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Jean-Enrique Martinez-Legaz*) and Ivan Singer**)

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*) *Department of Applied Mathematics and Anakysis University of Barcelona, Gran Via, 585, Barcelona 08071,*

**) *Department of Mathematics, The National Institute for Scientific and Technical Creation, Bd. Pacii 220, 79622 Bucharest, Romania.*

The structure of hemi-spaces in R^n

Jean-Enrique Martínez-Legaz

Department of Applied Mathematics and Analysis

University of Barcelona

Gran Via, 585

Barcelona 08071, Spain

and

Ivan Singer

Department of Mathematics

INCREST

Bd.Păcii 220

Bucharest 79622, Romania

O = script

Abstract

We study hemi-spaces (i.e., convex sets with convex complements) in R^n. We give several geometric characterizations of hemi-spaces and several ways of representing them with the aid of linear operators and lexicographical order. We obtain a metric-affine classification of hemi-spaces, in terms of their "rank" and "type", and a "decomposition theorem". We also give some characterizations of affine transformations which preserve a hemi-space.

§0. Introduction

A set H in R^n is called (see e.g.[5]) a hemi-space, if both H and its complement $R^n \setminus H$ are convex. Actually, this definition makes sense in other structures, too, in which the notion of "convex set" is defined in some way (e.g., in arbitrary linear spaces, partially ordered sets, etc., or, more generally, in the case of axiomatically defined "convex sets" [5],[16],[1]), but here we shall consider only hemi-spaces in R^n; we shall study hemi-spaces in some other structures elsewhere (in preparation).

The main use of hemi-spaces in linear spaces has been the application of various particular classes of hemi-spaces, such as closed half-spaces in locally convex spaces, "semi-spaces" in linear spaces ([2],[3]) and "generalized half-spaces" in R^n ([11],[12]) and of general hemi-spaces in linear spaces (e.g. [2],[3],[8]) to the separation of convex sets. Furthermore, hemi-spaces can be also applied to the study of quasi-affine (i.e., simultaneously quasi-convex and quasi-concave) functions, or, in other words, functions $h:R^n \to \bar{R}=[-\infty,+\infty]$, whose level sets $S_c(h)=\{y\in R^n \mid h(y)\leq c\}$ are hemi-spaces, for all $c\in R$). For example, let us observe that for a set $H\subseteq R^n$, denoting by χ_H the "indicator function" of H (defined by $\chi_H(y)=0$ for $y\in H$ and $\chi_H(y)=+\infty$ for $y\in R^n\setminus H$), the following statements are equivalent:

1°. H is a hemi-space.

2°. There exists $d\in R$ such that χ_H+d is quasi-affine.

3°. For all $d\in R$, χ_H+d is quasi-affine.

Indeed, this follows from the above definitions, and from the fact that for any $c,d\in R$ we have $S_c(\chi_H+d)=\emptyset$ (the empty set) if $-\infty<c<d$, and $S_c(\chi_H+d)=H$ if $d\leq c<+\infty$.

Although various particular classes of hemi-spaces in R^n (such as those mentioned above) have been studied in the literature (see e.g.[2],[3],[6],[8],[11],[12]), for general hemi-spaces such a study has begun only recently (see [9] and [10], where the term "convex half-space" is used, instead of "hemi-space"). In the present paper, as announced in [12], §0, one of our main tools for the study of hemi-spaces in R^n, will be the lexicographical separation of two convex subsets of R^n by linear operators (see theorem 0.1 below). Therefore, our methods and results are different from those of the papers [9] and [10], with the exception of the equivalence $1^0 \Leftrightarrow 2^0$ of theorem 2.1, as mentioned in remark 2.2 a) below (in fact, we have learned of [9] and [10] only after the present paper had been completed).

In §§1-3 we shall give several geometric characterizations of hemi-spaces and several ways of representing hemi-spaces with the aid of linear operators and lexicographical order. Furthermore, in §2 we shall obtain a metric-affine classification of hemi-spaces in R^n, and in §3 we shall introduce and study the "linear manifold associated to a hemi-space", which will lead us, in particular, to a "decomposition theorem" for hemi-spaces. In §4 we shall give some characterizations of affine transformations preserving a hemi-space. Finally, in §5 (Appendix) we shall give a theorem on separation of p subsets of R^n by hemi-spaces, and some results which, in R^n, extend a theorem given by V. Klee for semi-spaces of linear spaces ([6], theorem 2.2).

We conclude this Introduction by recalling some notions, notations and results, which will be used in the sequel.

Unless otherwise stated, whenever we shall write R^n (where $R=(-\infty,+\infty)$) or $\bar{R}^n$ (where $\bar{R}=[-\infty,+\infty]$), we shall understand that $n\geq 1$; however, we shall also work with $R^0=\{0\}$ and $\bar{R}^0=\{0\}$ (e.g., in R^r of theorem 2.1 and $\bar{R}^r$ of proposition 2.5, r may also be 0). The elements of R^n will be considered column vectors, and the superscript T will mean transpose. We recall that $x=(\xi_1,\dots,\xi_n)^T\in\bar{R}^n$ is said to be "lexicographically less than" $y=(\eta_1,\dots,\eta_n)^T\in\bar{R}^n$ (in symbols, $x <_L y$) if $x\neq y$ and if for $k=\min\ \{i\in\{1,\dots,n\}\mid \xi_i\neq\eta_i\}$ we have $\xi_k<\eta_k$. We write $x \leq_L y$ if $x <_L y$ or $x=y$. The notations $y >_L x$ and $y \geq_L x$, respectively, will be also used. We shall denote by $\{e_j\}_1^n$ the unit vector basis of R^n. When $\{e'_j\}_1^n$ is an arbitrary basis of R^n, we shall also consider (only in theorem 4.4 below) the lexicographical order on R^n "in the basis" $\{e'_j\}_1^n$ ", defined similarly to the above, with $x=(\xi_1,\dots,\xi_n)^T=\sum_{j=1}^n \xi_j e_j$ and $y=(\eta_1,\dots,\eta_n)^T=\sum_{j=1}^n \eta_j e_j$ replaced by $x=\sum_{j=1}^n \xi_j e'_j$ and $y=\sum_{j=1}^n \eta_j e'_j$ respectively.

We shall denote by $\mathcal{L}(R^n)$, $\mathcal{U}(R^n)$ and $\mathcal{O}(R^n)$, the families of all linear operators, all isomorphisms, and all linear isometries $v:R^n\to R^n$ respectively, and by $\mathcal{L}(R^n, R^r)$ the family of all linear operators $u:R^n \to R^r$. We shall identify each $u\in\mathcal{L}(R^n, R^r)$ with its $r\times n$ matrix with respect to the unit vector bases of R^n and R^r, that is, we shall write

$$u = (m_{ij}) = (m_1,\dots,m_r)^T, \tag{0.1}$$

where $m_i^T = (m_{i1},\dots,m_{in})$ $(i=1,\dots,r)$ are the rows of (m_{ij}); we shall also use this identification, in particular, when $r=n$, i.e., for each $v\in\mathcal{L}(R^n)$.

We recall that a set $H\subset R^n$ is called [2] a <u>semi-space at x_o</u>, if H is a maximal convex set (with respect to inclusion) such that $x_o\notin S$. By [15], lemma 1.1, a set $H\subset R^n$ is a semi-space (at some x_o) if and only if there exist $v\in\mathcal{O}(R^n)$ and $z\in R^n$ (namely, $z=v(x_o)$) such that

$$H = \{y\in R^n \mid v(y) <_L z\}. \tag{0.2}$$

More generally, any set of the form (0.2), with $v\in\mathcal{L}(R^n)$ and $z\in\overline{R}^n$ (instead of $v\in\mathcal{O}(R^n)$ and $z\in R^n$) is called ([11], [12]) a <u>generalized half-space.</u> As has been observed in [11], the empty set $\emptyset$ and the whole space R^n are generalized half-spaces, and, if $n\geq 2$, then all closed half-spaces and all open half-spaces are generalized half-spaces (for suitable v and z in (0.2)). Clearly, all generalized half-spaces and all complements of generalized half-spaces are hemi-spaces; in §1 we shall show that these are the only hemi-spaces, and we shall obtain further relations between these concepts.

Let us recall the following "lexicographical separation theorem" (where co G_i denotes the convex hull of G_i), which we shall use in the sequel:

<u>Theorem 0.1</u> ([12], theorem 2.1). <u>For any sets $G_1, G_2 \subset R^n$, the following statements are equivalent:</u>

1°. $\mathrm{co}\, G_1 \cap \mathrm{co}\, G_2 = \emptyset$.

2°-4°. <u>There exists</u> $v\in\mathcal{L}(R^n)$, $v\in\mathcal{U}(R^n)$, $v\in\mathcal{O}(R^n)$, <u>respectively, such that</u>

$$v(y_1) <_L v(y_2) \qquad (y_1\in G_1,\ y_2\in G_2). \tag{0.3}$$

5°. <u>There exists either a generalized half-space $H'\subset R^n$ such that</u>

$$G_1 \subseteq H', \qquad G_2 \subseteq R^n\setminus H', \tag{0.4}$$

<u>or a generalized half-space $H''\subset R^n$ such that</u>

$$G_1 \subseteq R^n\setminus H'', \quad G_2 \subseteq H''. \tag{0.5}$$

<u>Remark 0.1.</u> After the appearance of [12], we have learned that the term "lexicographic separation" and a theorem on lexicographical separation of two disjoint convex sets in an arbitrary linear space (different from theorem 0.1 above, but related to it, in the particular case of R^n), are due to V. Klee ([7], Section 2.4).

For two subsets A and B of R^n, we shall use, following Pontryagin [14], the notation

$$A \stackrel{*}{-} B = \{y \in R^n | y+B \subseteq A\}. \tag{0.6}$$

For a linear subspace S of R^n, we shall denote by $S^{\perp}$ its orthogonal complement in R^n. By a linear manifold M we shall mean a translate of a linear subspace S, or, equivalently, a set M such that the relations $y_1, y_2 \in M$, $\lambda \in R$ imply $\lambda y_1 + (1-\lambda) y_2 \in M$. We shall identify the conjugate space $(R^n)^*$ of R^n with R^n in the usual way (with the aid of the scalar product), and thus the adjoint u^* of a linear operator $u \in \mathcal{L}(R^n, R^r)$ will belong to $\mathcal{L}(R^r, R^n)$; we shall use the well-known fact that $u^*(R^r) = (\mathrm{Ker}\ u)^{\perp}$ (where $\mathrm{Ker}\ u = \{y \in R^n | u(y) = 0\}$). Finally, we shall denote by I the identity operator on R^n.

