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ISSN 0250 3638

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PREPRINT SERIES IN MATHEMATICS

No.38/1988

BUCURESTI

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July 1988

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1. INTRODUCTION

The purpose of this paper is to give a time dependent scattering theory for operators of the form $p_o(D)+V$, where $p_o(D)$ is a convolution operator with the symbol not satisfying the condition $\lim_{|\xi|\to\infty} p_o(\xi)=\infty$ and V is a short range perturbation.

The methods used here are essentially the same as those used in [1] and [5]. However, if in [1] and [5] the homogeneity property of the symbol of the free hamiltonian was intensely used, in this more general case the constructions of the auxiliary operators must be made with care, such that the results we obtain should not be affected by the absence of the homogeneity. As we shall see, these constructions are natural and give operators with nice commutation properties with functions of the free hamiltonian.

HYPOTHESES

I. The free hamiltonian H_o is a self-adjoint operator on the Hilbert space $\mathcal{H}=L^2(R^n)$, with the domain $\mathcal{D}(H_o)=\{u\in\mathcal{H}; p_o\hat{u}\in\mathcal{H}\}$,

$H_o u = p_o \hat{u}$, where $\hat{u}$ is the Fourier transform of u and p_o is a real valued function which satisfies:

(i) $p_o : \mathbb{R}^n \to \mathbb{R}$ is a continuous function.

(ii) If we denote by S the following set $\{\xi \in \mathbb{R}^n;\ p_o$ is not C^∞ in any neighborhood of ξ, or $\nabla p_o(\xi) = 0\}$, then $\overline{p_o(S)}$ is a countable subset of $\mathbb{R}$.

(iii) For any compact interval $I \subset \mathbb{R} \setminus \overline{p_o(S)}$ we have

$$\inf \{ |\nabla p_o(\xi)|;\ \xi \in p_o^{-1}(I) \} > 0 .$$

(iv) (local compactness). For any compact interval $I \subset \mathbb{R} \setminus \overline{p_o(S)}$ and for each $r > 0$, the operator

$$F(|x| < r) E_o(I)$$

is compact. Here $F(M)$ denotes the indicator function of the set M and $E_o(I)$ denotes the spectral projection for H_o onto the interval I.

II. Let $V : \hat{\mathcal{D}} \to \hat{\mathcal{D}}$ be a symmetric operator such that

(v) For some $\varepsilon > 0$ the operator $V \phi(H_o) \langle x \rangle^{1+\varepsilon}$ has a bounded extension to the whole of $\mathcal{H}$ for each ϕ in $C_o^\infty(\mathbb{R})$.

We used the notations: $\hat{\mathcal{D}}$ for the image of $\mathcal{D}$ (the space of test functions defined on $\mathbb{R}^n$) by the Fourier transform and $\langle x \rangle = (1 + |x|^2)^{1/2}$, $x \in \mathbb{R}^n$.

III. (vi) The operator $H_o + V$ with the domain $\hat{\mathcal{D}}$ has a self-adjoint extension H.

(vii) For any $\phi \in C_o^\infty(\mathbb{R})$ the operator

$\phi(H)-\phi(H_o)$

is a compact one.

The main result is the following

THEOREM 1.1. Assume that the hypotheses (i)-(vii) are satisfied. Then

(a) The wave operators $W_\pm = \text{s-}\lim_{t\to\pm\infty} e^{iHt}e^{-iH_o t}E_{ac}(H_o)$ exist;

(b) Range $W_\pm = \mathcal{H}_c(H)$, the continuous subspace of H;

(c) $\sigma_{sc}(H)=\Phi$;

(d) Any eigenvalue of H not in $\overline{p_o(S)}\cup\{0\}$ is of finite multiplicity. The eigenvalues of H can accumulate only at points of $\overline{p_o(S)}\cup\{0\}$.

Before proving the main theorem we wish to make a few remarks about the hypotheses we made and about the connections between the present paper and other related on this subject.

REMARK 1.2. a) The free hamiltonian H_o is a convolution operator with a continuous real symbol which satisfies the conditions (i)-(iii). The growth conditions which are commonly imposed (see [3], [4], [7], [8], [9]) are replaced with the condition (iii). This condition can be read as follows:

(iii)' If the free energy lies in a compact interval disjoint from thresholds, then the velocity is bounded from below by a positive constant.

b) If we replace the condition (iii) by the stronger condition.

