,t,y,\xi)}a_{r,\pm}(y,\xi)u(y)dy)d\xi,$$

$$t\geq 0,\quad u\in\mathcal{S}(R^n)\ .$$

The number ρ will be fixed later sufficiently small. In the sequel we shall write $W(t,y,\xi)=W(\rho,t,y,\xi)$, $Y(t,y,\xi)=Y(\rho,t,y,\xi)$. Since Y satisfies the estimations from Lemma 2.2 on a neighbourhood of supp $a_{r,\pm}$, the functions $(y,\xi)\to e^{-iY(t,y,\xi)}a_{r,\pm}(y,\xi)$ satisfy the inequalities

$$|\partial_\xi^\alpha\partial_y^\beta e^{-iY(t,y,\xi)}a_{r,\pm}(y,\xi)|\leq C_{\alpha\beta}\langle t\rangle^{1-\varepsilon},\quad (y,\xi)\in R^{2n}\ ,\quad t\geq 0\ .$$

Therefore the operators defined by

$$F_{r,\pm}(t)u(x)=(2\pi)^{-n}\iint e^{i\langle x-y,\xi\rangle-iY(t,y,\xi)}a_{r,\pm}(y,\xi)u(y)dyd\xi$$

may be considered as pseudodifferential operators of order zero (and they admit extensions to continuous operators on $L^2(R^n)$).

We have

$$E_{r,\pm}(t)=e^{-itP(D)}F_{r,\pm}(t)=e^{-itH_o}F_{r,\pm}(t) \quad .$$

We want to prove that $F_{r,\pm}(t)$ and $E_{r,\pm}(t)$ are uniformly continuous operators in $L^2(R^n)$ with respect to $r\geq 1$ and $t\gtrless 0$. The proof of this statement is based on the following lemma from [7]:

LEMMA 4.2. Let Z be a C^∞ real function defined in $R_+ xR^n xR^n$ which satisfies:

(i) $||\partial_\xi \partial_y Z(t,y,\xi)||\leq\frac{1}{2}$, $(\forall)$ $(t,y,\xi)\in R_+ \times R^n x R^n$,

where $||\cdot||$ denotes the norm of the linear map in R^n;

(ii) $|\partial_\xi^\alpha \partial_y^\beta \partial_y Z(t,y,\xi)|\leq C_{\alpha,\beta}$, $(\forall)$ $(t,y,\xi)\in R_+ xR^n xR^n$, $\alpha+\beta\neq 0$.

We also suppose that $b:\Gamma xR_+ xR^n xR^n \to \mathbb{C}$ is a C^∞ function with respect to (y,ξ) $R^n xR^n$ (Γ is a set of parameters), such that for every τ in Γ there is a compact set $K(\tau)$ with the property that $b(\tau,t,y,\xi)=0$ if $\xi\notin K(\tau)$. In addition the function b satisfies the following estimations:

(iii) $|\partial_\xi^\alpha \partial_y^\beta b(\tau,t,y,\xi)|\leq C'_{\alpha,\beta}$, $(\forall)$ $(\tau,t,y,\xi)\in\Gamma xR_+ xR^n xR^n$.

Then the integral operator with kernel

$$I(\tau,t;x,y)=\int e^{i<x-y,\xi>-iZ(t,y,\xi)}b(\tau,t,y,\xi)d\xi$$

is a continuous operator in $L^2(R^n)$ and its norm is dominated by M $\sup\{C'_{\alpha,\beta};\ |\alpha|,|\beta|\leq p\}$ for p sufficiently large. The constant M depends only on a finite number of constants $C_{\alpha,\beta}$.