§1. Some representations and characterizations of hemi-spaces

In the sequel, we shall need the following lemma (the existence part of it was also assumed, implicitly, in [12]):

Lemma 1.1. For any set $G \subseteq \bar{R}^n$, the lexicographical infimum

$$\inf_L G = z = (\xi_j)_1^n \in \bar{R}^n \tag{1.1}$$

exists; namely, it is given by

$$\xi_k = \inf\{\gamma_k | g = (\gamma_j)_1^n \in G,\ \gamma_j = \xi_j\ (j=1,\dots,k-1)\} \quad (k=1,\dots,n). \tag{1.2}$$

Proof. For $z = (\xi_j) \in \bar{R}^n$ defined by (1.2) we have, clearly,

$$z \leq_L g \qquad (g \in G). \tag{1.3}$$

Assume now that $z' = (\xi_j')_1^n \in \bar{R}^n \setminus \{z\}$ satisfies

$$z' \leq_L g \qquad (g \in G); \tag{1.4}$$

we shall show that, in this case, $z' \leq_L z$, which will prove (1.1). Let

$$k = \min\{\ell \leq n | \xi_\ell' \neq \xi_\ell\}, \tag{1.5}$$

and take any $g = (\gamma_j)_1^n \in G$ such that $\gamma_j = \xi_j$ $(j=1,\dots,k-1)$. Then, by (1.5), we have $\xi_j' = \xi_j = \gamma_j$ $(j=1,\dots,k-1)$, whence, by (1.4), $\xi_k' < \gamma_k$. Therefore, by (1.2), we obtain

$$\xi_k' \leq \inf\{\gamma_k | g = (\gamma_j)_1^n \in G,\ \gamma_j = \xi_j\ (j=1,\dots,k-1)\} = \xi_k, \tag{1.6}$$

and hence, by (1.5), it follows that $z' <_L z$. On the other hand, if there is no $g = (\gamma_j)_1^n \in G$ such that $\gamma_j = \xi_j$ $(j=1,\dots,k-1)$, then, by (1.2) we have $\xi_k = \inf \emptyset = +\infty > \xi_k'$, whence, by (1.5), we obtain again that $z' <_L z$. ∎

Theorem 1.1. For a set $H \subseteq R^n$, the following statements are equivalent:

1°. H is a hemi-space.

2^o-4^o. There exist $y\in\mathcal{L}(R^n)$, $v\in\mathcal{U}(R^n)$, $v\in\mathcal{O}(R^n)$, respectively and $z\in\bar{R}^n$, such that either

$$H=\{y\in R^n \mid v(y) <_L z\}, \tag{1.7}$$

or

$$H=\{y\in R^n \mid v(y) \leqslant_L z\}. \tag{1.8}$$

5^o-7^o. There exists $v\in\mathcal{L}(R^n)$, $v\in\mathcal{U}(R^n)$, $v\in\mathcal{O}(R^n)$ respectively, such that either

$$H=\{y\in R^n \mid v(y) <_L \inf_L v(R^n\setminus H)\}, \tag{1.9}$$

or

$$H=\{y\in R^n \mid v(y) \leqslant_L \inf_L v(R^n\setminus H)\}. \tag{1.10}$$

8^o. H is either a generalized half-space, or the complement of a semi-space.

9^o. H is either a semi-space, or the complement of a generalized half-space.

10^o. H is either a generalized half-space, or the complement of a generalized half-space.

Proof. $1^o\Rightarrow 7^o$. If 1^o holds, then, by theorem 0.1, applied to $G_1=H$, $G_2=R^n\setminus H$, there exists $v\in\mathcal{O}(R^n)$ such that

$$v(y_1) <_L v(y_2) \qquad (y_1\in H,\ y_2\in R^n\setminus H). \tag{1.11}$$

Hence,

$$H\subseteq\{y\in R^n \mid v(y) \leqslant_L \inf_L v(R^n\setminus H)\}, \tag{1.12}$$

and, on the other hand, we have, clearly,

$$R^n\setminus H\subseteq\{y\in R^n \mid v(y) \geqslant_L \inf_L v(R^n\setminus H)\}. \tag{1.13}$$

Then, at least one of the lexicographical inequalities in (1.12), (1.13), must be strict; indeed, if not, then there exist $y_1\in H$ and $y_2\in R^n\setminus H$ such that $v(y_1)=$ $=\inf_L v(R^n\setminus H)=v(y_2)$, which contradicts (1.11). Consequently, both inclusions in (1.12) and (1.13) must be equalities.

The implications $7^o\Rightarrow 4^o$, $6^o\Rightarrow 3^o$ and $5^o\Rightarrow 2^o$ are obvious, with

$$z=\inf_L v(R^n\setminus H). \tag{1.14}$$

$4^o\Rightarrow 8^o$. If (1.7) holds, then H is a generalized half-space. If (1.8) holds with $z\in\bar{R}^n\setminus R^n$, then, since

$$\{y\in R^n \mid v(y)=z\}=\emptyset \qquad (z\in\bar{R}^n\setminus R^n), \tag{1.15}$$

we have again (1.7). Finally, if (1.8) holds and $z\in R^n$, then, by $v\in\mathcal{O}(R^n)$, H is the complement of a semi-space.

$4^o\Rightarrow 9^o$. If (1.8) holds, then H is the complement of the generalized half-space

$\{y\in R^n | v(y) >_L z\}$. If (1.7) holds and $z\in\bar{R}^n\setminus R^n$, then, by (1.15), we have again (1.8). Finally, if (1.7) holds and $z\in R^n$, then, by $v\in\mathcal{O}(R^n)$, H is a semi-space.

$10^\circ\Rightarrow 2^\circ$. If H is a generalized half-space, we have (1.7), while if $R^n\setminus H$ is a generalized half-space, we have $R^n\setminus H=\{y\in R^n | v(y) <_L z\}$, with suitable $v\in\mathcal{L}(R^n)$ and $z\in\bar{R}^n$, whence $H=\{y\in R^n | (-v)(y) \leq_L -z\}$.

The implications $8^\circ\Rightarrow 10^\circ$, $9^\circ\Rightarrow 10^\circ$, $7^\circ\Rightarrow 6^\circ\Rightarrow 5^\circ$ and $4^\circ\Rightarrow 3^\circ\Rightarrow 2^\circ$ are obvious. Finally, the implication $2^\circ\Rightarrow 1^\circ$ is immediate, using that, since $\leq_L$ is a total order on R^n, the pairs

$$\{y\in R^n | v(y) <_L z\},\ \{y\in R^n | v(y) \geq_L z\} \qquad (1.16)$$

and

$$\{y\in R^n | v(y) \leq_L z\},\ \{y\in R^n | v(y) >_L z\} \qquad (1.17)$$

are pairs of complementary sets. Thus, $1^\circ\Leftrightarrow\ldots\Leftrightarrow 10^\circ$. ■

Remark 1.1. a) As shown by the above proof of the implication $4^\circ\Rightarrow 8^\circ$, the representations (1.7) and (1.8) of H, even with the same $v\in\mathcal{L}(R^n)$ and $z\in\bar{R}^n$, are not mutually exclusive.

b) From theorem 1.1 there follows again [12], remark 2.1 f), according to which lexicographical separation is equivalent to the known (see e.g. [8], §17) separation by hemi-spaces. Indeed, since every generalized half-space and every complement of a generalized half-space is a hemi-space, from theorem 0.1, implication $1^\circ\Rightarrow 5^\circ$, it follows that for any sets $G_1, G_2\subset R^n$ with co $G_1\cap$co $G_2=\emptyset$ there exists a hemi-space $H\subset R^n$ such that

$$G_1\subseteq H, \quad G_2\subseteq R^n\setminus H \qquad (1.18)$$

(and, conversely, if there exists a hemi-space H satisfying (1.18), then co $G_1\subseteq H$, co $G_2\subseteq R^n\setminus H$, whence co $G_1\cap$co $G_2=\emptyset$). On the other hand, assume that for any sets G_1, G_2 with co $G_1\cap$co $G_2=\emptyset$ there exists a hemi-space $H\subset R^n$ satisfying (1.18). Then, by theorem 1.1, H is either a generalized half-space or the complement of a generalized half-space, whence we obtain again theorem 0.1, implication $1^\circ\Rightarrow 5^\circ$.

c) Separation by hemi-spaces can be extended to p subsets of R^n (see §5).

Corollary 1.1. a) Every generalized half-space is either a semi-space, or the complement of a generalized half-space.

b) Every complement of a generalized half-space is either a generalized half-space, or the complement of a semi-space. ■

Theorem 1.2. For a set $H \subseteq R^n$, and for any $k \geq n+1$, the statements of theorem 1.1 are equivalent to

11°. There exist $u \in \mathcal{L}(R^n, R^k)$ and $x \in \bar{R}^k$, such that

$$H = \{y \in R^n \mid u(y) <_L x\}. \tag{1.19}$$

Proof. The implication $11^\circ \Rightarrow 1^\circ$ is obvious, by the definition of hemi-spaces.

$2^\circ \Rightarrow 11^\circ$. Assume first (1.7) and let (where $-\infty$ denotes the element $(-\infty,\dots,-\infty)^T \in \bar{R}^{k-n}$)

$$u = (v^T, 0)^T \in \mathcal{L}(R^n, R^k), \quad x = (z^T, -\infty^T)^T \in \bar{R}^k. \tag{1.20}$$

Then, by (1.7), we have

$$u(y) = (v(y)^T, 0)^T <_L (z^T, -\infty^T)^T = x \qquad (y \in H),$$

whence the inclusion $\subseteq$ in (1.19). Conversely, if $y \in R^n$, $u(y) <_L x$, then either $v(y) <_L z$ or $v(y) = z$ and $0 <_L -\infty$; but, the second case is impossible, whence, by (1.7), we obtain $y \in H$.

Assume now (1.8) and let (where $+\infty$ denotes the element $(+\infty,\dots,+\infty)^T \in \bar{R}^{k-n}$)

$$u = (v^T, 0)^T \in \mathcal{L}(R^n, R^k), \quad x = (z^T, +\infty^T)^T \in \bar{R}^k. \tag{1.21}$$

Then, by (1.8), we have

$$u(y) = (v(y)^T, 0)^T <_L (z^T, +\infty^T)^T = x \qquad (y \in H),$$

whence the inclusion $\subseteq$ in (1.19). Conversely, if $y \in R^n$, $u(y) <_L x$, then $v(y) \leq_L z$, whence, by (1.8), we obtain $y \in H$. ∎

Remark 1.2. a) In other words, condition 11° can be expressed as follows: H is the solution set of a linear lexicographical strict inequality $u(y) <_L x$, where $u \in \mathcal{L}(R^n, R^k)$, $x \in \bar{R}^k$. Similar remarks can be also made for other results of this paper.

b) Theorem 1.2 remains also valid for $<_L$ replaced by $\leq_L$ in (1.19). Indeed, one has only to add, in the $\subseteq$ parts of the proof, that $\{y \in R^n \mid u(y) <_L x\} \subseteq \{y \in R^n \mid u(y) \leq_L x\}$ for all $x \in \bar{R}^k$, and to replace, in the $\supseteq$ parts, $u(y) <_L x$ by $u(y) \leq_L x$ (and $0 <_L -\infty$ by $0 \leq_L -\infty$). See also remark 2.2 c) below.

c) One cannot replace $k \geq n+1$ by $k \geq n$, in theorem 1.2, implication $2^\circ \Rightarrow 11^\circ$. Indeed, if we have (1.8) for some $v \in \mathcal{O}(R^n)$ and $z \in R^n$ (i.e., if H is the complement of a semi-space), then, by theorem 1.3 below, we cannot have (1.19) with $u \in \mathcal{L}(R^n, R^n)$ and $x \in \bar{R}^n$.