(iii)" $\lim\limits_{\substack{|\xi|\to\infty \\ \xi\notin S}} (|p_o(\xi)|+|\nabla p_o(\xi)|)=\infty$,

then the local compactness property of H_o (i.e. condition (iv)) is fulfiled (see the appendix).

c) In the same way one can prove a similar theorem with the condition (v) replaced by the condition

(v)' For some $\varepsilon>0$ the operator $\phi(H)V\langle x\rangle^{1+\varepsilon}$ has a bounded extension to the whole of $\mathcal{H}$ for each ϕ in $C_o^\infty(R)$.

This condition is allways true when V is a symmetric H_o-compact operator and there is an $\varepsilon>0$ such that the operator

$$(H_o+i)^{-1}V\langle x\rangle^{1+\varepsilon}$$

has a bounded extension.

d) By taking into account the above remarks one can compare this paper with [7] and [8].

<u>Proof of Theorem 1.1</u>. (a) The proof of the existence of wave operators is standard (see [3], [4], [9]), so that we only sketch it.

By Cook's argument, it suffices to show that

$$\int_{-\infty}^{\infty}||V\phi(H_o)e^{-iH_ot}u||dt<\infty$$

for any $\phi\in C_o^\infty(\mathbb{R})$ and $u\in\mathcal{S}(\mathbb{R}^n)$ with supp $\hat{u}$ a compact set, disjoint from S. By using (v) this follows from

$$\int_{-\infty}^{\infty}||\langle x\rangle^{-1-\varepsilon}e^{-iH_ot}u||dt<\infty \quad ,$$

which can be proved for instance by writing

$$<x>^{-1-\varepsilon}e^{-iH_ot}u=<x>^{-1-\varepsilon}F(|x|\leqq\delta t)e^{-iH_ot}u+<x>^{-1-\varepsilon}F(|x|\geqq\delta t)e^{-iH_ot}u=B+C$$

and estimating $||B||$ by means of stationary phase method and $||C||$ by $c_{\delta,\varepsilon}<t>^{-1-\varepsilon}$.

The other parts of Theorem 1.1 will be proved below by means of time dependent methods.

2. PRELIMINARIES

In this section we shall make the constructions already announced. As a consequence they will provide operators which will serve to prove some propagation estimates which are the main tools in the proof of Theorem 1.1.

We pass now to define the operators which we mentioned at the begining. Let $\gamma\epsilon C_o^\infty(\mathbb{R}\backslash\overline{p_o(S)})$. We define the smooth vector field v in phase space by

$$(2.1)\quad v(\xi)=p_o(\xi)\gamma(p_o(\xi))|\nabla p_o(\xi)|^{-2}\nabla p_o(\xi),\quad \xi\epsilon\mathbb{R}^n .$$

Then the condition (iii) implies that there exists $0<c<\infty$ such that

$$(2.2)\qquad |v(\xi)|\leqq C,\quad \forall\quad \xi\epsilon\mathbb{R}^n .$$

From this relation it follows that the Cauchy problem

$$(2.3)\qquad \begin{aligned}&(d/d\alpha)\Gamma(\alpha,\xi)=v(\Gamma(\alpha,\xi))\\ &\Gamma(0,\xi)=\xi\end{aligned}$$

defines a group of C^∞-diffeomorphisms $\Gamma(\alpha,.):\mathbb{R}^n\to\mathbb{R}^n$. To this group of diffeomorphisms $\{\Gamma(\alpha,.)\}_{\alpha\epsilon\mathbb{R}}$ we associate a group of unitary operators $\{V(\alpha)\}_{\alpha\epsilon\mathbb{R}}$ on $L^2(\mathbb{R}^n,d\xi)$ by

(2.4) $$(V(\alpha)\psi)(\xi)=|\det \partial\Gamma(\alpha,\xi)/\partial\xi|^{1/2}\psi(\Gamma(\alpha,\xi)), \quad \psi\in L^2(\mathbb{R}^n,d\xi) .$$

If we denote by $\mathcal{F}$ the Fourier transform on $L^2(\mathbb{R}^n)$, then we obtain another group of unitary operators on $L^2(\mathbb{R}^n,dx)$ defined by