REMARK 4.3. The fact that the estimations (i) and (ii) hold for all y,ξ in R^n is essential for the proof of this lemma given in [7]. But we can prove that the conclusion of Lemma 4.2 remains valid if

$$Z(t,y,\xi)=Y(t,y,\xi),$$

$$\text{supp } b(\tau,t,\cdot,\cdot)\subset D_+(\sigma_o,c,C), \qquad (\forall)\ \tau\in\Gamma,\ \ t>0$$

Indeed, from Lemma 2.2 we obtain that for ρ small enough the estimations (i) and (ii) are satisfied on $D_+(\sigma,c/2,2C)$, when $\sigma_o<\sigma<1$. The kernel of the integral operator remains the same if we replace $Y(t,y,\xi)$ with $h_1(\xi)Y(t,y,\xi)$, where

$$h_1(\xi)=g_1(|P'(\xi)|)f(\frac{P(\xi)}{C(1+|P'(\xi)|)}),$$

$g_1(\lambda)=g(2\lambda)$; g and f are the functions defined at the beginning of this section. Therefore we can suppose that $Y(t,y,\xi)$ satisfies the estimations (i) and (ii) from the Lemma 4.2 on the set

$$\{(t,y,\xi);\ t\geq 0,\ y\cdot P'(\xi)\geq -\sigma|y||P'(\xi)|\} \quad .$$

We want to split our operator into a sum of operators and to apply the proof of Lemma 4.2 to each term of this sum. In order to do this we choose an appropriate C^∞ partition of unity on S^{n-1}, $\{\chi_j\}_{j=1}^q$. We suppose that if $\omega_1,\omega_2\in\text{supp }\chi_j$, then $|\omega_1-\omega_2|<(\sigma-\sigma_o)/4$. Let ψ_j be such that $\psi_j\chi_j=\chi_j$ and $|\omega_1-\omega_2|<(\sigma-\sigma_o)/2$ if $\omega_1,\omega_2\in\text{supp }\omega_j$. We define

$$b_j(\tau,t,y,\xi)=\chi_j(\frac{P'(\xi)}{|P'(\xi)|})b(\tau,t,y,\xi),$$

$$Y_j(t,y,\xi)=\psi_j(\frac{P'(\xi)}{|P'(\xi)|})Y(t,y,\xi) \quad .$$

Then

$$\int e^{i<x-y,\xi>-i\Upsilon(t,y,\xi)}b(\tau,t,y,\xi)d\xi=\sum_{j=1}^{q}\int e^{i<x-y,\xi>-i\Upsilon_j(t,y,\xi)}\cdot$$

$$\cdot b_j(\tau,t,y,\xi)d\xi \quad .$$

The functions $\Upsilon_j(t,y,\xi)$ satisfy the estimations from Lemma 2.2 on the sets

$$C^j=\{(y,\xi);y\cdot P'(\eta_j)\geq-\frac{\sigma+\sigma_o}{2}|y||P'(\eta_j)|\} ,$$

where $\omega_j=P'(\eta_j)/|P'(\eta_j)|$ is in supp χ_j. We also have that

$$\text{supp } b_j(\tau,t,\cdot,\cdot)\subset\tilde{C}^j=\{(y,\xi);\ y\cdot\omega_j\geq-\frac{3\sigma_o+\sigma}{4}|y|\}$$

Now, for j fixed, there exists a family $\{\omega_j^k\}_{k=0}^p$ such that if

$$S_k=\{y;\ y\cdot\omega_j^k>0\} ,$$

then

$$\pi_y\tilde{C}_j\subset\bigcup_{k=0}^{p}S_k\ ,\quad S_k\subset\pi_yC^j\ ,\quad (\forall)\quad k=0,1,\ldots p\ .$$

(ω_j^o can be taken to be equal to ω_j).

Let $\{\theta_k\}_{k=0}^p$ be a partition of the unity in the neighbourhood of $\pi_y\tilde{C}^j$, positive homogeneous of degree zero and subordinated to the covering $\{\delta_k\}_{k=1}^p$ of $\pi_y\tilde{C}_j$. Finally we set

$$b_{jk}(\tau,t,y,\xi)=\theta_k(y)b_j(\tau,t,y,\xi)\ .$$

Then

$$\text{supp } b_{jk}(\tau,t,\cdot,\cdot)\subset\{(y,\xi);\ y\cdot\omega_j^k>0\}$$

and Υ_j (t,y,ξ) satisfies the estimations from Lemma 2.2 on the set $\{(y,\xi);\ y\cdot\omega_j^k \geqq 0\}$.

The proof of Lemma 4.2 applies to this case.

LEMMA 4.4. $\sup_{r>1}\sup_{t\gtrless 0} \|E_{r,\pm}(t)\| < \infty$.