Lemma 1.2. Let H be a generalized half-space. Then, for every $y \in H$ there exist $d = d(y) \in R^n \setminus \{0\}$ and $\varepsilon = \varepsilon(y, d) > 0$, such that

$$y + \lambda d \in H \qquad (\lambda \in R, |\lambda| < \varepsilon). \tag{1.22}$$

Proof. Let

$$H = \{y \in R^n \mid v(y) <_L z\}, \tag{1.23}$$

with $v=(m_1,\dots,m_n)^T\in\mathcal{L}(R^n)$ and $z=(\xi_1,\dots,\xi_n)^T\in\bar{R}^n$, and let $y\in H$. Then, by (1.23), there exists $i\in\{1,\dots,n\}$ such that

$$m_j^T y=\xi_j \quad (j=1,\dots,i-1),\quad m_i^T y<\xi_i \,. \tag{1.24}$$

If $i=1$, let $d\in R^n\setminus\{0\}$ be arbitrary. If $i>1$, then, since $i-1\leq n-1$, we can choose $d\in R^n\setminus\{0\}$ such that

$$m_j^T d=0 \qquad (j=1,\dots,i-1). \tag{1.25}$$

Finally, let

$$\varepsilon=\begin{cases} +\infty & \text{if } m_i^T d=0 \text{ or } \xi_i=+\infty \text{ (or both)},\\ \dfrac{\xi_i-m_i^T y}{|m_i^T d|} & \text{otherwise.}\end{cases} \tag{1.26}$$

Then, by (1.24), $\varepsilon>0$. Furthermore, by (1.24) and (1.25), for each $\lambda\in R$ with $|\lambda|<\varepsilon$, we have

$$m_j^T(y+\lambda d)=m_j^T y+\lambda m_j^T d=\xi_j+\lambda.0=\xi_j \qquad (j=1,\dots,i-1),$$

$$m_i^T(y+\lambda d)=m_i^T y+\lambda m_i^T d\begin{cases}<m_i^T y+\varepsilon|m_i^T d|=\xi_i & \text{if } m_i^T d\neq 0,\\ =m_i^T y<\xi_i & \text{if } m_i^T d=0,\end{cases}$$

whence $v(y+\lambda d)<_L z$. Thus, by (1.23), we obtain (1.22). ∎

Now we can prove

Theorem 1.3. The complement of a semi-space is not a generalized half-space (i.e., the two cases in 8^0 of theorem 1.1 are mutually exclusive).

Proof. Let H be a semi-space (1.23), with $v\in\mathcal{U}(R^n)$ and $z\in R^n$, so

$$R^n\setminus H=\{y\in R^n \mid v(y)\geq_L z\}. \tag{1.27}$$

Then, since $v\in\mathcal{U}(R^n)$, we have $v(v^{-1}(z))=z$, whence $v^{-1}(z)\in R^n\setminus H$. Furthermore, if $d\in R^n\setminus\{0\}$, then, since $v\in\mathcal{U}(R^n)$, we have $v(d)\neq 0$, so either $v(d)<_L 0$, or $v(d)>_L 0$. Hence, by (1.23), in the first case we have $v(v^{-1}(z)+\lambda d)=z+\lambda v(d)<_L z$ $(\lambda>0)$, so $v^{-1}(z)+\lambda d\notin R^n\setminus H$ $(\lambda>0)$, while in the second case we have $v(v^{-1}(z)+\lambda d)=z+\lambda v(d)<_L z$ $(\lambda<0)$, so $v^{-1}(z)+\lambda d\notin R^n\setminus H$ $(\lambda<0)$. Hence, by lemma 1.2 (with $y=v^{-1}(z)$), $R^n\setminus H$ is not a generalized half-space. ∎

Remark 1.3. For another proof of theorem 1.3, see remark 2.4.

Corollary 1.2. A semi-space is not the complement of a generalized half-space (i.e., the two cases in 9^0 of theorem 1.1 are mutually exclusive).

Proof. This follows from theorem 1.3, by passing to complements. ∎

Corollary 1.3. Every generalized half-space H can be written in the form (1.7), with $v\in\mathcal{U}(R^n)$ (or, even $v\in\mathcal{O}(R^n)$) and $z\in\bar{R}^n$.

Proof. Since H is a hemi-space, by theorem 1.1, implication $1^\circ\Rightarrow 3^\circ$ (or $1^\circ\Rightarrow 4^\circ$), there exist $v\in\mathcal{U}(R^n)$ (respectively, $v\in\mathcal{O}(R^n)$) and $z\in\bar{R}^n$, such that we have either (1.7) or (1.8). Now, if (1.7) holds, then we are done. On the other hand, if (1.8) holds, then $z\in\bar{R}^n\setminus R^n$ (since otherwise we have (1.8) with $z\in R^n$, so H is the complement of a semi-space, in contradiction with theorem 1.3) and hence, by (1.15), we have again (1.7). ■

Remark 1.4. Most of the subsequent results on hemi-spaces, combined with theorems 1.1 and 1.3, yield, in a similar way, corresponding results on generalized half-spaces, which we shall omit.

Since a semi-space in R^n is a maximal convex set excluding some point of R^n, it may be of interest to prove

Theorem 1.4. A set $H\subseteq R^n$ is a hemi-space if and only if there exists a convex set $G\subseteq R^n$ such that H is a maximal convex set not intersecting G.

Proof. If H is a hemi-space, then, obviously, it is the largest convex set which does not intersect the convex set $G=R^n\setminus H$.

Conversely, assume now that H is a maximal convex set not intersecting a given convex set G. Then, by theorem 0.1, there exists $v\in\mathcal{L}(R^n)$ such that

$$v(y) <_L v(g) \qquad (y\in H,\ g\in G). \tag{1.28}$$

Hence,

$$H\subseteq H'=\{y\in R^n \mid v(y)\,\sigma\,\inf_L v(G)\}, \tag{1.29}$$

where $\inf_L v(G)$ exists (by lemma 1.1) and

$$\sigma \text{ is } \begin{cases} <_L, & \text{if } \inf_L v(G)\in v(G) \\ \leq_L, & \text{if } \inf_L v(G)\notin v(G). \end{cases}$$

Then, clearly, H' is a convex set, which does not intersect G. Hence, by the maximality of H and (1.29), we obtain $H=H'$. But, by (1.29) and theorem 1.1, implication $2^\circ\Rightarrow 1^\circ$, H' is a hemi-space, so H is a hemi-space. ■

Remark 1.5. For related results, in which the set G of theorem 1.4 is a linear manifold, see theorem 3.2, corollary 3.1 and remark 3.2 below.

§2. Further representations of hemi-spaces. Classification of hemi-spaces

Let first prove some geometric results, which we shall need in the proof of the uniqueness statements of theorem 2.1 below.

Proposition 2.1. Let

$$H=\{y\in R^n \mid u(y) \leq_L x\}, \quad M=\{y\in R^n \mid u(y)=x\}\neq\emptyset, \tag{2.1}$$

where $u=(m_1,\ldots,m_r)^T \in \mathcal{L}(R^n, R^r)$, $x=(\xi_1,\ldots,\xi_r)^T\in R^r$. Then M is the unique linear manifold such that $M\subseteq H$ and $H\setminus M$ is convex, or, equivalently, such that $M\subseteq H$ and $H\setminus M$ is a hemi--space.

Proof. Clearly, M is a linear manifold, $M\subseteq H$, and the set

$$H\setminus M=\{y\in R^n \mid u(y) <_L x\} \tag{2.2}$$

is convex (even a hemi-space). Assume now that N is a linear manifold such that $N\subseteq H$ and $H\setminus N$ is convex. Let us first show that

$$N\subseteq M. \tag{2.3}$$

Indeed, if $N\not\subseteq M$, then, for each $y\in N\setminus M$, let

$$\ell(y)=\min\{i\leq r \mid (m_1,\ldots,m_i)^T y\neq(\xi_1,\ldots,\xi_i)^T\}, \tag{2.4}$$

and choose $y'\in N\setminus M$ such that $\ell=\ell(y')=\min_{y\in N\setminus M} \ell(y)$, whence

$$(m_1,\ldots,m_{\ell-1})^T y=(\xi_1,\ldots,\xi_{\ell-1})^T \qquad (y\in N). \tag{2.5}$$

Then, by $y'\in N\subseteq H$, (2.5) and (2.4), we obtain

$$m_\ell^T y'<\xi_\ell . \tag{2.6}$$

Observe now that m_ℓ^T is constant on N, i.e., there exists $c_\ell\in R$ such that

$$N\subseteq\{y\in R^n \mid m_\ell^T y=c_\ell\}; \tag{2.7}$$

indeed, otherwise there would exist y_1, $y_2\in N$ such that $m_\ell^T y_1\neq m_\ell^T y_2$, whence (since N is a linear manifold) also an element

$$\tilde{y}\in\{\lambda y_1+(1-\lambda)y_2\in R^n \mid \lambda\in R\}\subseteq N\subseteq H$$

such that $m_\ell^T\tilde{y}>\xi_\ell$, which, together with (2.5), contradicts (2.1). Note also that, by $y'\in N$, (2.7) and (2.6), we have

$$c_\ell<\xi_\ell . \tag{2.8}$$

Since $M\neq\emptyset$, take and $y_o\in M$, and let

$$y'' = 2y' - y_o. \tag{2.9}$$

Then, by $y'\in N$, (2.7), $y_o\in M$, (2.1) and (2.8),

$$m_\ell^T y''=2m_\ell^T y'-m_\ell^T y_o=2c_\ell-\xi_\ell<2c_\ell-c_\ell=c_\ell<\xi_\ell ,$$

whence, by (2.5) and (2.7), we obtain $y''\in H\setminus N$. Furthermore, by $y_o\in M\subseteq H$ and by $m_\ell^T y_o=\xi_\ell>c_\ell$ and (2.7), we have $y_o\in H\setminus N$. Hence, by (2.9) and the convexity of $H\setminus N$, we obtain

$$y' = \frac{1}{2}(y'' + y_o)\in H\setminus N,$$

in contradiction with our choice of y'. This proves (2.3).