(2.5) $$U(\alpha)=\mathcal{F}^{-1}V(\alpha)\mathcal{F} \quad \text{on} \quad L^2(\mathbb{R}^n,dx) .$$

Let now $A=A_{H_o,\gamma}$ be the self-adjoint operator on $\mathcal{H}=L^2(\mathbb{R}^n,dx)$ such that

$$U(\alpha)=e^{-iA\alpha}$$

By taking into account the definition of $U(\alpha)$ one obtains in a straightforward manner the following

LEMMA 2.1. $\hat{\mathcal{D}}$ is a core of A and

(2.6) $$A=\sum_{j=1}^{n}(v_j(D)x_j+x_jv_j(D))/2 \quad \text{on } \hat{\mathcal{D}} .$$

Next we shall establish some relations which give the commutators $i[f(H_o),A]$ and $i[f(H_o),(A+i)^{-m}]$.

LEMMA 2.2. Let $f\in C^1(\mathbb{R})$ be a bounded function. Then the form $i[f(H_o),A]=i(f(H_o)A-Af(H_o))$ defined on $\mathcal{D}(A)$ has a bounded extension and

(2.7) $$i[f(H_o),A]=H_of'(H_o)\gamma(H_o)$$

Proof. The proof of this lemma is elementary and it is based on the relation

$$U(\alpha)f(H_o)U(-\alpha)=\mathcal{F}^{-1}M_{f\circ p_o\circ\Gamma(\alpha,.)}\mathcal{F} .$$

Q.E.D.

LEMMA 2.3. Let $m \in \mathbb{N}$ and let $f \in C^{\infty}(\mathbb{R})$ be a bounded function. Then

$$(A+i)^{-m} f(H_o) = \{ \sum_{k=0}^{m} \binom{m}{k} (A+i)^{-k} f_k(H_o) \} (A+i)^{-m} \tag{2.8}$$

with $f_k \in C^{\infty}(\mathbb{R})$ given by

$$f_k(\lambda) = (-i\lambda\gamma(\lambda)(d/d\lambda))^k f(\lambda)$$

<u>Proof</u>. The proof of this lemma is also elementary and is made by induction.

$$(A+i)^{-1} f(H_o) = f(H_o)(A+i)^{-1} + (A+i)^{-1} [f(H_o), A] (A+i)^{-1} =$$

$$= \{ f_o(H_o) + (A+i)^{-1} f_1(H_o) \} (A+i)^{-1} \qquad \text{(by (2.7))}$$

Assume that the statement is true for m. Then

$$(A+i)^{-m-1} f(H_o) = (A+i)^{-1} \{ \sum_{k=0}^{m} \binom{m}{k} (A+i)^{-k} f_k(H_o) \} (A+i)^{-m} \quad .$$

But (2.7) of Lemma 2.2 implies that

$$(A+i)^{-1} f_k(H_o) = \{ f_k(H_o) + (A+i)^{-1} f_{k+1}(H_o) \} (A+i)^{-1} \quad .$$

So we obtain

$$(A+i)^{-m-1} f(H_o) = \{ \sum_{k=0}^{m} \binom{m}{k} (A+i)^{-k} \, f_k(H_o) + (A+i)^{-1} f_{k+1}(H_o) \} (A+i)^{-m-1}$$

$$= \{ \sum_{k=0}^{m} \binom{m}{k} (A+i)^{-k} f_k(H_o) + \sum_{k=1}^{m+1} \binom{m}{k-1} (A+i)^{-k} f_k(H_o) \} \cdot (A+i)^{-m-1}$$

Since $\binom{m}{k} + \binom{m}{k-1} = \binom{m+1}{k}$, it follows that

$$(A+i)^{-m-1} f(H_o) = \{ \sum_{k=0}^{m+1} \binom{m+1}{k} (A+i)^{-k} f_k(H_o) \} (A+i)^{-m-1} \quad .$$

Q.E.D.

One can use the above results to prove another needed lemma.

LEMMA 2.4. For $0 \leqq \alpha \leqq 2$,

$$\langle A\rangle^{\alpha}(H_o+i)^{-1}\langle x\rangle^{-\alpha} \equiv J\alpha$$

is a bounded operator on $\mathscr{H}$. Here $\langle A\rangle=(1+A^2)^{1/2}=|A+i|$.

