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by

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by V.Nistor

1. Introduction

In a pioneering work J.Diximier and A.Douady classified fields of elementary C^*-algebras ([12]).Fields of AF-C^*-algebras were considered by several authors,[1],[2],[10],[12],but the classification problem has not been solved completely.

If $X = S^n$ then the isomorphism classes of homogeneous locally trivial fields of C^*-algebras with fiber A are in one-to-one correspondence with $\pi_{n-1}(\mathrm{Aut}(A))/\pi_0(\mathrm{Aut}(A))$ ([15]) and $\pi_k(\mathrm{Aut}(A))$ has been computed for a large class of AF-C^*-algebras([20],[26]). A similar device holds if (X,x_0) is a pointed compact connected CW-complex.If we denote by $[X,\mathrm{Aut}(A)]$ the set of homotopy classes of basepoint preserving mappings $X \to \mathrm{Aut}(A)$,Aut(A) being pointed by the identity automorphism,then isomorphism classes of homogeneus locally trivial fields of C^*-algebras with fiber A are in one-to-one correspondence with $[X,\mathrm{Aut}(A)]/\pi_0(\mathrm{Aut}(A))$.

The main result of this paper is the determination of $[X,\mathrm{Aut}(A)]$ up to an extension of groups if A is an AF-C^*-algebra satisfying certain technical conditions as in 4.1. We also determine the kernel of this extension.

The techniq of proof generalysis the techniq developed in [20] and the resulting exact sequence has close trends with the exact sequence of the Universal Coefficient Theorem for Kasparov's KK-groups,[24].

An important fact is that $\mathrm{Aut}^0(A) \to \mathrm{End}^0(A)$ is a weak homotopy equivalence ([20]).

Let us suppose that A is simple and (X,x_0) is a H'-space ([27], chapter III) then our results are complete and give isomorphisms $[X,\mathrm{Aut}(A)] \xrightarrow{\sim} KK^0(A, C_0(X\setminus\{x_0\}, A))$ if $1 \notin A$ and $[X,\mathrm{Aut}(A)] \xrightarrow{\sim} KK^1(A', C_0(X\setminus\{x_0\}, A))$ if $1 \in A$, here A' denotes the mapping cone of the inclusion $\mathbb{C} \to A$.

The next section contains notational conventions. Section 3 contains results about filtered modules, morphisms and extensions of filtered modules. The definitions and the results of this section are in the spirit of those in [20]. Their purpose is to give a satisfactory framework for the groups appearing in the exact sequence. The objects we introduce and the theorem we prove reduce to well known ones if the filtrations are trivial and this endeed happens if A is simple. The reader interested only in this case may very well skip this sections. Section four containes preliminary results concerning cancellatio and comparability of projections. The results we obtain are crucial in turning K-theory data in homotopy information, they are in the spirit of the programs of [4] and [23]. Sections five and six contain the exact sequence in the general case and the determination of $[X,\mathrm{Aut}(A)]$ for A simple and X a H'-space. The last section contains a brief discution of the Samelson product. It is proved that in general there exists no natural grou structure on the set of isomorphism classes of locally trivial fields of C^*-algebras with fiber A. This contrasts with the results of J. Diximier and A. Douady ([12]).

2. Notations and conventions

In this section we shall fix same notations and make some conventions to be used in the sequel.

$K_i, i \in \{0,1\}$ will denote the K-theory functors ([5],[25]). $M(A)$ is the multiplier C^*-algebra of A ([21]). We shall denote by $U(A)$ the set of those unitaries μ in $M(A)$ such that $\mu - 1 \in A$. S is the suspension factor in the category of pointed topological spaces or in the category of C^*-algebras.

By <u>ideal</u> we shall mean <u>closed two-sided ideal</u>.

Let A,B be C^*-algebras, we shall denote by $Hom(A,B)$ the set of all $*$-homomorphisms $A \to B$ with the topology of pointwise norm convergence.

We shall denote by id_θ the identity morphism of an object θ.

1 denotes the unit of various C -algebras, if A has no unit, 1 denotes the unit of $M(A)$.

$\mathcal{K}$ is the C^*-algebra of compact operators on a separable Hilbert space.

If A is a C^*-algebra, A^+ denotes the algebra with adjoint unit.

3. Filtered modules, morphisms and extensions of filtered modules

Let Ω be a complete lattice, R a commutative ring with unit.

3.1. Definition An Ω-filtered module E over a ring R is a left R-module E with a family of submodules $(E_\omega)_{\omega\in\Omega}$ such that $\omega\to E_\omega$ is a morphism of lattices. An Ω-filtered $\mathbb{Z}$-module will be called simply an Ω-filtered abelian group.

3.2. Let E, F be Ω-filtered R-modules. A morphism $f:E\to F$ such $f(E_\omega)\subset F_\omega$ will be called compatible.
The set of compatible morphisms $f:E\to F$ will be denoted by $\mathrm{Hom}_{R,c}(E,F)$.

3.3. Let $0\to E\to F\to G\to 0$ be an exact sequence of Ω-filtered R-modules with compatible morphisms.

Definition The above exact sequence will be called a compatible extension of G by E if $E_\omega = F_\omega \cap E$ and $E_\omega/F_\omega \to G_\omega$ is an isomorphism.
Two compatible extensions $0\to E\to F_j\to G\to 0, j\in\{0,1\}$ are equivalent if there exists a commutative diagram of compatible morphisms

$$\begin{tikzcd} 0 \arrow[r] & E \arrow[r] \arrow[d, equal] & F_0 \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & E \arrow[r] & F_1 \arrow[r] & G \arrow[r] & 0 \end{tikzcd}$$

A compatible extension $0\to E\to F\xrightarrow{f} G\to 0$ will be called trivial if there exists a compatible morphism $f_1:G\to F$ such that $f\circ f_1=\mathrm{id}_G$.

The pointed set of equivalence classes of compatible extensions of G by E will be denoted by $\mathrm{Ext}_{R,c}(G,E)$.

3.4. We want to show that $\mathrm{Ext}_{R,c}(G,E)$ is a group with the Baer sum as operation and with the trivial extension as neutral element.

Let $g \in G$. Denote by $\Omega_g = \{\omega \in \Omega,\ g \in E_\omega\}$.
Then Ω_g is a complete sublattice of Ω. Denote by $\omega(g)$ the least element of Ω_g and call it <u>support</u> of g.
For each $g \in G, g \neq 0$ choose $f(g) \in F_{\omega(g)}$ such that $\rho(f(g)) = g$, if $0 \to E \to F \xrightarrow{\rho} G \to 0$ is a compatible extension.

Let $f(0) = 0, \xi(g_1,g_2) = f(g_1) + f(g_2) - f(g_1 + g_2)$, $\zeta(r,g) = rf(g) - f(rg)$. Then $(\xi,\zeta) \in Z^1_{R,c}(G,E)$ if we denote by $Z^1_{R,c}(G,F)$ the group of pairs (ξ,ζ), $\xi : G\times G \to E$, $\zeta : R\times G \to E$ satisfying:

(1) $\xi(g_1,g_2) + \xi(g_1 + g_2,g_3) = \xi(g_1,g_2+g_3) + \xi(g_2,g_3)$,

(2) $\xi(g_1,g_2) = \xi(g_2,g_1)$,

(3) $\xi(0,g) = \xi(g,0) = 0, \quad (0,g) = (1,g) = 0$,

(4) $\zeta(r_1r_2,g) = \zeta(r_1,r_2g) + r_1\zeta(r_2,g)$

(5) $r\xi(g_1,g_2) = \xi(rg_1,rg_2) + \zeta(r,g_1) + \zeta(r,g_2) - \zeta(r,g_1+g_2)$

(6) $\xi(g_1,g_2) \in E_\omega, \zeta(r,g) \in E_\omega$ for any $g_1,g_2,g \in G_\omega, r \in R$

$(g,g_1,g_2 \in G,\ r,r_1,r_2 \in R)$.

Denote by $B^1_{R,c}(G,E) \subset Z^1_{R,c}(G,E)$ the group of these pairs $(\xi,\zeta) \in Z^1_{R,c}(G,E)$ such that

$$\xi(g_1,g_2) = e(g_1) + e(g_2) - e(g_1+g_2),$$

$$\zeta(r,g) = -e(rg) + re(g)$$

for some function $e : G \to E, e(0) = 0, e(G_\omega) \subset E_\omega, (\forall)\ \omega \in \Omega$.

Let $x \in \mathrm{Ext}_{R,c}(G,E)$. Define (ξ,ζ) as above, two different choises of f give elements of $Z^1_{R,c}(G,E)$ wich differ by an element of $B^1_{R,c}(G,E)$. This shows that we obtain a well defined function $c:\mathrm{Ext}_{R,c}(G,E)\to Z^1_{R,c}(G,E)/B^1_{R,c}(G,E)$.

Conversely, given (ξ,ζ) satisfying (1) to (6) then let $G\times E = F$ with opperations

$$(g_1,e_1) + (g_2,e_2) = (g_1+g_2,e_1+e_2+ \xi(g_1,g_2))$$

$$r(g,e) = (rg,re+\zeta(r,g)),e,e_1,e_2\in E,g,g_1,g_2\in G.$$

and filtration $F_\omega = G_\omega\times E_\omega$. Note that F is an Ω-filtered R-module due to (1)-(6) and satisfies an exact sequence $0\to E_\omega\to F_\omega\to G_\omega\to 0$ for any $\omega\in\Omega$ and if we let $f:G\to F$ be given by $f(g) = (g,0)$ then $\xi(g_1,g_2) = f(g_1) + f(g_2) - f(g_1 + g_2)$ and $\zeta(r,g) = rf(g) - f(rg)$.

Since to Baer sum of extension there corresponds the sum of cycles we get that c is the desired isomorphism (see [18]). It is obvious that $\mathrm{Ext}_{R,c}(G,E)\simeq\{0\}$ if G is a free R-module.