In order to prove the opposite inclusion, let $y \in M$ be arbitrary. Take $\bar{y} \in N \subseteq M$ and let

$$y' = 2\bar{y} - y. \tag{2.10}$$

If $y' \in N$, then, since N is a linear manifold, we have $y = 2\bar{y} - y' \in N$. On the other hand, if $y' \notin N$, then, since M is a linear manifold, we have $y' = 2\bar{y} - y \in M \subseteq H$, so $y' \in H \setminus N$. Since $\frac{1}{2}(y+y') = \bar{y} \notin H \setminus N$, we obtain, by the convexity of $H \setminus N$, that $y \notin H \setminus N$, which, by $y \in M \subseteq H$, yields $y \in N$. Thus, $M \subseteq N$, which, together with (2.3), yields $N = M$. ∎

Proposition 2.2. *Let*

$$\dot{H} = \{y \in R^n \mid u(y) <_L x\}, \quad M = \{y \in R^n \mid u(y) = x\} \neq \emptyset, \tag{2.11}$$

where $u = (m_1, \ldots, m_r)^T \in \mathcal{L}(R^n, R^r)$ *and* $x = (\xi_1, \ldots, \xi_r)^T \in R^r$. *Then M is the largest linear manifold such that* $H \cap M = \emptyset$ *and* $H \cup M$ *is convex, or, equivalently, the only linear manifold such that* $H \cap M = \emptyset$ *and* $H \cup M$ *is a hemi-space.*

Proof. Clearly, we have $H \cap M = \emptyset$ and the set

$$H \cup M = \{y \in R^n \mid u(y) \leq_L x\} \tag{2.12}$$

is convex (even a hemi-space). Assume now, a contrario, that there exists a linear manifold N such that $H \cap N = \emptyset$ and $H \cup N$ is convex, and such that $N \not\subseteq M$. For each $y \in N \setminus M$, let

$$\ell(y) = \min\{i \leq n \mid (m_1, \ldots, m_i)^T y \neq (\xi_1, \ldots, \xi_i)^T\}, \tag{2.13}$$

and choose $y' \in N \setminus M$ such that $\ell = \ell(y') = \min_{y \in N \setminus M} \ell(y)$, whence

$$(m_1, \ldots, m_{\ell-1})^T y = (\xi_1, \ldots, \xi_{\ell-1})^T \quad (y \in N). \tag{2.14}$$

Then, by $y' \in N$, $H \cap N = \emptyset$ and (2.14), we obtain

$$m_\ell^T y' > \xi_\ell. \tag{2.15}$$

Observe now that m_ℓ^T is constant on N, i.e., there exists $c_\ell \in R$ such that

$$N \subseteq \{y \in R^n \mid m_\ell^T y = c_\ell\}; \tag{2.16}$$

indeed, otherwise there would exist $y_1, y_2 \in N$ such that $m_\ell^T y_1 \neq m_\ell^T y_2$, whence (since N is a linear manifold) also an element

$$\tilde{y} \in \{\lambda y_1 + (1-\lambda) y_2 \in R^n \mid \lambda \in R\} \subseteq N$$

such that $m_\ell^T \tilde{y} < \xi_\ell$, whence, by (2.14), $\tilde{y} \in H$, in contradiction with $H \cap N = \emptyset$. Note also that, by $y' \in N$, (2.16) and (2.15), we have

$$c_\ell > \xi_\ell. \tag{2.17}$$

Since $M \neq \emptyset$, take any $y_o \in M$, and let

$$y'' = 2y_o - y'. \tag{2.18}$$

Then, by $y_o \in M$, (2.11), $y' \in N$, (2.16) and (2.17), we have

$$m_\ell^T y'' = 2 m_\ell^T y_o - m_\ell^T y' = 2\xi_\ell - c_\ell < 2\xi_\ell - \xi_\ell = \xi_\ell,$$

whence $y''\in H\subseteq H\cup N$; also, $y'\in N\subseteq H\cup N$. Hence, by (2.18) and the convexity of $H\cup N$, we obtain

$$y_o=\frac{1}{2}(y'+y'')\in H\cup N. \tag{2.19}$$

However, by $y_o\in M$, (2.11) and (2.17), we have $m_\ell^T y_o=\xi_\ell<c_\ell$, whence, by (2.16), we obtain $y_o\notin N$. Also, by $y_o\in M$ and $H\cap M=\emptyset$, we have $y_o\notin H$, in contradiction with (2.19).

Finally, assume that N is a linear manifold such that $H\cap N=\emptyset$ and $H\cup N$ is a hemi-space. Then, $N\subseteq R^n\setminus H$ and the set

$$(R^n\setminus H)\setminus N=(R^n\setminus H)\cap(R^n\setminus N)=R^n\setminus(H\cup N) \tag{2.20}$$

is convex. Since

$$R^n\setminus H=\{y\in R^n|u(y)\geqslant_L x\}=\{y\in R^n|-u(y)\leqslant_L -x\}, \tag{2.21}$$

by proposition 2.1 we have that $M'=\{y\in R^n|-u(y)=-x\}$ is the unique linear manifold such that $M'\subseteq R^n\setminus H$ and $(R^n\setminus H)\setminus M'$ is convex. Hence, by the above properties of N and by (2.11), we obtain

$$N=M'=\{y\in R^n|-u(y)=-x\}=M. \blacksquare \tag{2.22}$$

<u>Remark 2.1.</u> In propositions 2.1 and 2.2, a sufficient condition in order that $M\neq\emptyset$, is that

$$\text{rank } u=r. \tag{2.23}$$

By theorem 2.1, implication $1^o\Rightarrow 17^o$, the assumption (2.23) is not restrictive.

Now we shall show that, under the assumption (2.1) (respectively, (2.11), there cannot hold a result like proposition 2.2 (respectively, 2.1).

<u>Lemma 2.1. For a set H of the form</u> (2.1), <u>a linear manifold</u> $N\subset R^n$ <u>does not intersect H if and only if there exist</u> $\ell\in\{1,\dots,r\}$ <u>and</u> $c_\ell>\xi_\ell$, <u>such that</u>

$$N\subseteq\{y\in R^n|m_j^T y=\xi_j\ (j=1,\dots,\ell-1), m_\ell^T y=c_\ell\}. \tag{2.24}$$

<u>Proof.</u> If there exist $\ell\in\{1,\dots,r\}$ and $c_\ell>\xi_\ell$ such that (2.24) holds, then for each $y\in N$ we have

$$(m_1,\dots,m_{\ell-1}, m_\ell)^T y=(\xi_1,\dots,\xi_{\ell-1}, c_\ell)^T >_L (\xi_1,\dots,\xi_{\ell-1},\xi_\ell)^T, \tag{2.25}$$

whence also $u(y)>_L x$, so $y\notin H$.

Conversely, assume that N is a linear manifold satisfying $H\cap N=\emptyset$. Let

$$\ell=\min\{i\leq r|N\nsubseteq\{y\in R^n|m_j^T y=\xi_j\ (j=1,\dots,i)\}\}. \tag{2.26}$$

Note that the set in the right hand side of (2.26) contains at least i=r, since otherwise we would have

$$N\subseteq\{y\in R^n|m_j^T y=\xi_j\ (j=1,\dots,r)\}=\{y\in R^n|u(y)=x\}\subseteq H,$$

in contradiction with $H\cap N=\emptyset$; thus, ℓ of (2.26) is well defined.

Observe now that m_ℓ^T is constant on N, i.e., there eixsts $c_\ell \in R$ such that we have (2.16); indeed, this follows with the same argument as in the proof of proposition 2.2 (since (2.5) holds now by the definition (2.26) of ℓ).

Now, by (2.5) and (2.16), we have (2.24), whence, by $H\cap N=\emptyset$ and (2.1), we obtain $c_\ell \geq \xi_\ell$; but, by (2.16) and (2.26), we have $c_\ell \neq \xi_\ell$, and hence $c_\ell > \xi_\ell$. ■

Proposition 2.3. If we have (2.1), then there is no linear manifold $N\subset R^n$ such that $H\cap N=\emptyset$ and $H\cup N$ is convex.

Proof. Assume, a contrario, that N is a linear manifold in R^n, such that $H\cap N=\emptyset$ and $H\cup N$ is convex. Take any $y_o\in M$, $y_1\in N$, and let ℓ and c_ℓ be as in lemma 2.1. Then, by (2.1) and (2.24), we have

$$m_j^T y_o=\xi_j=m_j^T y_1 \qquad (j=1,\dots,\ell-1), \tag{2.27}$$

$$m_\ell^T y_o=\xi_\ell<c_\ell=m_\ell^T y_1 , \tag{2.28}$$

whence

$$m_j^T \frac{y_o+y_1}{2}=\xi_j \qquad (j=1,\dots,\ell-1), \tag{2.29}$$

$$\xi_\ell < m_\ell^T \frac{y_o+y_1}{2} < c_\ell . \tag{2.30}$$

Hence, by (2.1) and (2.24), we obtain $\frac{y_o+y_1}{2}\notin H\cup N$, which, together with $y_o\in M\subseteq H\subseteq H\cup N$ and $y_1\in N\subseteq H\cup N$, contradicts the convexity of $H\cup N$. ■

Lemma 2.2. For a set H of the form (2.11), a linear manifold $N\subset R^n$ is contained in H if and only if there exist $\ell\in\{1,\dots,k\}$ and $c_\ell<\xi_\ell$ such that we have (2.24).

Proof. Since $N\subseteq H$ if and only if $(R^n\setminus H)\cap N=\emptyset$, the conclusion follows from lemma 2.1 applied to the set

$$R^n\setminus H=\{y\in R^n \mid (-u)(y) \leqslant_L -x\}. \quad ■ \tag{2.31}$$

Proposition 2.4. If we have (2.11), then there is no linear manifold $N\subset R^n$ contained in H and such that $H\setminus N$ is convex.

Proof. Assume, a contrario, that N is a linear manifold in R^n, contained in H and such that $H\setminus N$ is convex. Take any $y_o\in N$, $y_1\in M$, and let ℓ and c_ℓ be as in lemma 2.2. Then, by (2.24) and (2.11), we have

$$m_j^T y_o=\xi_j=m_j^T y_1 \qquad (j=1,\dots,\ell-1), \tag{2.32}$$

$$m_\ell^T y_o=c_\ell<\xi_\ell=m_\ell^T y_1 , \tag{2.33}$$

whence

$$m_j^T(y_o \pm \frac{1}{2}(y_1-y_o)) = m_j^T y_o \pm \frac{1}{2}(m_j^T y_1 - m_j^T y_o) = \xi_j \qquad (j=1,\dots,\ell-1), \tag{2.34}$$

$$m_\ell^T(y_o \pm \frac{1}{2}(y_1-y_o)) = m_\ell^T y_o \pm \frac{1}{2}(m_\ell^T y_1 - m_\ell^T y_o) =$$

$$= c_\ell \pm \frac{1}{2}(\xi_\ell - c_\ell) < c_\ell + (\xi_\ell - c_\ell) = \xi_\ell \,. \tag{2.35}$$

Hence, by (2.11) and (2.24), we obtain $y_o \pm \frac{1}{2}(y_1-y_o) \in H\setminus N$, which, since $y_o \in N$, contradicts the convexity of $H\setminus N$. ∎

Theorem 2.1. For a set $H \subseteq R^n$, the statements of theorems 1.1 and 1.2 are equivalent to the following statements:

12°-14°. There exist $(r,\sigma) \in (\{0,1,\dots,n\} \times \{<_L\}) \cup \{(n, \leq_L)\}$ (unique in 13°, 14°), $z=(\xi_1,\dots,\xi_r, \xi_{r+1}, +\infty,\dots,+\infty)^T \in R^r \times \{\pm\infty\} \times \{+\infty\}^{n-r-1}$ (unique in 14°, and with ξ_{r+1} being unique in 13°, if $r<n$), and $v \in \mathcal{L}(R^n)$, respectively $v \in \mathcal{U}(R^n)$, respectively $v=(m_1,\dots,m_n)^T \in \mathcal{O}(R^n)$ (with $m_1,\dots,m_r$ being uniquely determined in 14°), such that

$$H = \{y \in R^n \mid v(y) \sigma z\}. \tag{2.36}$$

15°. There exist $\sigma \in \{<_L, \leq_L\}$, $z \in R^n$ and $v \in \mathcal{L}(R^n)$, such that we have (2.36).