<u>Proof</u>. As we have seen, $E_{r,\pm}(t)=e^{-itP(D)}F_{r,\pm}(t)$. The operators $F_{r,\pm}(t)$ are continuous on $L^2(R^n)$ for every $t\gtrless 0$. Let ϕ be in $C_o^\infty(R^n)$, $\phi(\xi)=1$ for $|\xi|\leqq 1$. Then $\phi(D/s)F_{r,\pm}(t)\to F_{r,\pm}(t)$ strongly when $s\to\infty$. But the operators $\phi(D/s)F_{r,\pm}(t)$ satisfy the properties listed in Remark 4.3.

Q.E.D.

The proof of the relations (2) and (2)' is based on the fact that

$$\lim_{r\to\infty} \left|\int_o^{\pm\infty} \left\|\frac{d}{dt}\psi(H)e^{itH}E_{r,\pm}(t)\right\| |dt| = 0 \qquad (5)$$

for every ψ in $C_o^\infty(R)$. Indeed, for u in $\mathcal{S}$ we have

$$\frac{d}{dt}(\psi(H)e^{itH}E_{r,+}(t)u) = e^{itH}\psi(H)V_S E_{r,+}(t)u+$$

$$+\psi(H)e^{itH}(V_L(x)-V(\rho,t,x))E_{r,+}(t)u+$$

$$+(2\pi)^{-n}\psi(H)e^{itH}\iint e^{i<x,\xi>-iW(t,y,\xi)}\left[V(\rho,t,x)-V(\rho,t,W'_\xi(t,y,\xi)\right]\cdot$$

$$\cdot a_{r,+}(y,\xi)u(y)dyd\xi+(2\pi)^{-n}\psi(H)e^{itH}\iint e^{i<x,\xi>-iW(t,y,\xi)}\cdot$$

$$\cdot\left[V(\rho,t,W'_\xi(t,y,\xi))-V(\rho,t,W'_\xi(m_Q-1,\rho,t,y,\xi)\right]\cdot a_{r,+}(y,\xi)u(y)dyd\xi\ .$$

LEMMA 4.5. (Schur). Let K be a continuous function defined in $R^n \times R^n$ such that $\sup_y \int |K(x,y)|dx \leq C$, $\sup_x \int |K(x,y)|dy \leq C$. Then the integral operator with kernel K is a continuous operator in $L^2(R^n)$ with norm $\leq C$.

The proof of this lemma is not difficult and can be found in [4].

LEMMA 4.6. Let η be in $C_o^\infty(R^n)$, $0 \leq \eta \leq 1$, $\eta(x)=1$ for $|x| \leq \frac{1}{2}$, $\eta(x)=0$ for $|x| \geq 1$. Then there exists a constant $b>0$ such that for every m in N the following estimation holds:

$$||\eta(\frac{x}{b(r+|t|)})E_{r,\pm}(t)|| \leq C(m)(1+r+|t|)^{-m} \quad , \ t \gtrless 0 \ .$$

<u>Proof</u>. The kernel of $\eta(\frac{x}{b(r+t)})E_{r,+}(t)$ is

$$K(r,t;x,y) = (2\pi)^{-n} \int e^{i<x,\xi>-iW(t,y,\xi)} \eta(\frac{x}{b(r+t)}) a_{r,+}(y,\xi)d\xi \ ,$$

where the integral is an oscillatory one. On the support of the function to integrate the following inequality holds

$$|y+tP'(\xi)+Y'_\xi(t,y,\xi)| \geq c(|y|+t|P'(\xi)|) \ .$$

Hence there exists a constant b such that

$$|x-y-tP'(\xi)-Y'_\xi(t,y,\xi)| \geq c_1(r+ta) \tag{6}$$

and

$$|x-y-tP'(\xi)-Y'_\xi(t,y,\xi)| \geq c_2(|x|+|y|+t<\xi>^\delta) \tag{7}$$

for $(x,y,\xi) \in \operatorname{supp} \eta(\frac{x}{b(r+t)}) a_{r,+}(y,\xi)$.