3.5. Let $\varphi:G_1\to G$, $\psi:E\to E_1$, be compatible morphisms then we obtain morphisms $Z^1_{R,c}(G,E)\to Z^1_{R,c}(G_1,E)$, $B^1_{R,c}(G,E)\to B^1_{R,c}(G_1,E)$, $Z^1_{R,c}(G,E)\to Z^1_{R,c}(G,E_1)$, $B^1_{R,c}(G,E)\to B^1_{R,c}(G,E_1)$ defined by $\xi\to\xi\circ(\varphi\times\varphi)$, $\zeta\to\zeta\circ(\mathrm{id}_R\times\varphi)$ and $\xi\to\psi\circ\xi$, $\zeta\to\psi\circ\zeta$.
This shows that $\mathrm{Ext}_{R,c}(\cdot,\cdot)$ is contravariant in the second variable in the category of Ω-filtered R-modules with compatible morphisms, and covariant in the first variable.

We shall denote by $\varphi^*:\mathrm{Ext}_{R,c}(G,E)\to\mathrm{Ext}_{R,c}(G_1,E)$ and $\psi_*:\mathrm{Ext}_{R,c}(G,E)\to\mathrm{Ext}_{R,c}(G,E_1)$ the morphisms defined by φ and ψ.

3.6. <u>Lemma</u> Let F,E_n be Ω-filtered R-modules, $\varphi_n:E_n\to E_{n+1}$, $E = \varinjlim(E_n,\varphi_n)$. Then there exists an exact sequence

$$0\to\varprojlim{}^1(\mathrm{Hom}_{R,c}(E_n,F),\varphi_n^*)\to\mathrm{Ext}_{R,c}(E,F)\to\varprojlim(\mathrm{Ext}_{R,c}(E_n,F),\varphi_n^*)\to 0$$

Proof.

Let $X_n \in \mathrm{Ext}_{R,c}(E_n,F)$ such that $\varphi_n^*(X_{n+1}) = X_n$. There exists an infinite commutative diagram

$$\begin{tikzcd}
\cdots & \cdots \arrow[d, equal] & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \\
0 \arrow[r] & F \arrow[r] \arrow[d, equal] & G_n \arrow[r] \arrow[d, "\psi_n"] & E_n \arrow[r] \arrow[d, "\varphi_n"] & 0 \\
0 \arrow[r] & F \arrow[r] \arrow[d, equal] & G_{n+1} \arrow[r] \arrow[d] & E_{n+1} \arrow[r] \arrow[d] & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{tikzcd}$$

Such that $0 \to F \to G_n \to E_n \to 0$ represents X_n in $\mathrm{Ext}_{R,c}(E_n,F)$. Let $G = \lim(G_n,\psi_n)$. Then there exists an exact sequence $0 \to F \to G \to E \to 0$ whose image in $\mathrm{Ext}_{R,c}(E_n,F)$ is X_n.

Let us identify the kernel of $\mathrm{Ext}_{R,c}(E,F) \xrightarrow{\eta} \varprojlim(\mathrm{Ext}_{R,c}(E_n F),\varphi_n^*)$. To this end denote by $\chi_n : E_n \to E$ the obvious morphism. Suppose that $0 \to F \to G \to E \to 0$ defines an element $X \in \mathrm{Ext}_{R,c}(E,F)$ such that $\chi_n^*(X) = 0$ for any χ. Then exists a compatible morphism $\tau_n : E_n \to G$ making the following diagram commutative

$$\begin{tikzcd}
 & & & E_n \arrow[dl, "\tau_n"'] \arrow[d] & \\
0 \arrow[r] & F \arrow[r] & G \arrow[r] & E \arrow[r] & 0
\end{tikzcd}$$

Let $\lambda = (\lambda_n)_{n\in N} \in \bigoplus_{n\in N} \mathrm{Hom}_{R,c}(E_n,F)$, $\lambda_n = \tau_{n+1}\circ\varphi_n - \tau_n$. Denote by $d: \bigoplus_{n\in N} \mathrm{Hom}_{R,c}(E_n,F) \to \bigoplus_{n\in N} \mathrm{Hom}_{R,c}(E_n,F)$ the morphisms $d((f_n)_{n\in N}) = (f_{n+1}\circ\varphi_n - f_n)_{n\in N}$. Then two different choises of τ_n define sequences λ differing by an element in Imd. This shows that there exists a well defined morphism $\ker\eta \to \varprojlim^1(\mathrm{Hom}_{R,c}(E_n,F),\varphi_n^*)$ which turns out to be an isomorphis m.

3.7. We shall need also an other group, the group of extensions with order unit. It is defined as in [20].

Let E,G be Ω-filtered modules, $u \in G$ an element such that $u \in G_\omega$ if and only if $\omega = \sup\Omega$ ($\sup\Omega$ is the largest element in Ω).

By a compatible extension with order unit of G by E we shall mean a compatible extension $0 \to E \to F \to G \to 0$ such that F has a

order unit v and $f(\mathcal{V}) = u$. We shall write in this case $0 \to E \to$
$\to (F,\mathcal{V}) \to (G,\mathcal{U}) \to 0$.

Two compatible extensions with order unit
$0 \to E \to (F_j, v_j) \to (G,\mathcal{U}) \to 0$ $j \in \{0,1\}$ are equivalent if and only if there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & E & \to & (F_0, v_0) & \to & (G,\mathcal{U}) & \to & 0 \\ & & \| & & \downarrow f & & \| & & \\ 0 & \to & E & \to & (F_1, v_1) & \to & (G,\mathcal{U}) & \to & 0 \end{array}$$

such that f is a compatible morphism and $f(v_0) = v_1$.

A compatible extension with order unit
$0 \to E \to (F,\mathcal{V}) \to (G,\mathcal{U}) \to 0$ will be called <u>trivial</u> if there exists a compatible morphism $f_1 : G \to F$ such that $f_1(\mathcal{U}) = \mathcal{V}$ and $f \circ f_1 = id_G$.

We shall denote by $Ext^{\mathcal{U}}_{R,c}(G,E)$ the set of equivalence classes of compatible extensions with order unit of G by E. We omitt R when $R = \mathbb{Z}$.

3.8. <u>Propozition</u> a) $Ext^{\mathcal{U}}_{R,c}(G,\ )$ is a covariant functor from the category of Ω-filtered R-modules with compatible morphisms to Ens.

b) $Ext^{\mathcal{U}}_{R,c}(\ ,E)$ is a contravariant functor from the category of Ω-filtered R-modules with unit preserving compatible morphism to Ens.

c) $Ext^{\mathcal{U}}_{R,c}$ is an abelian group in a natural way.

d) There exists an exact sequence:

$$0 \to Hom(G/R\mathcal{U},E) \to Hom_{R,c}(G,E) \to Hom_{R,c}(G,E) \to Hom_R(R\mathcal{U},E) \to$$
$$\xrightarrow{\varepsilon_G} Ext^{\mathcal{U}}_{R,c}(G,E) \to Ext_{R,c}(G,E) \to 0$$

Proof.(Sketch) a) Let $f:E\to E_1$ be a compatible morphism between the Ω-filtered R-modules E and E_1.Let $0\to E\to(F,\mathcal{V})\to$ $\to(G,\mathcal{U})\to 0$ be a compatible extension with order unit, denote by X its class in $\mathrm{Ext}^n_{R,c}(G,E)$.There exists by 3.5 a commutative diagram of compatible morphisms

$$\begin{array}{ccccccccc} 0 & \to & E & \to & F & \to & G & \to & 0 \\ & & \downarrow f & & \downarrow f' & & \| & & \\ 0 & \to & E_1 & \to & F_1 & \to & G & \to & 0 \end{array}$$

Then $f_*(X)$ is defined to be the class of

$0\to E\to(F_1,f'(v))\to(G,\mathcal{U})\to 0$ in $\mathrm{Ext}^{\mathcal{U}}_{R,c}(G,E_1)$.

b) Let $\varphi:(G_1,\mathcal{U}_1)\to(G,\mathcal{U})$ be a unit preserving compatible morphisms, G,G_1 being Ω-filtered R-modules with order units.Let $X\in\mathrm{Ext}^n_{R,c}(G,E)$ be represented by $0\to E\to(F,v)\to(G,\mathcal{U})\to 0$ Let $F_1\subset F\oplus G_1$ be the submodule consisting of those pairs (f,g_1) such that $h(f) = \varphi(g_1)$.Let $\mathcal{V}_1 = (\mathcal{V},\mathcal{U}_1)$ be the order unit of F_1 then $\varphi^*(X)$ is represented by $0\to E\to(F_1,\mathcal{V}_1)\to(G_1,\mathcal{U}_1)\to 0$

c) Let $0\to E\to(F_j,\mathcal{V}_j)\to(G,\mathcal{U})\to 0$ represent $X_j\in\mathrm{Ext}^n_{R,c}(G,E)$, $j\in\{0,1\}$.Denote by $d_2:G\to G\times G$ the diagonal map: $d_2(g) = (g,g)$, and by $\sigma_2:E\times E\to E$ the "addition" map: $\sigma_2(e_1,e_2)$ $= e_2 + e_2$.Let $X\in\mathrm{Ext}^n_{R,c}(G\times G,E\times E)$ be represented by

$$0\to E\oplus E\to(F_1\oplus F_2,(v_1,v_2))\to(G\oplus G,(\mathcal{U},\mathcal{U}))\to 0$$

Then $x_1 + x_2$ is defined to be $d_2^*(\sigma_{2*}(X)) = \sigma_{2*}(d_2^*(X))$.(The last relation is proved as lemma III.1.6. of [18])

d) The morphism in $\varepsilon_G:\mathrm{Hom}_R(R_n,E)\to\mathrm{Ext}^n_{R,c}(G,E)$ is defined as follows.Let $e\in E$ and denote by f_e the morphism $f_e(r) = re$.Then $\varepsilon_G(f_e)$ is the class of $0\to E\to(E\oplus G,(-e,\mathcal{U}))\to$ $\to(G,\mathcal{U})\to 0$.$\mathrm{Ext}^{\mathcal{U}}_{R,c}(G,E)\to\mathrm{Ext}_{R,c}(G,E)$ is defined by "forgeting" the units.The exactness is obvious.