16°-18°. There exist $(r,\tau) \in \{0,1,\dots,n\} \times \{<_L, \leq_L\}$ (unique in 17° and 18°), $x \in R^r$ (unique in 18°) and $u \in \mathcal{L}(R^n, R^r)$, respectively $u \in \mathcal{L}(R^n, R^r)$ with rank $u=r$, respectively $u \in \mathcal{L}(R^n, R^r)$ with $uu^* = I$ (unique in 18°), such that

$$H = \{y \in R^n \mid u(y) \tau x\}. \tag{2.37}$$

19°-20°. There exist unique $(r,\tau) \in \{0,1,\dots,n\} \times \{<_L, \leq_L\}$, an element $y_o \in R^n$, and a basis $\{e'_j\}_1^n$, respectively, an orthonormal basis $\{e'_j\}_1^n$, of R^n, with coefficient functionals $\{\Psi_j\}_1^n \subset (R^n)^*$ (with $\{e'_j\}_j^r$ and $\{\Psi_j(y_o)\}_1^r$ uniquely determined in 20°), such that

$$H = \{y \in R^n \mid (\Psi_1(y),\dots,\Psi_r(y),0,\dots,0)^T \tau (\Psi_1(y_o),\dots,\Psi_r(y_o),0,\dots,0)^T\}. \tag{2.38}$$

21°. There exist (unique) $(r,\tau) \in \{0,1,\dots,n\} \times \{<_L, \leq_L\}$, an element $y_o \in R^n$, and $v \in \mathcal{O}(R^n)$ (with $\{v^{-1}(e_j)\}_1^r$ and the first r coordinates of y_o in the basis $\{v^{-1}(e_j)\}_1^n$ of R^n being uniquely determined), such that

$$H = y_o + v^{-1}(H_o), \tag{2.39}$$

where H_o is the hemi-space

$$H_o = \{y' = (\eta'_j) \in R^n \mid (\eta'_1,\dots,\eta'_r,0,\dots,0)^T \tau 0\}. \tag{2.40}$$

Proof. Let us first note that, in 12°-14°, $r=0$ means that $z=(\pm\infty,+\infty,\dots,+\infty)^T$, and thus, for $\xi_1=+\infty$ (respectively, $\xi_1=-\infty$), (2.36) yields $H=R^n$ (respectively, $H=\emptyset$). Furthermore, in 16°-18°, $r=0$ corresponds to $R^0=\{0\}$, so $x=0$ in (2.37), and the only $u \in \mathcal{L}(R^n,R^0)$ is $u\equiv 0$, whence (2.37) with τ being $\leq_L$ (respectively, $<_L$) yields $H=R^n$ (respectively,

$H=\emptyset$). Finally, in 19°-21°, $r=0$ means that $(\eta_1,\dots,\eta_r,0,\dots,0)^T$ of (2.40) is 0, and a similar remark holds for (2.38), whence, by (2.45) below, formula (2.40) or (2.38), with τ being $\leq_L$ (respectively, $<_L$), yields $H=R^n$ (respectively, $H=\emptyset$).

$4^{\circ}\Rightarrow 20^{\circ}$. If $v\in\mathcal{O}(R^n)$ and $z=(\xi_1,\dots,\xi_n)^T\in\bar{R}^n$ are as in 4°, let

$$e'_j=v^{-1}(e_j) \qquad (j=1,\dots,n). \tag{2.41}$$

Then, by $v\in\mathcal{O}(R^n)$, $\{e'_j\}_1^n$ is an orthonormal basis of R^n. If $z\in\bar{R}^n\setminus R^n$, then, by (1.15), we are in case (1.7) of 4°. Define

$$r=\min\{i\leq n\mid \xi_i\notin R\}-1, \tag{2.42}$$

$$\tau \text{ is}\begin{cases}<_L, & \text{if } \xi_{r+1}=-\infty\\ \leq_L, & \text{if } \xi_{r+1}=+\infty.\end{cases} \tag{2.43}$$

$$y_o=\sum_{j=1}^{r}\xi_j e'_j\in R^n . \tag{2.44}$$

Then, since

$$y=\sum_{j=1}^{n}\Psi_j(y)e'_j \qquad (y\in R^n), \tag{2.45}$$

we have, by (2.41),

$$v(y)=\sum_{j=1}^{n}\Psi_j(y)v(e'_j)=\sum_{j=1}^{n}\Psi_j(y)e_j=(\Psi_1(y),\dots,\Psi_n(y))^T \qquad (y\in R^n). \tag{2.46}$$

Hence, $v(y) <_L (\xi_1,\dots,\xi_r,-\infty,\dots)^T$ if and only if $(\Psi_1(y),\dots,\Psi_r(y),0,\dots,0)^T <_L (\xi_1,\dots,\xi_r,0,\dots,0)^T$, and, $v(y) <_L (\Psi_1(y),\dots,\Psi_r(y),+\infty,\dots)^T$ if and only if $(\Psi_1(y),\dots,\Psi_r(y),0,\dots,0)^T \leq_L (\xi_1,\dots,\xi_r,0,\dots,0)^T$, which, by (2.44), (1.7) and (1.8), proves (2.38).

Finally, if $z\in R^n$, then, taking $r=n$, $\tau\in\{<_L, \leq_L\}$ the same as in (1.7), respectively (1.8), and y_o as in (2.44) (with $r=n$), and using (2.46) (with $r=n$), we obtain, again, (2.38).

$19^{\circ}\Rightarrow 17^{\circ}$. If 19° holds, define $u\in\mathcal{L}(R^n, R^r)$ and $x\in R^r$ by

$$u(e'_j)=\begin{cases}e_j, & \text{if } j\in\{1,\dots,r\}\\ 0, & \text{if } j\in\{r+1,\dots,n\}.\end{cases} \tag{2.47}$$

$$x=u(y_o). \tag{2.48}$$

Then, rank $u=r$ and, by (2.45), we have

$$u(y)=\sum_{j=1}^{n}\Psi_j(y)u(e'_j)=\sum_{j=1}^{r}\Psi_j(y)e_j=(\Psi_1(y),\dots,\Psi_r(y),0,\dots,0)^T \qquad (y\in R^n),$$

$$x=u(y_o)=(\Psi_1(y_o),\dots,\Psi_r(y_o),0,\dots,0)^T ,$$

which, by (2.38), proves (2.37) (with the same r and τ).

$16^{\circ}\Rightarrow 15^{\circ}$. If (r,τ), $x\in R^r$ and $u\in\mathcal{L}(R^n, R^r)$ are as in 16°, define $\sigma\in\{<_L, \leq_L\}$, $z\in R^n$ and $v\in\mathcal{L}(R^n)$ by

$$\sigma \text{ is } \begin{cases} <_L \text{ (in } R^n\text{), if } \tau \text{ is } <_L \text{ (in } R^k) \\ \leq_L \text{, if } \tau \text{ is } \leq_L \text{,} \end{cases} \tag{2.49}$$

$$z = (x^T, 0, \dots, 0)^T \in R^n, \tag{2.50}$$

$$v(y) = (u(y)^T, 0, \dots, 0)^T \in R^n \qquad (y \in R^n). \tag{2.51}$$

Then, by (2.37), we have (2.36).

Thus, since the implications $20^o \Rightarrow 19^o$, $17^o \Rightarrow 16^o$ and $15^o \Rightarrow 12^o \Rightarrow 2^o$ are obvious, we have proved the equivalences $4^o \Leftrightarrow 20^o \Leftrightarrow 19^o \Leftrightarrow 17^o \Leftrightarrow 16^o \Leftrightarrow 15^o \Leftrightarrow 12^o$.

$20^o \Rightarrow 14^o$. If 20^o holds, define $z = (\xi_1, \dots, \xi_r, \xi_{r+1}, +\infty, \dots, +\infty)^T \in R^r \times \{\pm\infty\} \times \{+\infty\}^{n-r-1}$ and $v \in \mathcal{O}(R^n)$ by

$$\xi_j = \Psi_j(y_o) \qquad (j = 1, \dots, r), \tag{2.52}$$

$$\xi_{r+1} = \begin{cases} -\infty & \text{if } \tau \text{ is } <_L \\ +\infty & \text{if } \tau \text{ is } \leq_L \text{,} \end{cases} \tag{2.53}$$

$$v(e_j') = e_j \qquad (j = 1, \dots, n). \tag{2.54}$$

Then, by the above proof of the implication $4^o \Rightarrow 21^o$, we have (2.46). Hence, when $r \in \{0, 1, \dots, n-1\}$, we have $v(y) <_L (\Psi_1(y_o), \dots, \Psi_r(y_o), -\infty, +\infty, \dots, +\infty)^T$ if and only if $(\Psi_1(y), \dots, \Psi_r(y), 0, \dots, 0)^T <_L (\Psi_1(y_o), \dots, \Psi_r(y_o), 0, \dots, 0)^T$, and, we have $v(y) <_L (\Psi_1(y_o), \dots, \Psi_r(y_o), +\infty, \dots, +\infty)^T$ if and only if $(\Psi_1(y), \dots, \Psi_r(y), 0, \dots, 0)^T \leq_L (\Psi_1(y_o), \dots, \Psi_r(y_o), 0, \dots, 0)^T$, which, by (2.53), (2.52) and (2.38), proves (2.36) (with σ being $<_L$).

Finally, when $r=n$, from (2.46) we obtain, again, (2.36), with $\sigma = \tau$ of (2.38).

$13^o \Rightarrow 17^o$ (respectively, $14^o \Rightarrow 18^o$). Let (r, σ), $z = (\xi_1, \dots, \xi_n)^T$ and $v = (m_1, \dots, m_n)^T \in \mathcal{U}(R^n)$ (respectively, $\mathcal{O}(R^n)$) be as in 13^o (respectively, 14^o). If $z \in \bar{R}^n \setminus R^n$, then, by (1.15), we have (2.36), with σ being $<_L$. In this case, define τ by (2.43), and let

$$u = (m_1, \dots, m_r)^T \in \mathcal{L}(R^n, R^r), \tag{2.55}$$

$$x = (\xi_1, \dots, \xi_r)^T \in R^r, \tag{2.56}$$

with r of 13^o (respectively, 14^o). Then, $v(y) <_L (\xi_1, \dots, \xi_r, -\infty, +\infty, \dots +\infty)^T$ if and only if $u(y) <_L x$, while $v(y) <_L (\xi_1, \dots, \xi_r, +\infty, \dots, +\infty)^T$ if and only if $u(y) \leq_L x$, which, by (2.36) (with σ being $<_L$), proves (2.37). On the other hand, if $z \in R^n$, then $r=n$, and hence, taking $u=v$, $x=z$ and $\tau=\sigma$, from (2.36) we obtain, again, (2.37). Furthermore, in both cases, $v \in \mathcal{U}(R^n)$ (respectively, $v \in \mathcal{O}(R^n)$) implies rank $u=r$ (respectively, $uu^* = I$).