<u>Proof</u>. We need only to prove the case $\alpha=2$ and then use the complex interpolation. Thus we must to prove that

$$A^2(H_o+i)^{-1}\langle x\rangle^{-2}$$

is bounded. For a suitable function f, we obtain from Lemma 2.2 that

$$Af(H_o)=iH_o\gamma(H_o)f'(H_o)+f(H_o)A \ .$$

By iterating this formula we get

$$A^2f(H_o)=-H_o\gamma(H_o)(\gamma(H_o)f'(H_o)+H_o\gamma'(H_o)f'(H_o)+H_o\gamma(H_o)f''(H_o)) +$$
$$+2iH_o\gamma(H_o)f'(H_o)A+f(H_o)A^2 \ .$$

By taking $f(\lambda)=(\lambda+i)^{-1}$ we obtain the conclusion of Lemma 2.4 by using the explicit formula for A (Lemma 2.1).

Q.E.D.

We can now prove the basic estimate which we shall use in the proof of the asymptotic completeness. Since we shall work with functions belonging to the space $C_o^{\infty}((a,b))$, where (a,b) is an open interval such that $[a,b]\subset \mathbb{R}^+\setminus\overline{p_o(S)}\cup\{0\}$ or $[a,b]\subset \mathbb{R}^-\setminus\overline{p_o(S)}\cup\{0\}$, we shall consider, as an auxiliary operator, the self-adjoint operator $A=A_{H_o,\gamma}$ associated to a function

$\gamma\in C_0^\infty((\alpha,\beta))$, $0\leqq\gamma\leqq 1$, $\gamma=1$ in a neighborhood of $[a,b]$. Here (α,β) is another open interval such that $[a,b]\subset(\alpha,\beta)$ and $[\alpha,\beta]\subset \mathbb{R}^+\setminus\overline{p_0(S)}\cup\{0\}$ in the first case and $[\alpha,\beta]\subset\mathbb{R}^-\setminus\overline{p_0(S)}\cup\{0\}$ in the second case. Let P^+ and P^- be the spectral projectors of A on the positive and negative parts of its spectrum, $\langle A\rangle$ the usual operator $(1+A^2)^{1/2}=|A+i|$ and $\chi^\pm$ the indicator function of $\mathbb{R}^\pm\setminus\{0\}$. Then we have the following

THEOREM 2.5. Let $0\leqq\mu'<\mu$. Assume that (a,b) is an open interval such that $[a,b]\subset\mathbb{R}^+\setminus p_0(S)\cup\{0\}$. Let $g\in C_0^\infty((a,b))$. Then there is a constant $c=c(g,\mu,\mu')$ such that

$$||\chi^\pm(t)\langle A\rangle^{-\mu}e^{-iH_0t}g(H_0)P^\pm||\leqq c|t|^{-\mu'} \quad . \tag{2.9}$$

<u>Proof</u>. 1° The proof of this theorem follows in almost the same way as the proof of Theorem 2.1 of [1] or the proof of Lemma 2.1 of [5]. However, the absence of homogeneity of p_0 requires some changes which we shall point out at the right time.

We shall give the proof in the case $t>0$. As a result of the first three steps of the proof of Theorem 2.1 of [1] we obtain that

$$\langle A\rangle^{-m}e^{-iH_0t}g(H_0)P^+=m'!(it)^{-m'}e^{\varepsilon t}(2\pi i)^{-1}\cdot \tag{2.10}$$

$$\cdot\int_{-\infty}^{\infty}e^{-iEt}\langle A\rangle^{-m}(H_0-E-i\varepsilon)^{-m'-1}g(H_0)P^+dE \quad ,$$

$$||\int_{\mathbb{R}\setminus[a,b]}e^{-iEt}\langle A\rangle^{-m}(H_0-E-i\varepsilon)^{-m'-1}g(H_0)P^+dE||\leqq c_0(g,m,m'), \tag{2.11}$$

$$\forall\ 0\leqq\varepsilon\leqq 1,$$

and we see that we are reduced to study the family of operators

$$\{\langle A\rangle^{-m}(H_o-E-i\varepsilon)^{-m'-1}P^+\}_{\substack{E\in[a,b]\\ \varepsilon\in(0,1]}} .$$

(Here we use Lemma 2.3 to prove that $\langle A\rangle^{-m}g(H_o)\langle A\rangle^{m}$ is a bounded operator for $m\in\mathbb{R}$).