Therefore we can introduce the differential operator

$$L=-i\frac{x-y-tP'(\xi)-Y_\xi'(t,y,\xi)}{|x-y-tP'(\xi)-Y_\xi'(t,y,\xi)|^2}\cdot\partial_\xi \quad .$$

A repeated integration by parts gives

$$K(r,t;x,y)=\int e^{i<x,\xi>-iW(t,y,\xi)}\eta(\frac{x}{b(r+t)})({}^tL)^k a_{r,+}(y,\xi)d\xi \quad .$$

Taking k large and applying Lemma 4.5 we obtain the desired estimation.

Q.E.D.

LEMMA 4.7. $\lim_{r\to\infty}\int_o^\infty||\psi(H)V_S E_{r,+}(t)||dt=0$.

Proof. We have

$$||\psi(H)V_S E_{r,+}(t)||\leq||\psi(H)V_S\theta(\frac{x}{b(r+t)})||\cdot||E_{r,+}(t)||+$$

$$+||\psi(H)V_S||\cdot||(1-\theta(\frac{x}{b(r+t)}))E_{r,+}(t)|| \quad .$$

Now the assumptions (v) and (x) from the introduction are used to estimate the first term of the sum and the assumption (x) and Lemma 4.6 for the estimation of the second term.

Q.E.D.

LEMMA 4.8. $\lim_{r\to\infty}\int_o^\infty||(V_L(x)-V(\rho,t,x))E_{r,+}(t)||dt=0$.

Proof. Since $1-\chi(\rho x)\chi(\frac{x\log<t>}{<t>})$ is equal to zero for

$|x|\geq\max\{\frac{<t>}{\log<t>},\ \rho^{-1}\}$ there exists a constant $t_o>0$ such that

$$(1-\chi(\rho x)\chi(\frac{x\log<t>}{<t>}))\eta(\frac{x}{b(r+t)})=1-\chi(\rho x)\chi(\frac{x\log<t>}{<t>}) \qquad (8)$$

for $t\geq t_o$. From Lemma 4.6 we deduce that

$$\lim_{r\to\infty} \int_{t_o}^{\infty} ||(V_L(x)-V(\rho,t,x))E_{r,+}(t)||dt=0 .$$

On the other hand, if $t \leqq t_o$, we can find a constant $r_o>0$ such that (8) holds for $r \geqq r_o$. We can apply again Lemma 4.6.

Q.E.D.

The following lemma is a consequence of Remark 4.3 and of Lemma 2.2.

LEMMA 4.9. If

$$A_{r,+}(t)u(x)=(2\pi)^{-m}\iint e^{i<x,\xi>-iW(t,y,\xi)}\Big[V(\rho,t,W'_\xi(t,y,\xi))-$$

$$-V(\rho,t,W'_\xi(m_o-1,\rho,t,y,\xi)\Big]\cdot a_{r,+}(y,\xi)u(y)dyd\xi$$

then

$$\lim_{r\to\infty} \int_{o}^{\infty} ||A_{r,+}(t)||dt=0 .$$

LEMMA 4.10. If

$$B_{r,+}(t)u(x)=(2\pi)^{-n}\iint e^{i<x,\xi>-iW(t,y,\xi)}\Big[V(\rho,t,x)-V(\rho,t,W'_\xi(t,y,\xi)\Big]\cdot$$

$$\cdot a_{r,+}(y,\xi)u(y)dyd\xi \quad ,$$

then

$$\lim_{r\to\infty} \int_{o}^{\infty} ||B_{r,+}(t)||dt=0 .$$

<u>Proof</u>. We decompose the domain of integration in two sub-domains. If $|x-y-tP'(\xi)| \leqq c(|x|+|y+tP'(\xi)|)$ and if $c<1$ is sufficiently small, then $|x-y-tP'(\xi)| \leqq \frac{1}{2}|y+tP'(\xi)|$. Let χ_1 be in $C_o^\infty(R)$, $\chi_1(\lambda)=1$ for $|\lambda| \leqq (\frac{c}{2})^2$, $\chi_1(\lambda)=0$ for $|\lambda| \geqq c^2$. We define

$$\tilde{\chi}(x,y,\xi)=\chi_1\left(\frac{|x-y-tP'(\xi)|^2}{|x|^2+|y+tP'(\xi)|^2}\right) \quad ,$$

$$a_1(x,y,\xi)=\tilde{\chi}(x,y,\xi)a_{r,+}(y,\xi) \quad ,$$

$$a_2(x,y,\xi)=a_{r,+}(y,\xi)-a_1(x,y,\xi) \quad .$$

Accordingly to these definitions, the operator $B_{r,+}(t)$ splits in a sum of two operators, $B_{r,1}(t)$ and $B_{r,2}(t)$.