$20^o \Rightarrow 21^o$. If we have 20^o, define $v \in \mathcal{O}(R^n)$ by (2.54). Then, since

$$v^{-1}(y') = v^{-1}\Big(\sum_{j=1}^n \eta_j' e_j\Big) = \sum_{j=1}^n \eta_j' v^{-1}(e_j) = \sum_{j=1}^n \eta_j' e_j' \qquad (y' = (\eta_j') \in R^n), \tag{2.57}$$

we have

$$\Psi_i(v^{-1}(y'))=\eta'_i \qquad (y'=(\eta'_j)\in R^n,\ i=1,\dots,n), \tag{2.58}$$

whence, for H_o of (2.40),

$$v^{-1}(H_o)=\{v^{-1}(y')\,|\,y'=(\eta'_j)\in R^n,\ (\eta'_1,\dots,\eta'_r,0,\dots,0)^T\tau 0\}=$$

$$=\{y\in R^n\,|\,(\Psi_1(y),\dots,\Psi_r(y),0,\dots,0)^T\tau 0\}. \tag{2.59}$$

Hence, by (2.38), we obtain

$$y_o+v^{-1}(H_o)=\{y_o+y\in R^n\,|\,(\Psi_1(y_o+y),\dots,\Psi_r(y_o+y),0,\dots,0)^T\tau$$

$$\tau(\Psi_1(y_o),\dots,\Psi_r(y_o),0,\dots,0\}^T = H,$$

i.e., (2.39) (with the same (r,τ) and y_o, as in 20^o).

$21^o\Rightarrow 20^o$. If we have 21^o, define $\{e'_j\}_1^n$ by (2.41). Then, by $v\in\mathcal{O}(R^n)$, $\{e'_j\}_1^n$ is an orthonormal basis of R^n. Finally, by (2.39) and the computations of the above proof of the implication $20^o\Rightarrow 21^o$, we obtain (2.38) (with the same (r,τ) and y_o, as in 21^o).

Thus, since the implications $14^o\Rightarrow 13^o$ and $18^o\Rightarrow 17^o$ are obvious, we have proved all equivalences stated in theorem 2.1. Let us prove now the uniqueness statements of the theorem.

<u>Uniqueness of</u> (r,τ) <u>in</u> 17^o (<u>and hence</u> 18^o). Assume that we have two representations (2.37), say,

$$H=\{y\in R^n\,|\,u_1(y)\tau_1 x_1\}=\{y\in R^n\,|\,u_2(y)\tau_2 x_2\}, \tag{2.60}$$

where $(r_i,\ \tau_i)\in\{0,1,\dots,n\}\times\{<_L,\ \leq_L\}$, $x_i\in R^{r_i}$ and $u_i\in\mathcal{L}(R^n,\ R^{r_i})$, with rank $u_i=r_i$ ($i=1,2$), and let

$$M_i=\{y\in R^n\,|\,u_i(y)=x_i\} \qquad (i=1,2). \tag{2.61}$$

Then, since rank $u_i=r_i$, we have $M_i\neq\emptyset$ ($i=1,2$). If τ_1 is $\leq_L$, then, by proposition 2.3, there exists no linear manifold N such that $H\cap N=\emptyset$ and $H\cup N$ is convex, and hence, by $M_2\neq\emptyset$ and proposition 2.2, we obtain that τ_2 is $\leq_L$, too. Hence, by $M_1\neq\emptyset$ and proposition 2.1, it follows that $M_1=M_2$, and therefore, since by rank $u_i=r_i$ we have $r_i=\operatorname{codim} M_i$ ($i=1,2$), we obtain $r_1=r_2$. On the other hand, if τ_1 is $<_L$, then the argument is similar, using propositions 2.4, 2.1 and 2.2.

<u>Uniqueness of</u> x <u>and</u> u <u>in</u> 18^o. Assume that we have (2.60), with $\tau_1=\tau_2=\tau$, $x_i\in R^r$ and with $u_1=(m_1,\dots,m_r)^T$, $u_2=(\tilde m_1,\dots,\tilde m_r)^T\in\mathcal{L}(R^n,\ R^r)$ satisfying $u_iu_i^*=I$ ($i=1,2$). If $u_1\neq u_2$, let

$$\ell=\min\{i\leq n\,|\,\tilde m_i\neq m_i\}. \tag{2.62}$$

Observe now that, by $\|m_\ell\|=\|\tilde m_\ell\|=1$ and $m_\ell\neq\tilde m_\ell$, we have the strict inequality $m_\ell^T m_\ell<$ $<\|m_\ell\|\|\tilde m_\ell\|=1$, and hence we can define

$$d = \frac{1}{1-m_\ell^T \tilde{m}_\ell}(m_\ell - \tilde{m}_\ell) \in R^n. \tag{2.63}$$

Then, by the orthonormality of the sets of rows of u_i $(i=1,2)$, we have

$$m_j^T d = 0 \qquad (j=1,\ldots,\ell-1), \tag{2.64}$$

$$m_\ell^T d = 1, \quad \tilde{m}_\ell^T d = -1. \tag{2.65}$$

On the other hand, by the above proof of the uniqueness of (r,τ) in 18^o, we have

$$\{y \in R^n \mid u_1(y) = x_1\} = \{y \in R^n \mid u_2(y) = x_2\} \neq \emptyset. \tag{2.66}$$

Let y_o be any element of (2.66). Then, by (2.64) and (2.65),

$$u_1(y_o+d) = u_1(y_o) + u(d) = x_1 + (0,\ldots,0,1,\ldots)^T >_L x_1,$$

$$u_2(y_o+d) = u_2(y_o) + u(d) = x_2 + (0,\ldots,0,-1,\ldots)^T <_L x_2,$$

whence, by (2.60), we obtain that $y_o+d \in H$ and $y_o+d \notin H$, which is impossible. This proves that $u_1 = u_2$, whence also $x_1 = u_1(y_o) = u_2(y_o) = x_2$.

<u>Uniqueness of</u> (r,σ) <u>in</u> 13^o <u>(and hence</u> 14^o), <u>of</u> ξ_{r+1} <u>in</u> 13^o, <u>when</u> $r<n$, <u>and of</u> z <u>and</u> $\{m_j\}_1^r$ <u>in</u> 14^o. These follow from the above proof of the implication $13^o \Rightarrow 17^o$ (respectively, $14^o \Rightarrow 18^o$), using the uniqueness of (r,τ) in 17^o (respectively, of (r,τ), x and u in 18^o).

<u>Uniqueness of</u> (r,τ) <u>in</u> 19^o <u>(and hence</u> 20^o). This follows from the above proof of the implication $19^o \Rightarrow 17^o$, using the uniqueness of (r,τ) in 17^o.

<u>Uniqueness of</u> $\{e_j'\}_1^r$ <u>and</u> $\{\Psi_j(y_o)\}_1^r$ <u>in</u> 20^o. This follows from the above proof of the implication $20^o \Rightarrow 14^o$, using the uniqueness of $\{m_j\}_1^r$ and $\{\xi_j\}_1^r$ in 14^o. Indeed, by (2.54) and $v=(m_1,\ldots,m_n)^T$, we have

$$m_j^T e_j' = 1 \qquad (j=1,\ldots,n), \tag{2.67}$$

which, since $\|m_j\| = \|e_j'\| = 1$ $(j=1,\ldots,n)$, implies

$$e_j' = m_j \qquad (j=1,\ldots,n), \tag{2.68}$$

and thus, since $m_1,\ldots,m_r$ are uniquely determined in 14^o, so are $e_1',\ldots,e_r'$ in 21^o. Finally, since $\xi_1,\ldots,\xi_r$ are uniquely determined in 14^o, so are, by (2.52), the numbers $\Psi_1(y_o),\ldots,\Psi_r(y_o)$ in 20^o.

<u>Uniqueness of</u> (r,τ), $\{v^{-1}(e_j)\}_1^r$ <u>and the first</u> r <u>coordinates of</u> y_o <u>in the basis</u> $\{v^{-1}(e_j)\}_1^n$ <u>of</u> R^n, <u>in</u> 21^o. These follow from the above proof of the implication $21^o \Rightarrow 20^o$, using the uniqueness of (r,τ), $\{e_j'\}_1^r$ and $\{\Psi_j(y_o)\}_1^r$ in 20^o. Thus, the proof of theorem 2.1 is complete. ∎

<u>Remark 2.2.</u> a) The equivalence $1^o \Leftrightarrow 20^o$, without the uniqueness properties in 20^o, has been also proved, with a different method, by M.Lassak ([9], theorem 1, statement

1^o), but he has not expressed 20^o in terms of the lexicographical order.

b) One can give simpler proofs of some of the implications of theorem 2.1, and hence of the whole chain of equivalences (e.g., one can replace $13^o \Rightarrow 17^o$ and $21^o \Rightarrow 20^o$ by the obvious implications $13^o \Rightarrow 1^o$ and $21^o \Rightarrow 1^o$), but the proofs of the implications selected above are also used in the proofs of the uniqueness statements of theorem 2.1.

c) Statement 15^o is the particular case $r=n$ of 12^o and of 16^o; also, 15^o shows that in 2^o (of theorem 1.1) one can replace $z \in \bar{R}^n$ by $z \in R^n$. However, similarly to remark 1.1 a), in the representation (2.36) (even with the same $v \in \mathcal{L}(R^n)$ and $z \in R^n$), σ is not uniquely determined, as shown e.g. by

$$H=\{y=(\eta_1,\eta_2)^T \in R^2 \mid (\eta_1,\ 0)^T <_L (1,1)^T\}=$$
$$=\{y=(\eta_1,\eta_2)^T \in R^2 \mid (\eta_1,\ 0)^T \leqslant_L (1,1)^T\}. \tag{2.69}$$

Furthermore, in 11^o of theorem 1.2 one can replace $x \in \bar{R}^k$ by $x \in R^k$, using, in the proof, 15^o (instead of 2^o) and -1 and 1 instead of $-\infty$ and $+\infty$, respectively. This proof works (with obvious changes) also for $<_L$ replaced by $\leqslant_L$ in (1.19).

d) It is well-known (see [2], [3], [8]) that H is a semi-space if and only if we have 21^o with $r=n$ and $\tau = <_L$ (and hence with H_o of (2.40) being a semi-space). Similarly, in the general case, condition 21^o will turn out to be useful in the sequel, since it says that, modulo a translation and a linear isometry, H_o of (2.40) is the general form of a hemi-space in R^n. In view of this special role of H_o, let us observe that for $H=H_o$, the (unique) $u \in \mathcal{L}(R^n,\ R^r)$ and $x \in R^r$ of 18^o are

$$u(y)=(\eta_1,\dots,\eta_r)^T \qquad (y=(\eta_j)_1^n \in R^n), \tag{2.70}$$

$$x=0. \tag{2.71}$$

Note also that 21^o gives, already in R^3, a simple example of a hemi-space which is neither R^3, nor $\emptyset$, nor a closed (open) half-space, nor a (complement of a) semi-space, namely, $H_o=\{y=(\eta_j)_1^3 \in R^3 \mid (\eta_1,\eta_2,\ 0)^T \tau\, 0\}$, with $\tau \in \{<_L,\ \leqslant_L\}$.

e) In 17^o the pair $(u,\ x)$ is unique up to a (unique) lexicographical order preserving linear isomorphism of R^r (for a study of lexicographical order preserving linear operators, see [13]), i.e., the equality

$$\{y \in R^n \mid u_1(y)\tau x_1\}=\{y \in R^n \mid u_2(y)\tau x_2\}, \tag{2.72}$$

with $\tau \in \{<_L, \leqslant_L\}$, $u_i \in \mathcal{L}(R^n,\ R^r)$, rank $u_i=r$, $x_i \in R^r$ $(i=1,2)$, holds if and only if there exists a (unique) lexicographical order preserving linear isomorphism $\ell \in \mathcal{U}(R^r)$ such that

$$u_2=\ell u_1,\quad x_2=\ell(x_1). \tag{2.73}$$

Indeed, assume that for $\tau\in\{<_L, \leq_L\}$ and (u_i, x_i) $(i=1,2)$ there exists $\ell\in\mathcal{U}(R^r)$ preserving the lexicographical order relation $\leq_L$ and satisfying (2.73). Then, the relations $y\in R^n$, $u_1(y)\tau x_1$ imply

$$u_2(y)=\ell(u_1(y))\tau\ell(x_1)=x_2$$

(since, by $\ell\in\mathcal{U}(R^r)$, ℓ preserves $\leq_L$ if and only if it preserves $<_L$); similarly, since $\ell^{-1}\in\mathcal{U}(R^r)$ preserves the lexicographical order, the relations $y\in R^n$, $u_2(y)\tau x_2$ imply $u_1(y)\tau x_1$, so (2.72) holds.