2° If $m,n\in\mathbb{N}$, $m>n$, $0<\varepsilon\leqq 1$, $a\leqq E\leqq b$, $0\leqq\theta\leqq\pi/2$, we define

$$F(\varepsilon,E,\theta)=\langle A\rangle^{-m}(H_oe^{-i\theta}-E-i\varepsilon)^{-n}e^{-\theta A}P^+$$

with

$$F(\varepsilon,E,0)=\langle A\rangle^{-m}(H_o-E-i\varepsilon)^{-n}P^+$$

and

$$\operatorname*{s-lim}_{\theta\to 0+} F(\varepsilon,E,\theta)=F(\varepsilon,E,0) .$$

Next we shall prove that the following estimate holds

$$(2.12)\quad ||(\partial/\partial\theta)F(\varepsilon,E,\theta)||\leqq c(g,m,n)||\langle A\rangle^{-m+1}(H_oe^{-i\theta}-E-i\varepsilon)^{-n}e^{-\theta A}P^+||$$

for $0<\varepsilon\leqq 1$, $a\leqq E\leqq b$, $0<\theta\leqq\delta$ with δ sufficiently small.

By using Lemma 2.2 it follows easily that

$$(\partial/\partial\theta)F(\varepsilon,E,\theta)=ine^{-i\theta}\langle A\rangle^{-m}H_o(1-\gamma(H_o))(H_oe^{-i\theta}-E-i\varepsilon)^{-n-1}e^{-\theta A}P^+$$

$$-\langle A\rangle^{-m}A(H_oe^{-i\theta}-E-i\varepsilon)^{-n}e^{-\theta A}P^+$$

To prove (2.12) we must estimate the norm of the operator

$$e^{-i\theta}\langle A\rangle^{-m}H_o(1-\gamma(H_o))(H_oe^{-i\theta}-E-i\varepsilon)^{-n-1}e^{-\theta A}P^+$$

which we shall write in the form

$$(A+i)^{m}\langle A\rangle^{-m}(A+i)^{-m}f(H_o)(H_oe^{-i\theta}-E-i\varepsilon)^{-n}e^{-\theta A}P^+$$

with

$$f(H_o)=H_o(1-\gamma(H_o))(H_o-Ee^{i\theta}-i\varepsilon e^{i\theta})^{-1} \quad .$$

Since we have $0<\alpha<a_1<a<b<b_1<\beta$, $[\alpha,\beta]\cap\overline{p_o(S)}=\phi$, supp $g\subset(a,b)$, supp $\gamma\subset(\alpha,\beta)$, $\gamma=1$ on (a_1,b_1), it follows from Lemma 2.3 by choosing δ sufficiently small that

$$(A+i)^{-m}f(H_o)=\{\sum_{k=0}^{m}\binom{m}{k}(A+i)^{-k}f_k(H_o)\}(A+i)^{-m}$$

with $\sum_{k=0}^{m}\binom{m}{k}(A+i)^{-k}f_k(H_o)$ uniformly bounded for $0<\varepsilon\leqq 1$, $a\leqq E\leqq b$ and $0\leqq\theta\leqq\delta$, so the proof of (2.12) is complete.

3° Following now the arguments of the steps 3°, 4°, 5° and 6° of the proof of Theorem 2.1 of [1], one proves that for every $(m,m')\in\mathbb{N}\times\mathbb{N}$, $m>m'+1$ there exists $c=c(g,m,m')$ such that

$$(2.13) \qquad ||\langle A\rangle^{-m}e^{-iH_ot}g(H_o)P^+||\leqq ct^{-m'} \quad .$$

From this relation we deduce that for every $(m,m')\in\mathbb{R}\times\mathbb{N}$, $m>m'+2$ there is $c=c(g,m,m')$ such that (2.13) holds.

Now the general case follows by interpolation.

Q.E.D.