4 Preliminary results

4.1 We shall fix from now on an AF-C^*-algebra A with the following proprierties:

a) Let $I \subset J \subset A$ be ideals $I \neq J$, then J/I is not type $\overline{\underline{I}}$

b) Either $1 \in A$ or A is <u>completely nonunital</u> in the sense that for any projection $e \in A$, $(1 - e)A(1 - e)$ is full in A.

4.2. <u>Definition</u> ([20]) definition 2.2). Let (G,G_+) be an ordered group, we say that (G,G_+) has large dominators if for any $g \in G_+$ and $n \in \mathbb{N}$ there exists $g_1 \in G_+$ and $m \in \mathbb{N}$ such that $ng_1 \leqslant g \leqslant mg_1$.

4.3. <u>Proposition</u> Let A be an AF-C^*-algebra. Then A satisfies 4.1.a) if and only if $K_0(A)$ has large denominators.

<u>Proof</u>. Suppose that $\pi : A \to B(\mathcal{H})$ is an irreducible representation such that $\mathcal{K}(\mathcal{H}) \subset \pi(A)$ ($\mathcal{K}(\mathcal{H})$ denotes the algebra of compact operators on $\mathcal{H}$).

Using L.Brown's lifting projection theorem for AF-C^*-algebras ([7], [13]) we find a projection $e \in A$ such that $\pi(e)$ is a rank one projection in $\mathcal{K}(\mathcal{H})$. Then $g=[e]$ does not satisfy the conditions of definition 2.1.

Conversely, let e be a projection in $M_q(A)$ for some $q \in \mathbb{N}$. Denote by J' the ideal generated by e in $M_q(A)$. Fix $n \in \mathbb{N}$ and let $J=eJ'e=\overline{\cup J_k}$ eith J_k finite dimensional. Denote by I_k the ideal of J_k consisting of those factors of J_k having dimension $\geqslant n$. It follows that $I_k \subset I_{k+1}$ and hence $I=\overline{\bigcup I_k}$ is an ideal of J. Denote as in [11] by $r(n)$ the last integer m with the propriety that $\sum_{\sigma \in S_m} \mathrm{sgn}(\sigma) a_{\sigma(1)} \cdots a_{\sigma(m)} = 0$ for any $a_1, \ldots, a_m \in M_n(\mathbb{C})$ (S_m is the symetric group of order m). It follows that for any $x_1, \ldots, x_m \in J/I$, $m \geqslant r(n)$, $\sum_{\sigma \in S_m} \mathrm{sign}(\sigma) x_{\sigma(1)} \cdots x_{\sigma(m)} = 0$ since this is true for x_j in the dense subalgebra J_k/I_k. The proof of [11], proposition 3.6.3. shows that J/I has only finite dimensional representations (of dimension $\leqslant n$).The assumption on A shows that I = J and hence $e \in J_k$ for same large k.Chose a minimal projection from each factor of J_k and denote by p their summ. Then $n[p] \leqslant [e] \leqslant m[p]$ for same large $m \in \mathbb{N}$.

4.4 Let $\mathcal{B} = (A(x), x \in X, \Gamma)$ be a locally trivial field of C^*-algebras such that $A(x) \simeq A$ for any $x \in X$, and X a compact space ([11], ch.X).$\mathcal{B}$ may be viewed as a fiber bundle with Aut(A) as structure group.Denote by $Aut^0(A)$ the connected component of the identity in Aut(A).Recall that $Aut^0(A) = \overline{Inn(A)}$ ([2]):

Denote by Ω the lattice of ideals of A, Ω can be identified with the lattice of the ideals of $A \otimes \mathcal{K}$.

Let $\varphi \in$ Aut (A), then $\varphi(\omega) = \omega$ for any $\omega \in \Omega$. Suppose that our bundle admits a restriction of the structure group to $Aut^0(A)$ (this allways happens if X is simply connected).Denote by ξ the associated $Aut^0(A)$ principal bundle.

The $Aut^0(A)$-equivariant inclusion $\omega \subset A$ gives rise to an inclusion $\xi[\omega] \subset \xi[A]$ of fiber bundles.(Our notation and terminology are taken from [15]).

Denote by $B(B_\omega)$ the C^*-algebra of continuous sections of $\xi[A] = \mathcal{B}\ (\xi[\omega])$, see [11], ch X We obtain an Ω-filtration of $K_0(B)$ by $K_0(B)_\omega = Ran(K\ (B\ ) \to K_0(B))$.

The following proposition is the key in translating homotopy information into K-theory algebraic language.

4.5. <u>Proposition</u> Let (X, x_0) be a pointed compact connected CW-complex, A, B, B_ω as above.Denote by $\eta : K_0(B) \to K_0(A)$ the morphism of "evaluation at x_0".

a) If $a, a' \in K_0(B)$, $\eta(a') \geqslant \eta(a)$ and $m(\eta(a') - \eta(a)) \geqslant \eta(a)$ for some $m \in \mathbb{N}$ then $a' \geqslant a$.

b) Let $a, a' \in K_0(B)$, $a = [e]$, $\eta(a) = \eta(a')$.Denote by ω the ideal generated by $e(x_0)$ in $A \otimes \mathcal{K}$.Then $a' \geqslant 0$ if and only if $a' - a \in K_0(B)_\omega$

c) Suppose ξ is trivial, then B has the cancellation propriety for projections and $\pi_j(U(B)) \to K_{j+1}(B)$ is an isomorphism for any $j \geq 0$.

Proof.

The idea of proof will be to regard elements $b \in B$ satisfying certain Proprieties as sections in a suitable defined fiber bundle.

We may assume that B is stable (i.e $B \simeq B \otimes \mathcal{K}$)

a) Let e, e' be projections in B representing a, a' in $K_0(B)$. We may assume that $e(x_0) = e'(x_0)$. Denote by $V(x) = \{v(x) \in A(x), v^*(x)v(x) = e(x), v(x)v^*(x) \leq e'(x)\}$, it is a sort of "generalised Stieffel manifold".

Let $V = \bigcup_{x \in X} V(x) \subset \xi[A]$ with the induced topology, then V becomes a locally trivial fiber bundle on X It is easy to prove that $U(e(x_0)Ae(x_0)) \ni u \to ue \in V(x_0)$ is a locally trivial fiber bundle (the proof is similar to lemma1.2 of [20]) with fiber $U((e(x_0)-e(x_0))A(e(x_0)-e(x_0)))$. The hypothesis shows that the ideals generated by $e(x_0)$ and $e(x_0)-e(x_0)$ in $A \otimes \mathcal{K}$ coincide. The exact sequence of homotopy groups ([27]) and proposition 2.4.b) of [20] shows that $\pi_k(V(x)) \simeq \pi_k(V(x_0)) \simeq \{0\}$ for any $k \geq 0$ and $x \in X$. A standard argument ([15], theorem 7.1. pag 21) shows that V has a cross-section. This cross-section defines a partial isometry from e to a subprojection of e'.

b) Let $e, e' \in B$ be projections such that $e(x_0) = e'(x_0)$. Then $e, e' \in B_\omega$ and hence $[e]-[e'] \in K_0(B)_\omega$.

Conversely, suppose that $a = [e]$, $a' - a \in K_0(B)_\omega$, ω being the ideal generated by $e(x_0)$ in A. Then $Aut^o(A)$ acts on ω^+. Let B be the C^*-algebra of continuous sections in $\xi[\omega^+]$. The split exact sequence $0 \to B_\omega \to \tilde{B}_\omega \xrightarrow{\chi_0} C(x) \to 0$ shows that the element

$a' = (a'-a) + a$ of $K_0(B_\omega)$ may be represented by $[e_1']-[e_1]$, e_1 and e_1' being projections in $\tilde{B}_\omega \otimes \mathcal{K}$ such that $\chi(e_1) = \chi(e_1)$, $(\chi = \chi_0 \otimes id_{\mathcal{K}})$. Denote by $\chi_x : \omega(x)^+ \otimes \mathcal{K} \to \mathcal{K}$ the quotient morphism ($\omega(x)^+$ is the fiber of $\xi[\omega^+]$ at x). Define $W(x) = \{ w(x) \in \omega(x)^+ \otimes \mathcal{K}$,

$\chi_x(w(x)) = \chi_x(e_1'(x)) (= \chi_x(e_1(x)))$,

$w(x)^* w(x) = e_1(x), w(x) w(x)^* \leqslant e_1'(x) \}$. It follow as in b) that $\pi_k(W_x) \simeq \{0\}$ and that $W = \bigcup_{x \in X} W_x$ has a cross-section. This shows that e_1 is equivalent to a subprojection of e_1 and hence $a' \geqslant 0$.

c) The cancellation propriety follows from standard results in topology. Ended, suppose that e_1, e_2 are projections in $M_q(B)$ such that $[e_1] = [e_2]$ in $K_0(B)$. We write $A = \overline{\cup A_n}$, A_n are finite dimensional C^*-algebras. Let d denote the dimension of X. We may suppose that $e_1, e_2 \in C(X, A_n)$ for same large n. Also, since $K_0(A)$ has large denominators we may suppose that the dimensions of the projections e_1 and e_2 in $K_0(C(X,A_n))$ are large enough (greater than d /2) and that $[e_1] = [e_2]$ in $K_0(C(X,A_n))$. This can be done by encreasing n if necessary. But stable isomorphic vector bundles of large dimension are isomorphic (see [15], theorem 8.1.7, page 100).

Let us observe that $s = tsr(C(X)) < \infty$, (see [22]) for definition and notations. It follows that $\pi_0(U(M_q(C(X))))$ is isomorphic to $K_1(C(X))$ for $q \geqslant s+2$ ([22], theorem 10.12), use also the fact that the topological stable rank and the Bass stable rank coincide for C^*-algebras [14]). This shows that $\pi_j(U(B)) \simeq \varinjlim \pi_j(U(C(X,A_n))) \simeq \varinjlim \pi_0(U(S^j C(X,A_n))) \simeq \varinjlim K_{j+1}(C(X,A_n)) \simeq K_{j+1}(B)$ since the dimension of the blocks of A_n encrease to ∞.