Now $|x-y-tP'(\xi)|$ is greater than $c_1(|x|+|y|+t|P'(\xi)|)$ on supp a_2, where c_1 is a positive constant sufficiently small. Thus the norm of $B_{r,2}(t)$ can be estimated as in Lemma 4.6. We have

$$||B_{r,2}(t)||\leq C(1+r+t)^{-2} \quad .$$

On the other hand, on supp a_1 we have that $|x-y-tP'(\xi)|\leq$ $\leq\frac{1}{2}|y+tP'(\xi)|$ and $|x|\leq C|y+tP'(\xi)|$. The kernel of $B_{r,1}(t)$ is equal to

$$i(2\pi)^{-n}\int e^{i<x,\xi>-iW(t,y,\xi)}\partial_\xi\cdot$$

$$\cdot\left[a_1(x,y,\xi)\int_0^1(\partial_x V)(\rho,t,\theta x+(1-\theta)W'_\xi(t,y,\xi))d\theta\right]d\xi \quad .$$

If we set

$$b_1(x,y,\xi)=\partial_\xi\left[a_1(x,y,\xi)\int_0^1(\partial_x V)(\rho,t,\theta x+(1-\theta)W'_\xi(t,y,\xi))d\theta\right]$$

then

$$|\partial_x^\alpha\partial_y^\beta\partial_\xi^\gamma b_1(x,y,\xi)|\leq C_{\alpha,\beta,\gamma}<y+tP'(\xi)>^{-1-|\alpha|-\varepsilon}$$

$$\leq C'_{\alpha,\beta,\gamma}(1+|x|+|y|+t|P'(\xi)|)^{-1-|\alpha|-\varepsilon} \quad .$$

Using once again the Taylor formula, we obtain

$$\int e^{i<x,\xi>-iW(t,y,\xi)} b_1(x,y,\xi)d\xi=$$

$$=\int e^{i<x,\xi>-iW(t,y,\xi)} b_1(W'_\xi(t,y,\xi),y,\xi)d\xi+$$

$$+i\int e^{i<x,\xi>-iW(t,y,\xi)} \partial_\xi \int_0^1 (\partial_x b_1)(\theta x+(1-\theta)W'_\xi(t,y,\xi),y,\xi)d\theta d\xi \ .$$

The Remark 4.3 can be used to estimate the first term of the sum. We denote with b_2 the amplitude of the second term

$$b_2(x,y,\xi)=\partial_\xi \int_0^1 (\partial_x b_1)(\theta x+(1-\theta)W'_\xi(t,y,\xi),y,\xi)d\theta \ .$$

Then

$$|\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma b_2(x,y,\xi)| \leq C_{\alpha\beta\gamma} <y+tP'(\xi)>^{-2-|\alpha|-\varepsilon} .$$

We proceed like before until we obtain a remainder $b_N(x,y,\xi)$ which satisfies the conditions from the lemma of Schur. Summing up we obtain that

$$||B_{r,1}(t)|| \leq C(1+t)^{-1-\varepsilon} r^{-\varepsilon} \ .$$

Q.E.D.

The following result is an immediate corollary of Lemmas 4.7-4.10.

LEMMA 4.11. (i) If ψ is in $C_0^\infty(R^n)$ then there exist the operators

$$W_{r,\pm}=s\text{-}\lim_{t\to\pm\infty} \psi(H)e^{itH}E_{r,\pm}(t) \ .$$

(ii) $\lim_{r\to\infty} ||\psi(H)P_{r,\pm}-W_{r,\pm}||=0$.

Now the proof of the theorem proceeds as in [7]. For the sake of completeness we shall indicate the main steps.

LEMMA 4.12. The relation (2) (for $\tilde{W}_+$ and $t>0$) holds.