For the case $[a,b]\subset\mathbb{R}^-\setminus\overline{p_o(S)}\cup\{0\}$ we have the following

COROLLARY 2.6. Let $0\leqq\mu'<\mu$. Assume that (a,b) is an open interval such that $[a,b]\subset\mathbb{R}^-\setminus\overline{p_o(S)}\cup\{0\}$. Let $g\in C_o^\infty((a,b))$. Then there is a constant $c=c(g,\mu,\mu')$ such that

$$(2.9)' \quad ||\chi^{\mp}(t)\langle A\rangle^{-\mu}e^{-iH_ot}g(H_o)P^{\pm}||\leqq c|t|^{-\mu'} \quad .$$

<u>Proof</u>. Apply the above theorem to the operator $-H_o$ (i.e.

$-p_o(D))$ and the functions $\check{g}$, $\check{\gamma}$, and observe that $A_{-H_o,\check{\gamma}} = A_{H_o,\gamma}$.

Here f means the function defined by $\check{f}(x)=f(-x)$.

Q.E.D.

3. ASYMPTOTIC COMPLETENESS

For the proof of the asymptotic completeness we need two compactness results. We start with the case of an open interval (a,b) such that $[a,b] \subset \mathbb{R}^+ \setminus \overline{p_o(S)} \cup \{0\}$.

LEMMA 3.1. Assume that the hypotheses (i)-(vii) are fulfilled, and let (a,b) be an open interval such that $[a,b] \subset \mathbb{R}^+ \setminus \overline{p_o(S)} \cup \{0\}$. Then for every $g \in C_o^\infty((a,b))$ the operators

$$(W_\pm - 1) g(H_o) P^\pm$$

are compact on $\mathcal{H}$.

<u>Proof</u>. We have

$$(W_+ - 1) g(H_o) P^+ = i \int_o^\infty e^{iHs} V e^{-iH_o s} g(H_o) P^+ ds$$

and for any $s>0$, $Ve^{-iH_o s} g(H_o) P^+$ is a compact operator as it follows from the hypotheses (iv) and (v).

Furthermore the integral

$$\int_o^\infty || V e^{-iH_o s} g(H_o) P^\pm || ds$$

is well defined since

$$||Ve^{-iH_o s} g(H_o) P^+|| = ||V\gamma(H_o)(H_o+i)^{-m} e^{-iH_o s}(H_o+i)^m g(H_o)P^+|| \leq$$

$$\leq ||V\gamma(H_o)\langle x\rangle^{1+\varepsilon}|| \; ||\langle x\rangle^{-1-\varepsilon}(H_o+i)^{-m}\langle A\rangle^{1+\varepsilon}|| \cdot$$

$$\cdot ||\langle A\rangle^{-1-\varepsilon} e^{-iH_o s}(H_o+i)^m g(H_o) P^+||$$

From Theorem 2.5 and the condition (v), it suffices to verify that $m>0$ can be chosen such that $\langle x\rangle^{-1-\varepsilon}(H_o+i)^{-m}\langle A\rangle^{1+\varepsilon}$ is a bounded operator on $\mathcal{H}$.

By Lemma 2.4 this is true for $m=1$, because we always may suppose that $\varepsilon\leqq 1$ in (v).

Q.E.D.

COROLLARY 3.2. Assume that the hypotheses (i)-(vii) are fulfilled, and let (a,b) be an open interval such that $[a,b]\subset\mathbb{R}^-\setminus\overline{p_o(S)}\cup\{0\}$. Then for every $g\in C_o^\infty((a,b))$ the operators

$$(W_{\mp}-1)g(H_o)P^{\pm}$$

are compact on $\mathcal{H}$.

Now it is clear that the conclusions of Theorem 1.1 can be obtained by using the Enss argument. Since there are many papers on this method ([1], [3], [9], [10]) we shall not repeat Enss' argument here.

APPENDIX

We commented in Remark 1.2 b) that the local compactness property of H_o is implied if we assume that the function p_o satisfies conditions (i), (ii) and (iii)". In this appendix we shall prove this compactness result which we shall state as

PROPOSITION A.1. Assume that $p_o:\mathbb{R}^n\to\mathbb{R}$ is a function which satisfies the conditions (i), (ii) and (iii)". Let $I\subset\mathbb{R}\setminus\overline{p_o(S)}$ be a compact interval and let $r>0$. Then

$$F(|x|\leqq r)E_o(I)$$

is a compact operator on $\mathcal{H}$.