Let $A = \cup A_n$ with A_n finite dimensional. Denote by $i_{m,n}$ the inclusion $A_n \to A_m$ and by i_n the inclusion $A_n \to A$. Let $\mathrm{Hom}^0(A_n, A)$ denote the connected component of i_n in $\mathrm{Hom}(A_n, A)$, pointed by i_n.

Let $B = C(X,A), J = C_0(X\setminus\{x_0\},A)$. Denote by $[X,\mathrm{Hom}^0(A_n,A)]$ the homotopy classes of base point preserving continuous functions $\varphi: X \to \mathrm{Hom}^0(A_n,A)$. Such a continuous function φ defines a morphism $\Phi_n(\varphi): A_n \to B$. We shall denote by $j_n: A_n \to B$ the morphism $\Phi_n(\varphi)$ for $\varphi(x) = i_n\ (\forall)\ x \in X$.

The following lemma shows the power of the previous proposition.

4.6. Lemma $[X,\mathrm{Hom}^0(A_n,A)] \ni [\varphi] \to K_0(\Phi_n(\varphi)) - K_0(j_n) \in \mathrm{Hom}_c(K_0(A_n),K_0(J))$ is well defined and bijective if the filtration of $K_0(J)$ is $K_0(J)_\omega = K_0(C_0(X\setminus\{x_0\},\omega))$, the filtration of $K_0(A_n)$ is $K_0(A_n)_\omega = K_0(i_n)^{-1}(K_0(\omega))$, and A does not have unit.

Proof.

Let $p_1,\dots,p_k$ be the minimal projections of A_n, denote by ω_ℓ the ideal generated by p_ℓ in A. It follows from proposition 4.5.b) that $K_0(\Phi_n(\varphi))([p_\ell]) - K_0(j_n)([p_\ell]) \in K_0(J)_{\omega_\ell}$. Conversely, suppose that $f \in \mathrm{Hom}_c(K_0(A_n),K_0(J))$ then it follows also from proposition 4.5.b) that there exists a projection $e'_\ell \in B$ such that $[e'_\ell] = [j_n(p_\ell)] + f([p_\ell])$. Suppose that $A_n = A_n^{(1)} \oplus \dots \oplus A_n^{(k)}$ and $A_n^{(q)}$ is a factor of type I_{m_q}. Using proposition 4.5.b) one obtaines by induction $m_1+m_2+\dots+m_k$ ortogonal projections $\tilde{e}^{(1)}_{11},\dots,\tilde{e}^{(1)}_{m_1m_1},\tilde{e}^{(2)}_{11},\dots,\tilde{e}^{(k)}_{m_km_k}$ such that $[\tilde{e}^{(\ell)}_{rr}] = [e'_\ell]$. It follows then from proposition 4.5.c) that $\tilde{e}^{(\ell)}_{11}$ is equivalent to $\tilde{e}^{(\ell)}_{rr}$, there exists $\tilde{e}^{(\ell)}_{r1}$ such that $\tilde{e}^{(\ell)*}_{r1}\tilde{e}^{(\ell)}_{r1} = \tilde{e}^{(\ell)}_{11}$ and $\tilde{e}^{(\ell)}_{r1}\tilde{e}^{(\ell)*}_{r1} = \tilde{e}^{(\ell)}_{rr}$, $r \in \{2,\dots,m_1\}$. Denote by $\tilde{e}^{(\ell)}_{rq} = \tilde{e}^{(\ell)}_{r1}(\tilde{e}^{(\ell)}_{q1})^*$, $q \in \{2,\dots m_1\}$ $\tilde{e}^{(\ell)}_{1r} = \tilde{e}^{(\ell)*}_{r1}$. Let $e^{(\ell)}_{rq}$ $\ell \in \{1,\dots,k\}, r,q \in \{1,\dots,m\}$ denote a matrix unit of A_n. There exists $\varphi: X \to \mathrm{Hom}^0(A_n,A)$, $\varphi_x(e^{(\ell)}_{rq}) = \tilde{e}^{(\ell)}_{rq}(x)$. Moreover any map $X \to \mathrm{Hom}^0(A_n,A)$ is homotopic to a base point preserving map. This shows the surjectivity of $[X,\mathrm{Hom}^0(A_n,A)] \to \mathrm{Hom}_c(K_0(A_n),K_0(J))$.

In order to prove that it is injective let us observe that if $\varphi,\psi \in \mathrm{Hom}(A_n,B)$ have the proprierty that $K_0(\varphi) = K_0(\psi)$ then it follows from proposition 4.5.c) using a standard trick of O.Bratteli that φ and ψ are unitary conjugated: $\varphi = ad_u \circ \psi$ with $u \in U(J)$.Let $e = \psi(1)$.Then there exists a unitary v in $U((1-e)J(1-e))$ such that uv is in the connected component of the identity (use proposition 4.5.c)) and $\varphi = ad_{uv} \circ \psi$ .It follows that φ is homotopic to ψ.

4.7 Remark Let us observe that if X is a H'-space ([27],ch III, 3) then $[X, Hom^0(A_n, A)]$ is a group and $[X, Hom^0(A_n, A)] \to$ $\to Hom_c(K_0(A_n), K_0(J))$ is actually a morphism.

5. The exact sequence

Denote by $Map(X, Aut^0(A))$ the space of base point preserving continuous mappings $X \to Aut^0(A)$.

5.1. There exists a commutative diagram

$$\begin{tikzcd} Map(X,Aut^0(A)) \arrow[d, "\Phi"'] \arrow[dr] & \\ Hom(A,B) \arrow[r, "T"] & End(B) = Hom(B,B) \end{tikzcd}$$

The first vertical arrow associates to a continuous function $X \ni x \to \varphi_x \in Aut(A)$ the morphism $\Phi(\varphi): A \to B$ given by $\Phi(\varphi)(a)(x) = \varphi_x(a)$. The hoorisontal arrow associates to a morphism $\psi: A \to B$ the morphism $T(\psi): C(X) \otimes A \to B$ defined by $T(\psi)(f \otimes a) = f\psi(a)$.

Passing to K-groups one obtaines the following commutative diagram

$$\begin{tikzcd} {[X, Aut^0(A)]} \arrow[d, "\alpha_i'"'] \arrow[dr, "\alpha_i"] & \\ Hom(K_0(A), K_0(B)) \arrow[r, "\mu_X"] & Hom_{K^0(X)}(K_0(B), K^0(B)) \end{tikzcd}$$

$(i = 0$ if $1 \notin A, i = 1$ if $1 \in A)$.

Since $K_0(B) \simeq K^0(X) \otimes K_0(A)$ it follows that $K_0(B)$ is a $K^0(X)$-module. This module structure can be described directly as follows, let $[e] \in K_0(B), [p] \in K^0(X)$ with $e \in M_q(B)$, $p \in M_r(C(X))$. Then $[p][e]$ is the class of $(p \otimes I_q)(I_r \otimes e) \in M_{rq}(B)$ in $K_0(B)$. This shows that $\alpha_i([\varphi])$ is endeed $K^0(X)$-linear.

$[X, Aut(A)]$ is a group with the law $[\varphi][\psi] = [\varphi \circ \psi]$.

It is clear from definition that $\alpha_i([\varphi][\psi]) = \alpha_i([\varphi])\alpha_i([\psi])$.

μ_X can be described by $\mu_X(f)(x \otimes z) = xf(z)$ for $f \in Hom(K_0(A), K_0(B))$, $x \in K^0(X), z \in K_0(A)$.

Let us denote by $G^i (G^{i'})$ the range of $\alpha_i (\alpha_i')$. Since $\mu_X: Hom(K_0(A), K_0(B)) \to Hom_{K^0(X)}(K_0(B), K_0(B))$ is bijective it follows that it is enough to determine $G^{i'}$.

Denote as before by Ω the lattice of ideals of A and observe that $K_0(A), K_j(B), K_j(J)$ have natural Ω-filtrations $(j \in \{0,1\})$.

If $\varphi \in \mathrm{Map}(X, \mathrm{Aut}(A))$ is constant denote by $\iota = \iota_X$ the embedding $K_0(A) \to K_0(B)$ defined by $K_o(\Phi(\varphi))$. It coincides with the composition of $K_0(A) \ni [e] \to [1] \otimes [e] \in K^0(X) \otimes K_0(A)$ with the isomorphism $K^0(X) \otimes K_o(A) \xrightarrow{\sim} K_0(B)$.

5.2. The following constructions are needed in order to determine the kernel of α_i.

Let $\varphi \in \mathrm{Map}(X, \mathrm{Aut}(A))$. Denote by $E_\varphi \subset M_\varphi \subset C([0,1] \times X, A)$ the C^*-algebras defined by $E_\varphi = \{ f, (\exists) a \in A$ such that $f(0,x) = a, f(t,x_0) = a, f(1,x) = \varphi_x(a)$ for any $x \in X, t \in [0,1] \}$ $M_\varphi = \{ f, f(1,x) = \varphi_x(f(0,x))$ for any $x \in X \}$. M_φ is the mapping torus of $\tilde{\varphi} = T \circ \Phi(\varphi) \in \mathrm{End}(B)$ (see [5]).

Let $\rho : E_\varphi \to A$, $\rho(f) = f(0,x_0)$, $\chi : M_\varphi \to B$, $\chi(f) = f|\{0\} \times X$.