Conversely, if (2.72) holds, then, by the above proof of the uniqueness of (r,τ) in 17^o, we have

$$\{y\in R^n|u_1(y)=x_1\}=\{y\in R^n|u_2(y)=x_2\}\neq\emptyset, \tag{2.74}$$

whence Ker u_1=Ker u_2. Hence, since rank $u_1=r$, there exists an unique $\ell\in\mathcal{U}(R^r)$ such that $\ell u_1=u_2$. Take any y_o in (2.74). Then,

$$x_2=u_2(y_o)=\ell(u_1(y_o))=\ell(x_1), \tag{2.75}$$

so (2.73) holds. Finally, take any $x\in R^r$ such that $x <_L 0$. Then, by rank $u_1=r$, there exists $y\in R^n$ such that

$$u_1(y)=x+u_1(y_o) <_L u_1(y_o)=x_1,$$

whence, by (2.72), $u_2(y) \leq_L x_2$. Hence, by (2.75),

$$\ell(x)=\ell(u_1(y-y_o))=u_2(y)-\ell(u_1(y_o))=u_2(y)-x_2 \leq_L 0,$$

which proves that ℓ preserves the lexicographical order $\leq_L$. ∎

Let us also observe that, by computing explicitly the unique $\ell\in\mathcal{U}(R^r)$ above, we obtain that (2.72) holds if and only if

(i) Ker u_1= Ker u_2,

(ii) $u_2u_1^*(u_1u_1^*)^{-1}$ preserves the lexicographical order,

(iii) $u_2u_1^*(u_1u_1^*)^{-1}(x_1)=x_2$.

Indeed, if (2.72) holds, then we have (i) and the unique $\ell\in\mathcal{U}(R^r)$, satisfying $\ell u_1=u_2$, is

$$\ell=u_2u_1^*(u_1u_1^*)^{-1}, \tag{2.76}$$

whence, since ℓ preserves the lexicographical order and satisfies $x_2=\ell(x_1)$, we obtain (ii) and (iii). In order to see that ℓ of (2.76) satisfies $\ell u_1=u_2$, note that $\ell u_1u_1^*=u_2u_1^*$, whence $\ell u_1|_{u_1^*(R^r)}=u_2|_{u_1^*(R^r)}$; also, by (i), $\ell u_1|_{\text{Ker } u_1}=0=u_2|_{\text{Ker } u_2}=u_2|_{\text{Ker } u_1}$, so ℓu_1 coincides with u_2 on $u_1^*(R^k)$+Ker u_1=(Ker $u_1)^\perp$ + Ker $u_1=R^n$. Conversely, if we have (i)-(iii), define ℓ by (2.76). Then, by (i), we have Ker ℓ = {0}, so $\ell\in\mathcal{U}(R^r)$: for, if $x\in R^r$, $\ell(x)=0$, then $u_2(u_1^*(u_1 u_1^*)^{-1}(x))=0$, whence $x=u_1(u_1^*(u_1 u_1^*)^{-1}(x))=0$. Finally, by (ii) and (iii), ℓ preserves the lexicographical order and satisfies $\ell(x_1)=x_2$. ∎

f) Using similar arguments, one can prove that in 13°, with $v=(m_1,\dots,m_n)^T$ (respectively, in 19°), the pair $((m_1,\dots,m_r)^T, (\xi_j)_1^r)$ (respectively, the pair $(\{\Psi_j\}_1^r, \{\Psi_j(y_o)\}_1^r)$) are unique up to a lexicographical order preserving linear isomorphism of R^r. We omit the details.

g) In various proofs which we shall give in the sequel (and in [13]), separation of convex sets by hemi-spaces (see remark 1.1 b)), combined with the representation theorem 2.1 for hemi-spaces (which, in turn, has been obtained, ultimately, by applying the lexicographical separation theorem 0.1), will be a more powerful tool than a direct application of the lexicographical separation theorem 0.1.

<u>Definition 2.1</u>. For any hemi-space H in R^n, represented in one of the forms 17°-21° of theorem 2.1, we shall call the (unique) $r\in\{0,1,\dots,n\}$ and $\tau\in\{<_L, \leq_L\}$ of the representation, the <u>rank</u> and the <u>type</u> of the hemi-space H, respectively. We shall also use, when necessary, the notations

$$r=r(H), \qquad \tau=\tau(H). \tag{2.77}$$

When $\tau=\tau(H)$ is $<_L$ (respectively, $\leq_L$), we shall say that H is a <u>hemi-space of type</u> <u>$<_L$</u> (respectively, <u>of type $\leq_L$</u>).

<u>Remark 2.3.</u> a) From the uniqueness of (r,τ), u and x in 18° of theorem 2.1, it follows that <u>a hemi-space</u> H <u>is of type $<_L$</u> <u>if and only if it is not of type</u> $\leq_L$ (i.e., the types $<_L$ and $\leq_L$ are mutually exclusive).

b) $H\subseteq R^n$ <u>is a hemi-space of type</u> $\leq_L$ <u>if and only if</u> $R^n\setminus H$ <u>is a hemi-space of type</u> $<_L$. Indeed, we have $H=\{y\in R^n | u(y) \leq_L x\}$ (where $u\in\mathcal{L}(R^n, R^r)$, $x\in R^r$) if and only if $R^n\setminus H=\{y\in R^n | (-u)(y) <_L -x\}$.

c) By definition 2.1, every hemi-space of type $<_L$ is a generalized half-space. However, the converse is not true, as shown e.g. by the generalized half-space

$$\begin{aligned} H&=\{y=(\eta_1,\eta_2)^T\in R^2 | (\eta_1,\eta_2)^T <_L (0, +\infty)^T\}= \\ &=\{y=(\eta_1,\eta_2)^T\in R^2 | \eta_1\leq 0\}; \end{aligned} \tag{2.78}$$

indeed, the last term of (2.78) is a representation of H as in 17° of theorem 2.1, with $u\in\mathcal{L}(R^2, R^1)$ defined by

$$u(y)=\eta_1 \qquad (y=(\eta_1, \eta_2)\in R^2),$$

so H is a hemi-space of type $\leq_L$, whence not of type $<_L$.

d) Clearly, H is a semi-space if and only if it is a hemi-space of type $<_L$ and of rank $r(H)=n$. This fact, together with theorem 1.3, suggests to conjecture that the complement of a hemi-space of type $<_L$ is not a generalized half-space; however, (2.78) above provides a counter-example.

We have the following minimality property of the rank $r(H)$:

Proposition 2.5. Let $H\subseteq R^n$ be a hemi-space. Then, for any representation (2.37) of H, with $u\in\mathcal{L}(R^n, R^r)$ and $x\in\bar{R}^r$, we have

$$r\geq \text{rank } u\geq r(H). \tag{2.79}$$

Proof. Clearly, we have to prove only the second inequality. We may assume (replacing H, if necessary, by $R^n\setminus H$), that H is of type $<_L$. Then, by theorem 2.1, implication $1^o\Rightarrow 17^o$, there exist $u'\in\mathcal{L}(R^n, R^{r(H)})$, with rank $u'=r(H)$, and $x'\in R^{r(H)}$, such that

$$H=\{y\in R^n\,|\,u'(y)<_L x'\}; \tag{2.80}$$

also, by rank $u'=r(H)$ and $x'\in R^{r(H)}$, there exists $y_o\in R^n$ such that

$$u'(y_o)=x'. \tag{2.81}$$

Let $d\in\text{Ker } u$. Then $u(y_o\pm d)=u(y_o)\pm u(d)=u(y_o)$, whence, by $y_o\in R^n\setminus H$ and (2.37), we obtain $y_o\pm d\in R^n\setminus H$. Consequently,

$$x'\leq_L u'(y_o\pm d)=u'(y_o)\pm u'(d)=x'\pm u'(d),$$

whence $\pm u'(d)\geq_L 0$, which is equivalent to $u'(d)=0$. Thus, we have proved that $\text{Ker } u\subseteq \text{Ker } u'$, whence

$$\text{rank } u=n-\dim \text{Ker } u\geq n-\dim \text{Ker } u'=\text{rank } u'=r(H). \qquad \blacksquare$$

Corollary 2.1. Let $H\subseteq R^n$ be a hemi-space. Then, for any representation (2.37) of H, with $u\in\mathcal{L}(R^n, R^{r(H)})$ and $x\in\bar{R}^{r(H)}$, we have rank $u=r(H)$, $\tau=\tau(H)$ and $x\in R^{r(H)}$.

Proof. By proposition 2.5 (with $r=r(H)$), we have rank $u=r(H)$. Hence, if $x\in R^{r(H)}$, then, by (2.37) and the uniqueness of $\tau(H)$, we obtain $\tau=\tau(H)$. Therefore, let us prove that $x\in R^{r(H)}$. If $r(H)=0$, this is obvious (since $\bar{R}^0=\{0\}$), so let $x=(\xi_1,\dots,\xi_{r(H)})^T$, $r(H)\geq 1$; also, as in the proof of proposition 2.5, assume that H is of type $<_L$ and choose $u'\in\mathcal{L}(R^n, R^{r(H)})$ with rank $u'=r(H)$, $x'\in R^{r(H)}$ and $y_o\in R^n$ satisfying (2.80) and (2.81). Then, by the proof of proposition 2.5, we have $\text{Ker } u\subseteq \text{Ker } u'$, whence, since (by our assumptions on the ranks of u and u')

$$\dim \text{Ker } u=n-\text{rank } u=n-r(H)=n-\text{rank } u'=\dim \text{Ker } u',$$

we obtain

$$\text{Ker } u=\text{Ker } u'. \tag{2.82}$$

Let $i\in\{1,\dots,r(H)\}$, let e_i denote the i-th unit vector of $R^{r(H)}$, and by rank $u=$ $=r(H)$, choose

$$e_i'\in u^{-1}(e_i). \qquad (2.83)$$

Then, since $e_i'\notin \mathrm{Ker}\ u=\mathrm{Ker}\ u'$, we have $u'(e_i)\neq 0$, whence

$$s_i u'(e_i) <_L 0, \qquad (2.84)$$

for some $s_i\in\{1, -1\}$. Hence, by (2.81),

$$u'(y_o+s_ie_i')=u'(y_o)+s_iu'(e_i')=x'+s_iu'(e_i') <_L x',$$

and thus, by (2.80), $y_o+s_ie_i'\in H$. Therefore, by (2.83) and (2.37), we obtain $u(y_o)+$ $+s_ie_i=u(y_o+s_ie_i') <_L x$. Hence, if $\xi_i=\pm\infty$, then $u(y_o) <_L x$, and thus, by (2.37), $y_o\in H$; but then, by (2.81) and (2.80), $x'=u'(y_o) <_L x'$, which is impossible. This proves that $\xi_i\in R$, whence, since $i\in\{1,\dots,r(H)\}$ was arbitrary, $x\in R^{r(H)}$. ∎