We assume that $\mathbb{R}^n$ is divided into unit "cubes" C_k , $k\in\mathbb{N}$ so that

$$\mathbb{R}^n=\bigcup_k \overline{C}_k \quad \text{and} \quad C_k\cap C_\ell=\emptyset \quad \text{for } k\neq\ell .$$

Then, in order to prove Proposition A.1, it suffices to show (cf. Corollary 3 of [2]) that the following lemma is true.

LEMMA A.2. Assume that p_o satisfies (i), (ii) and (iii)". Let $I\subset\mathbb{R}\setminus\overline{p_o(S)}$ be a compact interval. Then

$$\lim_{k\to\infty} |C_k\cap p_o^{-1}(I)|=0 .$$

Here $|A|$ denotes the Lebesgue measure of the measurable set A.

Proof. If we denote by

$$\beta_k=\inf \{|\nabla p_o(\xi)|;\quad \xi\in p_o^{-1}(I)\cap C_k\}$$

then the compactness of I and the condition (iii)" imply that

$$\lim_{k\to\infty} \beta_k=\infty$$

Therefore the proof of the lemma is completed by the following estimate:

$$\text{(A.1)} \qquad |p_o^{-1}(I)\cap C_k|\leq n\sqrt{n}|I|\beta_k^{-1} , \quad k\in\mathbb{N} .$$

Let $B_j=|\xi\in\mathbb{R}^n\setminus S;\ |\nabla p_o(\xi)|\leq\sqrt{n}|\partial_j p_o(\xi)|\}$ and $\Phi_j:\mathbb{R}^n\setminus S\to\mathbb{R}^n$ defined by

$$\Phi_j(\xi)=(\xi_1,\ldots,\xi_{j-1},p_o(\xi),\xi_{j+1},\ldots,\xi_n)$$

for $j=1,\ldots,n$. Then Φ_j is a local diffeomorfism at every point in $p_o^{-1}(I)\cap B_j$.

Since $p_o^{-1}(I)\cap C_k=\bigcup_{j=1}^{n} p_o^{-1}(I)\cap C_k\cap B_j$, then (A.1) follows from

(A.1)' $\qquad |p_o^{-1}(I)\ C_k\cap B_j|\leqq\sqrt{n}|I|\beta_k^{-1},\quad k\in\mathbb{N},\quad j=1,\dots,n$.

This estimate can be obtained by making a change of variable. Let us write $p_o^{-1}(I)\cap C_k\cap B_j$ as a disjoint union $\bigcup_\ell \Phi_j^{-1}(C_{kI})\cap C_k\cap B_j\cap M_\ell$, where $C_{kI}=\pi_1(C_k)x\dots x\pi_{j-1}(C_k)xIx\pi_{j+1}(C_k)x\dots x\pi_n(C_k)$ and M_ℓ $\ell\in\mathbb{N}$ are disjoint measurable sets which have neighborhoods on which Φ_j is a diffeomorphism.

Then

$$p_o^{-1}(I)\cap C_k\cap B_j=\bigcup_\ell \Phi_j^{-1}(A_{kj\ell})$$

where $A_{kj\ell}$, $\ell\in\mathbb{N}$ are disjoint measurable subsets of C_{kI} such that $\Phi_j^{-1}(A_{kj\ell})=\Phi_j^{-1}(C_{kj})\cap C_k\cap B_j\cap M_\ell$, Φ_j is a diffeomorphism in a neighborhood of $\Phi_j^{-1}(A_{kj\ell})$ and

$$|\det(\Phi_j^{-1})'(\eta)|\leqq\sqrt{n}\beta_k^{-1}\quad \text{for } \eta\in A_{kj\ell}\ .$$

Hence

$$|p_o^{-1}(I)\cap C_k\cap B_j|=\sum_\ell\int_{\Phi_j^{-1}(A_{kj\ell})}d\xi=\sum_\ell\int_{A_{kj\ell}}|\det(\Phi_j^{-1})'(\eta)|d\eta\leqq$$

$$\leqq\sqrt{n}\beta_k^{-1}\int_{C_{kI}}d\eta=\sqrt{n}|I|\beta_k^{-1}$$

Q.E.D.

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