Then there exists a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & SJ & \to & E_\varphi & \xrightarrow{\rho} & A & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & SB & \to & M_\varphi & \xrightarrow{\chi} & B & \to & 0 \end{array}$$

Let us recall that the connecting morphisms of the K-theory exact sequence of the bottom row are the composition of $\mathrm{id} - K_j(\tilde{\varphi}) : K_j(B) \to K_j(B)$ and of $K_j(B) \xrightarrow{\sim} K_{j-1}(SB)$ (see [5], proposition 10.4.1)

Observe that if we denote by $K_*(\tilde{\varphi}) = K_0(\tilde{\varphi}) \oplus K_1(\tilde{\varphi}) : K_*(B) = K_0(B) \oplus K_1(B) \to K_*(B)$ then $K_*(\tilde{\varphi})$ is the unique $K^*(X) = K^0(X) \oplus K^1(X)$-linear extension of $K_0(\tilde{\varphi})$. Thus if $K_0(\tilde{\varphi}) = \mathrm{id}_{K_0(B)}$ then also $K_1(\tilde{\varphi}) = \mathrm{id}_{K_1(B)}$.

We obtain for any $\varphi \in \mathrm{Map}(X, \mathrm{Aut}(A))$ such that $\alpha_i([\varphi]) = \mathrm{id}$ a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 \to & K_1(J) & \to & K_0(E_\varphi) & \to & K_0(A) & \to 0 & & (1) \\ & \downarrow \simeq & & \downarrow & & \downarrow \iota & & & \\ 0 \to & K_1(B) & \to & K_0(M_\varphi) & \to & K_0(B) & \to 0 & & (2) \end{array}$$

If we denote $E_{\varphi,\omega} = E_\varphi \cap C([0,1]\times X,\omega)$, $M_{\varphi,\omega} = M_\varphi \cap C([0,1]\times X,\omega)$ then $K_0(E_\varphi)$ and $K_0(M_\varphi)$ have natural Ω-filtration. Moreover there exists an obvious morphism of $C(X)$ in the center of $M(M\varphi)$ giving a $K^0(X)$-module structure on $K_0(M\varphi)$.

Let us denote by $\gamma_0'(\varphi)$ the class of

(1) in $\mathrm{Ext}_c(K_0(A),K_1(J))$ and by $\gamma_0(\varphi)$ the class of

(2) in $\mathrm{Ext}_{K^0(X),c}(K_0(B),K_1(B))$ for $[\varphi]\in \ker \alpha_0$.

If A has a unit then $E\varphi$ and $M\varphi$ are also unital and the quotient morphisms ρ and χ are unit preserving. Let us note also that $K_0(E\varphi)$ and $K_0(M\varphi)$ have order units given by the classes of the units. This shows that if $\varphi \in \mathrm{Map}(X,\mathrm{Aut}(A))$, $\alpha_1([\varphi]) = \mathrm{id}_{K_0(A)}$ then we can define $\gamma_1(\varphi) \in \mathrm{Ext}^u_{K^0(X),c}(K_0(B),K_1(B))$ and $\gamma_1'(\varphi) \in \mathrm{Ext}^u_c(K_0(A),K_1(J))$ regarding (1) and (2) as compatible extensions with order unit.

Let $\varphi,\psi \in \mathrm{Map}(X,\mathrm{Aut}(A))$, $\tilde{\psi} = T_0\circ\Phi(\psi)$. Denote by σ_2 the composition of $SB \oplus SB \simeq C_0((0,1/2),B) \oplus C_0((1/2,1),B) \to SB$. Then, if we denote by $D = \{ f \in C([0,2],B), f(1) = \tilde{\psi}(f(0)), f(2) = \tilde{\varphi}(f(1))$,we obtain a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 \to & SB \oplus SB & \to & M_\psi \oplus M_\varphi & \to & B \oplus B & \to 0 \\ & \| & & \uparrow & & \uparrow \mathrm{id} & \\ 0 \to & SB \oplus SB & \longrightarrow & D & \longrightarrow & B & \longrightarrow 0 \\ & \swarrow \sigma_2 & & \downarrow & & \| & \\ 0 \to SB & & \longrightarrow & M_{\varphi\circ\psi} & \longrightarrow & B & \longrightarrow 0 \end{array}$$

If $K_0(\tilde{\varphi}) = K_0(\tilde{\psi}) = \mathrm{id}_B$ then the corresponding diagram of K_0-groups shows that $\gamma_i(\varphi\circ\psi) = \gamma_i(\varphi) + \gamma_i(\psi)$, $i \in \{0,1\}$. It is obvious that if φ is homotopic to the constant map $x \to \mathrm{id}_A$ then $\gamma_i(\varphi) = 0$. Also observe that there exists obvious morphisms

$r_0 : \mathrm{Ext}_{K^0(X),c}(K_0(B),K_1(B)) \to \mathrm{Ext}_c(K_0(A),K_1(J))$ and $r_1 : \mathrm{Ext}^u_{K^0(X),c}(K_0(B),K_1(B)) \to \mathrm{Ext}^u_c(K_0(A),K_1(J))$ obtained by composing the "forgetfull" morphism $\mathrm{Ext}_{K^0(X),c}(\cdot,\cdot) \to \mathrm{Ext}_c(\cdot,\cdot)$ $(\mathrm{Ext}^u_{K^0(X),c}(\cdot,\cdot) \to \mathrm{Ext}^u_c(\cdot,\cdot))$ with ι^* and using the isomorphism $K_1(J) \cong K_1(B)$. It follows that $\gamma'_i = r_i \circ \gamma_i$ is also a morphism. It also follows that $\gamma_0, \gamma'_0, \gamma_1$ and γ'_1 depend only on the class of φ in $[X, \mathrm{Aut}(A)]$. The preceding discution is partially included in the following lemma:

5.3. Lemma

a) There exist commutative diagram of morphisms:

$$\begin{tikzcd}
 & \ker \alpha_0 \arrow[dl, "\gamma_0"'] \arrow[dr, "\gamma'_0"] & \\
\mathrm{Ext}_{K^0(X),c}(K_0(B),K_1(B)) \arrow[rr, "r_0"] & & \mathrm{Ext}_c(K_0(A),K_1(J))
\end{tikzcd}$$

for A non unital, and

$$\begin{tikzcd}
 & \ker \alpha_1 \arrow[dl, "\gamma_1"'] \arrow[dr, "\gamma'_1"] & \\
\mathrm{Ext}^u_{K^0(X),c}(K_0(B),K_1(B)) \arrow[rr, "r_1"] & & \mathrm{Ext}^u_c(K_0(A),K_1(J))
\end{tikzcd}$$

for A unital. r_i is an isomorphism $i \in \{0,1\}$.

b) $\gamma_i([\psi]\xi[\psi]^{-1}) = \alpha_i([\psi])^{*-1} K_1(\tilde{\psi})_*(\gamma_i(\xi))$ for $\psi \in \mathrm{Map}(X, \mathrm{Aut}(\tilde{A}))$ $\xi \in \ker \alpha_i$.

Proof. Let $A = \overline{\cup A_n}$ with A_n finite dimensional.

Let us observe that there exists by lemma 3.6 a commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}_{K^0(X),c}(K_0(B),K_1(B)) \arrow[r, "h_0"] \arrow[d, "\cong"] & \mathrm{Ext}_c(K_0(A),K_1(J)) \arrow[d, "\cong"] \\
\varprojlim^1 \mathrm{Hom}_{K^0(X),c}(K^0(X) \otimes K_0(A_n), K_1(B)) \arrow[r, "\cong"] & \varprojlim^1 \mathrm{Hom}_c(K_0(A_n),K_1(J))
\end{tikzcd}$$

from which it follows that r_0 is also an isomorphism.

We get using lemma 3.8. a commutative diagram with exact rows:

$$
\begin{tikzcd}
{} \arrow[r] & \mathrm{Hom}_c(K_0(A),K_1(J)) \arrow[r] \arrow[d] & \mathrm{Hom}(\mathbb{Z},K_1(J)) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \mathrm{Hom}_{K^0(X),c}(K_0(B),K_1(B)) \arrow[r] & \mathrm{Hom}_{K^0(X)}(K^0(X),K_1(B)) \arrow[r] & {}
\end{tikzcd}
$$

$$
\begin{tikzcd}
{} \arrow[r] & \mathrm{Ext}^u_c(K_0(A),K_1(J)) \arrow[r] \arrow[d] & \mathrm{Ext}_c(K_0(A),K_1(J)) \arrow[r] \arrow[d] & 0 \\
{} \arrow[r] & \mathrm{Ext}^u_{K^0(X),c}(K_0(B),K_1(B)) \arrow[r] & \mathrm{Ext}_{K^0(X)}(K_0(B),K_1(B)) \arrow[r] & 0
\end{tikzcd}
$$

We obtain from the five lemma that r_1 is also an isomorphism.

The equality of b) follows from the commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & SB \arrow[r] \arrow[d, "S\tilde{\psi}"] & M \arrow[r] \arrow[d, "\varphi"] & B \arrow[r] \arrow[d, "\tilde{\psi}"] & 0 \\
0 \arrow[r] & SB \arrow[r] & M \arrow[r, "\psi\circ\varphi\circ\psi^{-1}"'] & B \arrow[r] & 0
\end{tikzcd}
$$

if $[\varphi] = \zeta$.

5.4. Proposition $G^{0} = \iota + \mathrm{Hom}_c(K_0(A),K_0(J))$ and γ'_0 is an isomorphism.

<u>Proof</u>. Let $A = \overline{\cup A_n}$ with A_n finite dimensional. Denote as before by $i_{m,n}: A_n \to A_m$ the inclusion of A_n in A_m and by i_n the inclusion of A_n in A. Let $\mathrm{Hom}^0(A_n,A)$ be the connected component of i_n in $\mathrm{Hom}(A_n,A)$, pointed by i_n . It is proved in [20], lemma 1.2. that the restriction $i_{m,n}: \mathrm{Hom}^0(A_m,A) \to \mathrm{Hom}^0(A_n,A)$ is a fibration, and hence $\mathrm{Map}(X,\mathrm{Hom}^0(A_m,A)) \to \mathrm{Map}(X,\mathrm{Hom}(A_n,A))$ is also a fibration ([27], theorem(7.10), pag 31).