<u>Proof</u>. It can be shown using Lemmas 2.4, 2.5 and Remark 4.3 that there exists

$$\omega_{r,+}=\text{s-}\lim_{t\to+\infty} e^{iW(t,D)}E_{r,+}(t)$$

and Ran $\omega_{r,+}\subset\mathcal{H}_{ac}(H_o)$.

Hence $W_{r,+}=\psi(H)\tilde{W}_+\omega_{r,+}$. Lemma 4.11 implies that

$$0=\lim_{r\to\infty}||\tilde{W}_+^*\psi(H)P_{r,+}-\tilde{W}_+^*\psi(H)\tilde{W}_+\omega_{r,+}||=$$

$$=\lim_{r\to\infty}||\tilde{W}_+\tilde{W}_+^*\psi(H)P_{r,+}-\tilde{W}_+\psi(H_o)\omega_{r,+}||=$$

$$=\lim_{r\to\infty}||\tilde{W}_+\tilde{W}_+^*\psi(H)P_{r,+}-W_{r,+}||=$$

$$=\lim_{r\to\infty}||\tilde{W}_+\tilde{W}_+^*\psi(H)P_{r,+}-\psi(H)P_{r,+}|| \quad .$$

At the second step we have used the fact that for ψ in $C_o^\infty(R\setminus\bar{C}_V)$ the range of $\psi(H_o)$ is contained in $\mathcal{H}_{ac}(H_o)=\text{Ran } \tilde{W}_+^*\tilde{W}_+$.

Q.E.D.

LEMMA 4.13. For every u in $L^2(R^n)$ and ψ in $C_o^\infty(R\setminus\bar{C}_V)$ we have that

$$\limsup_{r\to\infty}\ \sup_{t\gtrless0} ||(1-\tilde{W}_\pm\tilde{W}_\pm^*)\psi(H)h(D)\theta(\tfrac{\cdot}{r})e^{-itH}u||=0 \ .$$

<u>Proof</u>. We consider again only the case $t>0$. It is sufficient to show that

$$\lim_{r\to\infty} \sup_{t>0} ||(1-\tilde{W}_+\tilde{W}_+^*)\psi(H)P_{r,\pm}e^{-itH}u||=0.$$

The assertion for $P_{r,+}$ is contained in the conclusion of Lemma 4.12. In the case of $P_{r,-}$ we proceed as follows. As a consequence of Lemma 4.6 we obtain that

$$\lim_{r\to\infty} \sup_{t<0} ||E_{r,-}(t)^*u||=0 \quad .$$

On the other hand we know that

$$\lim_{r\to\infty} \sup_{t<0} ||\phi(H)P_{r,-}-\phi(H)e^{itH}E_{r,-}(t)||=0$$

for every ϕ in $C_o^\infty(R)$. Therefore

$$\lim_{r\to\infty} \sup_{t>0} ||P_{r,-}^*e^{-itH}\phi(H)u||=0, \quad (\forall)\ u\in L^2(R^n),\ (\forall)\quad \phi\in C_o^\infty(R).$$

Since ϕ is arbitrary in $C_o^\infty(R)$ and $||P_{r,-}-P_{r,-}^*||\leq Cr^{-1}$ we have that

$$\lim_{r\to\infty} \sup_{t>0} ||P_{r,-}e^{-itH}u||=0, \quad (\forall)\ u\in L^2(R^n) \ .$$

Q.E.D.

LEMMA 4.14. Let ψ be in $C_o^\infty(R\setminus\overline{C}_V)$ and u in $\mathcal{H}_c(H)$. Then

$$\lim_{T\to\infty} \frac{1}{T}\int_o^{\pm T} ||(1-\tilde{W}_\pm\tilde{W}_\pm^*)\phi(H)e^{-itH}u||dt=0 \quad .$$

The proof of this lemma is identical to that of Lemma 4 from [8]. It makes use of the RAGE theorem and of the fact that $\eta(x)(H_o+i)^{-1}$ is a compact operator for every η in $C_o^\infty(R^n)$ (see [2]).

The statements (b) and (c) of the theorem are simply corollaries of this lemma. The last statement results from the fact that $(1-\tilde{W}_+\tilde{W}_+^*)\psi(H)$ is a compact operator for every ψ in $C_0^\infty(R\setminus\overline{C}_V)$.

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