Denote $\mathrm{Hom}(A,A)$ by $\mathrm{End}(A)$ pointed by id_A and by $\mathrm{End}^0(A)$ the connected component of id_A in $\mathrm{End}(A)$. Then $\mathrm{Map}(X,\mathrm{Aut}^0(A)) \to \mathrm{Map}(X,\mathrm{End}^0(A))$ is a weak homotopy equivalence ([20], lemma 1.4). It is obvious that $\mathrm{Map}(X,\mathrm{End}^0(A))$ is homeomorphic to the inverse limit $\varprojlim \mathrm{Map}(X,\mathrm{Hom}^0(A_n,A))$.

The proof of Theorem(4.8) from [27], pag 433 shows that $H = \varprojlim^1 \pi_1(\mathrm{Map}(X,\mathrm{Hom}^0(A_n,A)))$ acts free on the pointed set $\pi_0(\mathrm{Map}(X,\mathrm{End}^0(A)))$ and that $\pi_0(\mathrm{Map}(X,\mathrm{End}^0(A))) \to \varprojlim \pi_0(\mathrm{Map}(X,\mathrm{Hom}^0(A_n,A)))$ gives a bijection

$$\pi_0(\mathrm{Map}(X,\mathrm{End}^0(A)))/H \to \varprojlim \pi_0(\mathrm{Map}(X,\mathrm{Hom}^0(A_n,A))).$$

Let us observe that $\pi_1(\mathrm{Map}(X,\mathrm{Hom}^0(A_n,A)))$ is naturaly isomorphic to $\left[SX,\mathrm{Hom}^0(A_n,A)\right] \simeq \mathrm{Hom}_c(K_0(A_n),K_1(J))$ by lemma 4.6.(use also remark 4.7.).It follows also from lemma 4.6. that there exists a commutative diagram

$$\begin{tikzcd}
\left[X,\mathrm{End}^0(A)\right] \arrow[r] \arrow[d, "\alpha_0-\iota"] & \varprojlim \pi_0(\mathrm{Map}(X,\mathrm{Hom}^0(A_n,A))) \arrow[d] \\
\mathrm{Hom}_c(K_0(A),K_0(J)) \arrow[r] & \varprojlim \mathrm{Hom}_c(K_0(A_n),K_0(J))
\end{tikzcd}$$

in which the bottom arrow is an isomorphism and the right vertical arrow is a bijection.

We obtain the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \varprojlim^1 \mathrm{Hom}_c(K_0(A_n),K_1(J)) \arrow[r] & \left[X,\mathrm{End}^0(A)\right] \arrow[r] & \iota+\mathrm{Hom}_c(K_0(A),K_0(J)) \arrow[r] & 0 \\
 & \mathrm{Ext}_c(K_0(A),K_1(J)) \arrow[u, "\simeq"] & \ker\alpha_0 \arrow[l, "\gamma_0'"'] \arrow[u] & &
\end{tikzcd}$$

in which the first vertical arrow is an isomorphism by lemma 3.6 and the top horrisontal line is an exact sequence of pointed sets.

The proof of this proposition will be concluded if we show that this diagram is commutative.

Let $\varphi \in \mathrm{Map}(X,\mathrm{Aut}^0(A))$, $[\varphi] \in \ker\alpha_0$.Our assumption shows that $\varphi|A_n \in \mathrm{Map}(X,\mathrm{Hom}^0(A_n,A))$ is homotopic to the function $x \to i_n$ via a homotopy $\psi_n \in \mathrm{Map}([0,1]\times X/[0,1]\times\{x_0\},\mathrm{Hom}^0(A_n,A))$ i.e. $\psi_n|_{\{1\}\times X} = \varphi|_{A_n}$, $\psi_n(0,x) = i_n$. ψ_n defines a morphism $f_n : A_n \to E$ such that $\rho\circ f_n = i_n$ (ρ is the quotient map $E_\varphi \to A$).

It follows from lemma 3.6. that $\gamma_0'([\varphi])$ is represented in $\varprojlim^1 \mathrm{Hom}_c(K_0(A_n),K_1(J))$ by the sequence

$(\lambda_n)_{n\in\mathbb{N}} = (K_0(f_{n+1})\circ K_0(i_{n+1,n}) - K_0(f_n))_{n\in\mathbb{N}}$ (we identify $K_0(SJ)$ to $K_1(J)$ by Bott periodicity).

Let us observe that we may define

$\eta_n' : [0,1]\times X \to \mathrm{Hom}^0(A_n,A)$ by $\eta_n'(t,x) = \psi_{n+1}(2t,x)/_{A_n}$ for $t \in [0,1/2]$ and by $\eta_n'(t,x) = \psi_n(2-2t,x)$ for $t \in [1/2,1]$ then

$\eta'_n(0,x) = \eta'_n(t,x_0) = \eta'_n(1,x) = i_n$ and η'_n factors to a mapping $\eta_n : SX \to \mathrm{Hom}^0(A_n,A)$ of pointed spaces. Then ([27], Theorem (4.8), pag 433) φ is represented in $\varprojlim^1 [SX,\mathrm{Hom}^0(A_n,A)]$ by the sequence $([\eta_n])_{n\in\mathbb{N}}$. Since $[\eta_n]$ is sent to $\lambda_n \in \mathrm{Hom}_c(K_0(A_n),K_1(J))$ under the isomorphism of lemma 4.6 (see also 4.7) the commutativity of the diagram follows.

We now turn to the unital case.

Let us note first that $A\otimes\mathcal{K}$ is completely non unital and that there exists a morphism $\mathrm{Aut}(A)\to\mathrm{Aut}(A\otimes\mathcal{K})$ given by $\eta\to\eta\otimes \mathrm{id}_{\mathcal{K}}$. We have denoted by $\mathcal{K}$, as usual, the C^*-algebra of compact operators on a separable Hilbert space.

Denote by $\sigma : [X,\mathrm{Aut}(A)] \to [X,\mathrm{Aut}(A\otimes\mathcal{K})]$ the corresponding morphism. The following lemma is folklore and identifies the range of this morphism. We sketch its proof for the convenience of the reader.

5.5. Lemma Let $\psi\in \mathrm{Map}(X,\mathrm{Aut}(A\otimes\mathcal{K}))$ then $[\psi]$ is in the range of σ if and only if $\alpha_0([\psi])([1]) = [1]$.

Proof. One implication is obvious.

Let $(e_{n,m})_{n,m\in\mathbb{N}}$ denote a matrix unit of $\mathcal{K}$. Denote as usual $B = C(X,A)$ and let $\tilde{\psi} : B\otimes\mathcal{K}\to B\otimes\mathcal{K}$ be the morphism de ined by ψ, $f_{n,m} = 1\otimes e_{n,m}\in B\otimes\mathcal{K}$. If $\tilde{\psi}(f_{n,m}) = f_{n,m}$ then it follows that there exists $\varphi\in \mathrm{Map}(X,\mathrm{Aut}(A))$ such that $\tilde{\psi} = \tilde{\varphi}\otimes \mathrm{id}_{\mathcal{K}}$ ($\tilde{\varphi}$ is the morphism $B\to B$ defined by φ), and it follows that ψ is even in the range of $\mathrm{Map}(X,\mathrm{Aut}(A))\to\mathrm{Map}(X,\mathrm{Aut}(A\otimes\mathcal{K}))$. In general, the assumption that $K_0(\tilde{\psi})([1]) = \alpha_0([\psi])([1]) = [1]$ shows that $\tilde{\psi}(f_{00})$ is equivalent to f_{00} (use proposition 4.5.). (If we identify $K_0(A)$ with $K_0(A\otimes\mathcal{K})$ by stability, then $[1] = [f_{00}]$.) Let $v\in B\otimes\mathcal{K}$ be such that $v^*v = f_{00}$, $vv^* = \tilde{\psi}(f_{00})$. Let $u = \sum_{n\in\mathbb{N}} \tilde{\psi}(f_{n0})vf_{0n}$, the convergence being in the strict topology of $M(B\otimes\mathcal{K})$, u is a unitary in $M(B\otimes\mathcal{K})$ and $\mathrm{ad}_u(f_{n,m}) = \tilde{\psi}(f_{n,m})$. Since the unitary group of $M(D\otimes\mathcal{K})$ is contractible for any C^*-algebra D([9],[19]) it follows that there exists an arc of unitaries $u_t^* \in M(B \otimes \mathcal{K})$ connecting u to $1 \in M(B \otimes \mathcal{K})$ and such that the value of u_t in x_0 is a multiple of the identity in $M(A \otimes \mathcal{K})$ (use the fact that $U(M(A \otimes \mathcal{K}))$ is a direct summand of $U(M(B \otimes \mathcal{K}))$. Then $t \to \mathrm{ad}_{u_t} \circ \psi$ is a homotopy of ψ in $\mathrm{Map}(X, \mathrm{Aut}(A \otimes \mathcal{K}))$ to a mapping in the range of $\mathrm{Map}(X, \mathrm{Aut}(A)) \to \mathrm{Map}(X, \mathrm{Aut}(A \otimes \mathcal{K}))$.

We get the following corollary:

5.6. Corollary a) $G^{1'} = G^{0'} \cap \{ \xi \in \mathrm{Hom}(K_1(A), K_0(B)),$
$\alpha_1(\xi)([1]) = [1] \in K_0(B) \}$

b) The restriction of σ to $\ker\alpha_1$ maps $\ker\alpha_1$ onto $\ker\alpha_0$.

Proof. Use proposition 5.4. and lemma 5.5.

We are left with the determination of the kernel of α_1.

5.7. Lemma γ_1' is an isomorphism.

Proof. We first prove that γ_1' is injective.

Suppose that there exists $\tau \in \mathrm{Hom}_c(K_0(A), K_0(E_\varphi))$ such that $K_0(\rho) \circ \tau = \mathrm{id}_{K_0(A)}$ and $\tau([1]) = [1]$.

Let $A = \overline{\bigcup A_n}$ with A_n finite dimensional and let $i_{m,n}, i_n$ have the same meaning as in the discution preading lemma 4.6. It is an immediate consequence of proposition 4.5. that there exists a morphism $\eta_n : A_n \to E_\varphi$ such that $\rho \circ \eta_n = i_n$ and $K_0(\eta_n) = \tau \circ K_0(i_n)$, moreover any two such morphisms are unitary conjugated. Using induction on n one can define morphisms $\eta_n : A_n \to E_\varphi$ as above such that $\eta_m|_{A_n} = \eta_n$. This can be done as follows. Suppose that we have defined $\eta_1, \ldots, \eta_n$ as above. Choose $\eta'_{n+1} : A_{n+1} \to E_\varphi$ arbitrar ly such that $\rho \circ \eta'_{n+1} = i_{n+1}$ and $K_0(\eta'_{n+1}) = \tau \circ K_1(i_n)$. Then there exists $u \in U(A)$ such that $\eta'_{n+1}|_{A_n} = \mathrm{ad}_u \circ \eta_n$. Let $\eta_{n+1} = \mathrm{ad}_{u^*} \circ \eta'_{n+1}$.

η_n collect to define a lifting $\eta : A \to E_\varphi$ for ρ: $\rho \circ \eta = \mathrm{id}_A$

Let $\psi:[0,1]\times X\to \mathrm{End}^0(A)$ be defined by $\psi(t,x)$ = the composition of $\eta:A\to B$ and of the "evaluation at (t,x)": $B\ni b\to b(t,x)\in A$. ψ defines an arc connecting φ to the constant mapping in $\mathrm{Map}(X,\mathrm{End}^0(A))$. Using the isomorphism $[X,\mathrm{Aut}^0(A)]\to[X,\mathrm{End}^0(A)]$ ([20], lemma 1.4) we obtain that $\ker\gamma_1'\simeq\{0\}$.

To prove the surjectivity of γ_1' consider the diagram

$$\begin{tikzcd}
K_1(J) \arrow[r, "Ad"] \arrow[d, "\simeq"] & \ker\alpha_1 \arrow[r] \arrow[d, "\gamma_1'"] & \ker\alpha_0 \arrow[r] \arrow[d, "\gamma_0'"] & 0 & \\
\mathrm{Hom}(\mathbb{Z},K_1(J)) \arrow[r] & \mathrm{Ext}^u_c(K_0(A),K_1(J)) \arrow[r] & \mathrm{Ext}_c(K_0(A),K_1(J)) \arrow[r] & 0
\end{tikzcd}$$

Ad is defined as follows. Let $u\in U(J)$, u is represented by a function $u:X\to U(A)$ such that $u(x_0)=1$. Let $Ad([u])$ be the class of $x\to ad_{u(x)}\to \mathrm{Aut}(A)$. It follows that the diagram is commutative and hence γ_1' is *onto* (a simple diagram chase).

We put together the results of this section in the following theorem.

5.8. Theorem Let (X,x_0) be a pointed compact connected CW-complex and A an AF-algebra which satisfies 4.1.a) and b). Let $i = 0$ if $1 \notin A$, $i = 1$ if $1 \in A$.

Denote by Ω the lattice of ideals of A, $B = C(X,A)$, $J = C_0(X \setminus \{x_0\}, A)$; $K_j(B)$ and $K_j(J)$ are Ω-filtered $K^0(X)$-modules ($j \in \{0,1\}$).

The range of α_0 is $G^0 = \mathrm{id}_{K_0(A)} + \mathrm{Hom}_{K^0(X),c}(\overbrace{K_0(J)}^{K_0(B),})$ and the range of α_1 is $G^1 = \{\eta \in G, \eta([1]) = [1]\}$.
The product in G^i is the composition of morphisms.

$$\gamma_0 : \ker\alpha_0 \longrightarrow \mathrm{Ext}_{K^0(X),c}(K_0(B),K_1(B)) \text{ and}$$
$$\gamma_1 : \ker\alpha_1 \longrightarrow \mathrm{Ext}^u_{K^0(X),c}(K_0(B),K_1(B))$$

are isomorphisms. We obtain exact sequences

$$0 \to \mathrm{Ext}_{K^0(X),c}(K_0(B),K_1(B)) \to [X,\mathrm{Aut}(A)] \to G^0 \to 0.$$

if $1 \notin A$, and

$$0 \to \mathrm{Ext}^u_{K^0(X),c}(K_0(B),K_1(B)) \to [X,\mathrm{Aut}(A)] \to G^1 \to 0$$

if $1 \in A$. These exact sequences are natural in (X,x_0); their kernels are determined by lemma 5.3.b).

Here are some consequence of the naturality in X of the exact sequence.

5.9. <u>Corollary</u> Let (X,x_0), (Y,y_0) be pointed compact connected CW-complexes. Suppose $f:(X,x_0) \to (Y,y_0)$ induces an isomorphism of the K-groups then $f^* : [Y,\mathrm{Aut}(A)] \to [X,\mathrm{Aut}(A))]$ is an isomorphism. If $K^0(f)$ is an isomorphism and $K^1(Y) \simeq \{0\}$ then the exact sequences of the preceding theorem split .

Proof. Denote by $G^i(X)(G^i(Y))$ the range of α_i in order to put in evidence the natural dependence of these groups on $X(Y)$.

Then there exist by the naturality of the exact sequences commutative diagrams

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Ext}_c(K_0(A),K^1(Y)) & \to & [Y,\mathrm{Aut}(A)] & \to & G^0(Y) & \to 0 \\ & \downarrow K^1(f)_* & & \downarrow f^* & & \downarrow G^0(f) & \\ 0 \to & \mathrm{Ext}_c(K_1(A),K^1(X)) & \to & [X,\mathrm{Aut}(A)] & \to & G^0(X) & \to 0 \end{array}$$

if $1 \notin A$ and

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Ext}^u_c(K_0(A),K^1(Y)) & \to & [Y,\mathrm{Aut}(A)] & \to & G^1(Y) & \to 0 \\ & \downarrow K^1(f)_* & & \downarrow f & & \downarrow G^1(f) & \\ 0 \to & \mathrm{Ext}^u_c(K_0(A),K^1(X)) & \to & [X,\mathrm{Aut}(A)] & \to & G^1(X) & \to 0 \end{array}$$

if $1 \in A$.

$G^0(f)$ is obtained from the commutative diagram

$$\begin{array}{ccc} \iota_Y + \mathrm{Hom}_c(K_0(A),K_0(C_0(Y\setminus\{y_0\},A))) & \xrightarrow{\mu_Y} & G_0(Y) \\ \downarrow K_0(f^*)_* & & \downarrow G^0(f) \\ \iota_X + \mathrm{Hom}_c(K_0(A),K_0(C_0(X\setminus\{x_0\},A))) & \xrightarrow{\mu_X} & G_0(X) \end{array}$$

X', Y', μ_X, μ_Y have the same meaning as in 5.1

$f^*: C_0(Y\setminus\{y_0\},A) \to C_0(X\setminus\{x_0\},A)$

is given by $b \to b \circ f$.

The first part of the corollary is a consequence of the Five Lemma. The second part follows from the fact that $\alpha_i: [Y,\mathrm{Aut}(A)] \to G^i(Y)$ and $G^i(f): G^i(Y) \to G^i(X)$ are isomorphism and hence $f^* \circ \alpha_i^{-1} \circ G^i(f)^{-1}$ is well defined and is the descried spliting

6. The case A simple and X a H'-space

Let A be a simple AF-C-algebra not stably isomorphic to $\mathcal{K}$, (X,x_0) a pointed compact connected CW complex which is also a H' space.

Let B and J have the meaning of the preceding sections and denote by A' the mapping cone of the inclusion $\mathbb{C} \to A$ if $1 \in A$. We shall prove that $[X,\mathrm{Aut}(A)]$ is naturaly isomorphic to $KK(A,J)$ if $1 \notin A$ or to $KK(A',SJ)$ if $1 \in A$.

For the definition and the basic proprieties of the KK-bifuncter the reader is refered to the original papers of G.G.Kasparov [16],[17] or to the book of B.Blackadar [5]. Our approach uses Cuntz's "quasihomomorphism picture" of KK-groups (see [8]or [5]).

We shall define first natural transformations $c_0:[X,\mathrm{Aut}(A)] \to KK(A,J)$ if $1 \notin A$ and $c_1:[X,\mathrm{Aut}(A)] \to KK(A',SJ)$ if $1 \in A$.

Let $\varphi_0 \in \mathrm{Map}(X,\mathrm{Aut}(A))$ denote the constant function. For $\varphi \in \mathrm{Map}(X,\mathrm{Aut}(A))$ we shall denote by $\Phi(\varphi) \in \mathrm{End}(B)$ the morphism defined by φ, i.e. $\Phi(\varphi)(a)(x) = \varphi_x(a)$ for any $a \in A, x \in X$. It follows that $\Phi(\varphi)(a) - \Phi(\varphi)(a) \in J$ for any $a \in A$ and hence the pair $(\Phi(\varphi),\Phi(\varphi_0))$ is a quasihomomorphism from A to J. We shall denote by $c_0([\varphi])$ the corresponding element in $KK(A,J)$ (see [5],[8]). If A is unital denote by $\psi,\psi_0:A' \to C([0,1]\times X,A) \to M(SJ)$ the morphisms defined as follows. Recall first that $A' = \{f:[0,1] \to A, f(0) = 0, f(1) \in \mathbb{C}\}$. Then $\psi(f)(t,x) = \varphi_x(f(t))$, $\psi_0(f)(t,x) = f(t)$ for any $f \in A$. It follows that $\psi(f) - \psi_0(f) \in SJ$ for any $f \in A$. We shall define $c_1([\psi])$ to be the class of the quasihomomorphism (ψ,ψ_0) in $KK(A,SJ)$ (see[5],[8])

6.1. Lemma c_i is a morphism $(i \in \{0,1\})$.

<u>Proof</u>. We shall prove the lemma for $i = 0$, for $i = 1$ the

proof is similar.

Denote by $\theta : X \to X \vee X$ the comultiplication of X and by $q_1, q_2 : X \vee X \to X$ the projection on the first or the second coordinate i.e. $q_1 = id_X \vee ct$, $q_2 = ct \vee id_X$ (here $ct : X \to X$ is the constant function $x \to x_0$). Then $q_1 \circ \theta$, $q_2 \circ \theta$ and id_X are homotopic via base point preserving homotopies (X is a H'space).

It is a well known fact that the multiplication in $[X, Aut(A)]$ may be defined also by $[\varphi] + [\psi] = [(\varphi \vee \psi) \circ \theta]$

There exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & J \oplus J \arrow[r] \arrow[d, "\nu"] & C(X \vee X, A) \arrow[r] \arrow[d, "\theta^*"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & J \arrow[r] & C(X,A) \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

The quotient maps are obtained from evaluation at the base point.

It follows from the assumptions on θ, q_1 and q_2 that ν is a homotopy equivalence on each factor. This shows that if $(\xi, \zeta) \ni KK(A,J) \oplus KK(A,J) \simeq KK(A, J \oplus J)$ then $\nu_*(\xi, \zeta) = \xi + \zeta$.
It follows from the definitions that

$\nu_*(c_0([\varphi]), c_0([\psi])) = c_0([(\varphi \vee \psi) \circ \theta])$ and hence
$c_0([\varphi][\psi]) = c_0([\varphi]) + c_0([\psi])$.

6.2. Theorem $c_0 : [X, Aut(A)] \to KK(A,J)$ if $1 \notin A$ and $c_1 : [X, Aut(A)] \to KK^1(A,J)$ if $1 \in A$ are isomorphisms.

<u>Proof</u>. Let ι be the composition $K_0(A) \ni \xi \to [1] \otimes \xi \in K^0(X) \otimes K_0(A) \simeq K_0(B)$.

There exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Ext_c(K_0(A), K_1(J)) \arrow[r] \arrow[d] & {[X, Aut(A)]} \arrow[r] \arrow[d, "c_0"] & Hom_c(K_0(A), K_0(J)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Ext(K_0(A), K_1(J)) \arrow[r] & KK(A,J) \arrow[r] & Hom(K_0(A), K_0(J)) \arrow[r] & 0
\end{tikzcd}$$

in which the top line is exact by proposition 5.4. and the bottom line is exact by the Universal Coeficient Theorem ([24]). The first vertical arrow is a morphism since it is the restriction of c_0. The third vertical arrow is also a morphism. This can be viewed as follows. Let $F_1, F_2 \in \mathrm{Hom}(K_0(A), K_0(J))$, $\mu_X : \mathrm{Hom}(K_0(A), K_1(B)) \to \mathrm{Hom}_{K^0(X)}(K_0(B), K_0(B))$ be as in 5.1., then

$$\mu_X(\iota + F_j)(\iota(a) + b_0) = \iota(a) + F_j(a) + b_0$$

for any $a \in K_0(A)$, $b_0 \in K_0(J)$ since $K^0(X)^2 = 0$. This shows that $\mu_X(\iota + F_1)\mu_X(\iota + F_2) = \mu_X(\iota + F_1 + F_2)$.

The filtration are trivial if A is simple and hence $\mathrm{Ext}_c(K_0(A), K_1(J)) \to \mathrm{Ext}(K_0(A), K_1(J))$ and $\mathrm{Hom}_c(K_0(A), K_1(J)) \to \mathrm{Hom}(K_1(A), K_0(J))$ are isomorphisms. This shows that c_0 is an isomorphism.

Let us prove now that c_1 is an isomorphism. Since $K_1(A') \simeq K_0(A)/\mathbb{Z}[1]$, $K_0(A') \simeq \{0\}$ we obtain using corollary 5.6.a) and lemma5.7. and the Universal Coeficient Theorem ([24]) that there exists a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^u_c(K_0(A), K_1(J)) \arrow[r] \arrow[d, "h"] & {[X, \mathrm{Aut}(A)]} \arrow[r] \arrow[d, "c_1"] & G_1 \arrow[r] \arrow[d, bend left] & 0 \\
0 \arrow[r] & \mathrm{Ext}(K_0(A)/\mathbb{Z}[1], K_1(J)) \arrow[r] & KK(A, SJ) \arrow[r] & \mathrm{Hom}(K_0(A)/\mathbb{Z}[1], K_0(J)) \arrow[r] & 0
\end{tikzcd}$$

Let us determine the morphism h.

Suppose that $\varphi \in \mathrm{Map}(X, \mathrm{Aut}^0(A))$, then $[\varphi] \in \ker \alpha_1$, if and only if $K_0(\Phi(\varphi)) = K_0(\Phi(\varphi_0)) = \iota$.

Then there exists a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & S^2J & \to & SE_\varphi & \to & SA & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & S^2J & \to & E'_\varphi & \to & A' & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathbb{C} & & \mathbb{C} & &
\end{array}$$

(We have denoted by E'_φ the mapping cone of the inclusion $\mathbb{C} \to E_\varphi$).
The corresponding diagram of K_1-groups shows that h associates to the class of the compatible extension with order unit

$$0 \to K_1(J) \to (K_0(E_\varphi), [1]) \to (K_0(A), [1]) \to 0$$

the class of

$$0 \to K_1(J) \to K_0(E_\varphi)/\mathbb{Z}[1] \to K_0(A)/\mathbb{Z}[1] \to 0$$

in $\operatorname{Ext}(K_0(A)/\mathbb{Z}[1], K_1(J))$. (see [5]) h is obviously an isomorphism if A is simple. The Five Lemma shows that c_1 is also an isomorphism.

7. The Samelson product

In this section we briefly study the effect of the Samelson product. It turns out that it does not vanish in general and hence the classifying space of $Aut^0(A)$ is not a H-space([27]). This shows that the set of isomorphism classes of locally trivial fields of C^*-algebras on X with fiber A cannot be given a natural group structure for any compact space X (see [27], pag.475(7.8)).

Let us recall the definition of the Samelson product ([27], pag.467) it is pairing

$$\langle \cdot , \cdot \rangle : [X, Aut(A)] \times [Y, Aut(A)] \to [X \wedge Y, Aut(A)]$$ defined by

$$\langle [\varphi], [\psi] \rangle = [\eta], \eta(x \wedge y) = \varphi(x)\psi(y)\varphi(x)^{-1}\psi(y)^{-1}$$

If $X = S^n, Y = S^m$ this gives a pairing

$$\pi_n(Aut(A)) \times \pi_m(Aut(A)) \to \pi_{n+m}(Aut(A))$$

Let us observe that $\alpha(\langle a,b \rangle)$ depends only on $\alpha(a)$ and $\alpha(b)$ (we omitt various subscripts of α) and it is defined by

$$(7.1) \quad j \circ \mu_{X \wedge Y}(\alpha(\langle a,b \rangle)) = (\mu'_X(a)\mu'_Y(b)\mu'_X(a)^{-1}\mu'_Y(b)^{-1}) \circ j$$

where $j : K_0(C(X\ Y,A)) \to K_0(C(X\ Y,A))$ is the obvious inclusion and $\mu'_X(a)$ is the extension of $\mu_X(\alpha(a)) : K_0(C(X,A)) \to K_0(C(X,A))$ to a $K^0(X \times Y)$-linear morphism

$\mu'_X(a) : K_0(C(X \times Y, A)) \to K_0(C(X \times Y, A))$. μ'_Y is defined similarily.

Moreover since $\ker\alpha$ is represented by approximatively inner loops we obtain the following result:

7.2. Proposition

a) $\alpha(\langle a,b \rangle)$ depends only on $\alpha(a)$ and $\alpha(b)$ and its formula is given by (7.1).

b) $\langle \ker\alpha, [Y, Aut(A)] \rangle$ and $\langle [X, Aut(A)], \ker\alpha \rangle$ are contained in $\ker\alpha$.

c) $\langle \ker\alpha, \ker\alpha \rangle = \{0\}$.

d) $\bigoplus_{n\geq 0}\pi_n(\mathrm{Aut}(A))$ with the Samelson product is gradedly isomorphic to $\mathrm{Aut}(K_0(A),\Sigma(A)) \oplus (\bigoplus_{k\geq 1}\mathrm{Ext}^{p(k)}(K_0(A),K_0(A)))$ with the product $\langle a,b\rangle' = aba^{-1}b^{-1}$ if a and b are of degree 0, $\langle a,b\rangle' = aba^{-1}-b$ if a is of degree 0 and b of degree ≥ 1, and $\langle a,b\rangle' = ab - ba$ if a,b are both of degree ≥ 1.

($\mathrm{Ext}^{p(k)}(K_0(A),K_0(A))$ denotes $\mathrm{Hom}_c(K_0(A),K_0(A))$ if k is even and $1 \notin A$, $\mathrm{Hom}_c(K_0(A),K_0(A)) \cap \mathrm{Hom}(K_0(A)/\mathbb{Z}[1],K_0(A))$ if k is even and $1 \in A$, $\mathrm{Ext}_c(K_0(A),K_0(A))$ if k is odd, $1 \notin A$, $\mathrm{Ext}_c^u(K_0(A),K_0(A))$ if k is odd and $1 \in A$, $\Sigma(A)$ is the scale of the ordered group $K_0(A)$).

Proof.

a),b) and c) are obvious and d) follows from theorem(5.8) using (7.1)(see also [20]).

5.8. d) gives a necessary condition on A in order to exist a natural group structure on the set of isomorphism classes of locally trivial fields of AF-algebras with fiber A. Indeed, if such a natural group structure would exist then every field on $S^n \vee S^m$ would have an extension on $S^n \times S^m$ thus forcing the vanishing of the Samelson product on $\pi_{n-1}(\mathrm{Aut}(A)) \times \pi_{m-1}(\mathrm{Aut}(A))$. (See [27] pag. 476 (7,10)). This cannot happen if A is simple and $\mathrm{Hom}(K_0(A),K_0(A))$ is not commutative.

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