Remark 2.4. From corollary 2.1 there follows again theorem 1.3. Indeed, if H is the complement of a semi-space, then $H=\{y\in R^n | v(y) \leqslant_L z\}$ for some $v\in\mathcal{U}(R^n)$, $z\in R^n$, so $r(H)=$ $=n$ and $\tau(H)$ is $\leqslant_L$. Hence, if H is also a generalized half-space, say, $H=\{y\in R^n | u(y) <_L x\}$, where $u\in\mathcal{L}(R^n)$, $x\in R^n$, then, by corollary 2.1, we obtain rank $u=r(H)=n$ (so $u\in\mathcal{U}(R^n)$), $x\in R^n$ and $<_L=\tau(H)$, in contradiction with the above observation that $\tau(H)$ is $\leqslant_L$. ∎

From theorem 2.1 it follows that the pairs $(r(H), \tau(H))$ yield a metric-affine classification of hemi-spaces. Indeed, we have

Theorem 2.2. For two hemi-spaces $H_1, H_2\subseteq R^n$, the following statements are equivalent:

1°. We have

$$(r(H_1), \tau(H_1))=(r(H_2), \tau(H_2)). \qquad (2.85)$$

2°-3°. There exist $v\in\mathcal{U}(R^n)$, $v\in\mathcal{O}(R^n)$, respectively, and $\bar{y}\in R^n$, such that

$$H_2 = \bar{y} + v^{-1}(H_1). \qquad (2.86)$$

Proof. 1°⇒3°. By theorem 2.1, implication 1°⇒21°, there exist y_o', $y_o''\in R^n$ and v_1, $v_2\in\mathcal{O}(R^n)$, such that

$$H_1=y_o'+v_1^{-1}(H_o), \quad H_2=y_o''+v_2^{-1}(H_o), \qquad (2.87)$$

with H_o of (2.40), where $r=r(H_1)=r(H_2)$, $\tau=\tau(H_1)=\tau(H_2)$ (thus, H_o is the same for both H_1 and H_2). Hence,

$$v_1(H_1-y_o')=H_o=v_2(H_2-y_o''),$$

whence

$$H_2=v_2^{-1}v_1(H_1)-v_2^{-1}v_1(y_o')+y_o'',$$

i.e, we have (2.86) with $\bar{y}=y''_o-v_2^{-1}v_1(y'_o)$ and $v=v_1^{-1}v_2\in\mathcal{O}(R^n)$.

The implication $3^o\Rightarrow 2^o$ is obvious.

$2^o\Rightarrow 1^o$. If 2^o holds and H_1 is represented in the form (2.87), then

$$H_2=\bar{y}+v^{-1}(y'_o+v_1^{-1}(H_o))=\bar{y}+v^{-1}(y'_o)+v^{-1}v_1^{-1}(H_o),$$

so H_2 satisfies (2.87) with $y''_o=\bar{y}+v^{-1}(y'_o)$, $v_2=v_1v$ and with the same H_o, whence we obtain 1^o. ∎

Let us show now how the representations of hemi-spaces, given in 12^o-14^o of theorem 2.1, can be also deduced directly from the representations 5^o-7^o of theorem 1.1, since this may present some interest for other applications as well. To this end, let us first prove

Lemma 2.3. For an element $z=(\xi_j)_1^n\in\bar{R}^n$, the following statements are equivalent:

1^o. There exists a subset G of R^n, such that we have (1.1).

2^o. There exists $r\in\{0,1,\dots,n\}$ such that $\xi_1,\dots,\xi_r\in R$ and

$$z=(\xi_1,\dots,\xi_r,\pm\infty,\ +\infty,\dots,\ +\infty)^T. \qquad (2.88)$$

Proof. $1^o\Rightarrow 2^o$. Assume 1^o, and let $r\in\{0,1,\dots,n\}$ be such that $\xi_{r+1}\in\bar{R}\setminus R$. Then, for $k\geq r+2$ we have $k-1\geq r+1$, whence, by lemma 1.1 and $g\in R^n$, we obtain

$$\xi_k=\inf\{\gamma_k|g=(\gamma_j)_1^n\in G,\ \gamma_j=\xi_j\ \ (j=1,\dots,k-1)\}=\inf\emptyset=+\infty.$$

$2^o\Rightarrow 1^o$. Assume that $z=(\xi_j)_1^n\in\bar{R}^n$ satisfies 2^o.

a) If $\xi_{r+1}=-\infty$, let

$$G=\{(\xi_1,\dots,\xi_r,\lambda,0,\dots,0)|\ \lambda\in R\}(\subseteq R^n). \qquad (2.89)$$

Then, by lemma 1.1, the lexicographical infimum

$$z'=\{\xi'_j\}_1^n=\inf_L G \qquad (2.90)$$

exists, and it is given by $\xi'_j=\xi_j$ $(j=1,\dots,r)$, $\xi'_{j+1}=\inf_{\lambda\in R}\lambda=-\infty$, $\xi'_k=\inf\emptyset=+\infty$ $(k=r+2,\dots,n)$, so $z'=z$, which proves (1.1).

b) If $\xi_{r+1}=+\infty$ and $r=0$, then we have

$$z=(+\infty,\dots,+\infty)^T=+\infty=\inf_L\emptyset,$$

i.e., (1.1) for $G=\emptyset$.

c) If $\xi_{r+1}=+\infty$ and $r\in\{1,\dots,n\}$, let

$$G=\{(\xi_1,\dots,\xi_{r-1},\lambda,0,\dots,0)^T|\lambda>\xi_r\}(\subset R^n). \qquad (2.91)$$

Then, by lemma 1.1, the lexicographical infimum (2.90) exists and it is given by $\xi'_j=\xi_j$ $(j=1,\dots,r-1)$, $\xi'_r=\inf_{\lambda>\xi_r}\lambda=\xi_r$, $\xi'_k=\inf\emptyset=+\infty$ $(k=r+1,\dots,n)$, so $z'=z$, which proves (1.1). ∎

Now, as announced before lemma 2.3, let us give the

Second proof of theorem 2.1, implication $1^\circ \Rightarrow 14^\circ$. If we have 1°, then, by theorem 1.1, implication $1^\circ \Rightarrow 7^\circ$, and lemma 2.3 (with $G=v(R^n \setminus H) \subseteq R^n$), there exist (r,σ), z and v as in 14° (indeed, if $r \in \{0,1,\dots,n-1\}$, then z of (2.88) belongs to $\bar{R}^n \setminus R^n$, whence, by (1.15), we have (2.36) with σ being $<_L$, while if r=n, then $z \in R^n$; so we have (2.36) with σ being the same as in (1.9) or (1.10), respectively. ∎

§3. Further geometric results on hemi-spaces

In addition to the geometric results of §1, we shall deduce now, from the preceding representation theorems, some further geometric results on hemi-spaces.

Theorem 3.1. For a set $H \subseteq R^n$, the following statements are equivalent:

1°. H is a hemi-space.

2°. $H + \{0,y\} = H \cup (H+y)$ is convex, for each $y \in R^n$.

3°. H + C is a hemi-space, for each set $C \subseteq R^n$.

4°. $H \divideontimes C$ is a hemi-space, for each set $C \subseteq R^n$.

Proof. $1^\circ \Rightarrow 3^\circ$. Let H be represented as in 15° of theorem 2.1, and let $C \subseteq R^n$. Then, by (2.36), we have

$$H+C=\{y+c \in R^n \mid y \in H,\ c \in C\}=\{y+c \in R^n \mid v(y)\sigma z,\ c \in C\}=$$
$$=\{y' \in R^n \mid \exists c \in C,\ v(y')\sigma z+v(c)\}. \tag{3.1}$$

Define

$$z' = z + \sup_L v(C), \tag{3.2}$$

$$\sigma' \text{ is } \begin{cases} \leq_L, & \text{if } \sigma \text{ is } \leq_L \text{ and } \sup_L v(C) \in v(C) \\ <_L & \text{otherwise.} \end{cases} \tag{3.3}$$

We shall show that

$$H + C = \{y' \in R^n \mid v(y')\sigma' z'\}, \tag{3.4}$$

which, by theorem 1.1, implication $2^\circ \Rightarrow 1^\circ$, will prove that H + C is a hemi-space. If $y' \in H+C$, then, by (3.1), there exists $c \in C$ such that

$$v(y')\sigma z+v(c). \tag{3.5}$$

Hence, if σ is $\leq_L$ and $\sup_L v(C) \in v(C)$, then, by (3.2), we have

$$v(y') \leq_L z+v(c) \leq_L z+\sup_L v(C)=z'$$

whence, by (3.3), $v(y')\sigma' z'$. If σ is $\leq_L$ and $\sup_L v(C) \notin v(C)$, then $v(c) <_L \sup_L v(C)$, whence, by (3.2),

$$v(y') \leq_L z+v(c) <_L z+\sup_L v(C)=z',$$

and hence $v(y') <_L z'$; thus, by (3.3), $v(y')\sigma' z'$. Finally, if σ is $<_L$, then, by (3.2),

$$v(y') <_L z+v(c) \leqslant_L z+\sup_L v(C)=z' ,$$

whence $v(y') <_L z'$; thus, for any $\sigma' \in \{<_L, \leqslant_L\}$, we have again $v(y')\sigma'z'$, which proves the inclusion $\subseteq$ in (3.4).

Conversely, if $y' \in R^n$, $v(y') <_L z'$, then there exists $c \in C$ such that $v(y') <_L z+v(c)$; indeed, otherwise $v(y') \geqslant_L z+v(c)$ $(c \in C)$, whence $v(y')-z \geqslant_L \sup_L v(C)$, so $v(y') \geqslant_L z+$ $+\sup_L v(C)=z'$, in contradiction with our assumption on y'. Hence, $v(y')\sigma z+v(c)$, and thus, by (3.1), $y' \in H+C$. On the other hand, if $y' \in R^n$ satisfies $v(y')=z'=z+\sup_L v(C)$ and σ' is $\leqslant_L$, then, by (3.3), σ is $\leqslant_L$ and $\sup_L v(C) \in v(C)$, whence there exists $c \in C$ such that $v(y')=z+v(c)$, so $v(y')\sigma z+v(c)$; thus, by (3.1), we have again $y' \in H+C$, which proves (3.4).

The implication $3^\circ \Rightarrow 2^\circ$ is obvious.

$2^\circ \Rightarrow 1^\circ$. Assume 2°. Then, by 2° for $y=0$, H is convex. Assume now, a contrario, that $R^n \setminus H$ is not convex, i.e., there exist $y_1, y_2 \in R^n \setminus H$ and $0<\lambda<1$ such that $(1-\lambda)y_1+\lambda y_2 \in H$. Then,

$$H + \{0, y_1-y_2\} \ni (1-\lambda)y_1+\lambda y_2+0=y_1+\lambda(y_2-y_1),$$
$$H + \{0, y_1-y_2\} \ni (1-\lambda)y_1+\lambda y_2+(y_1-y_2)=y_1+(1-\lambda)(y_1-y_2),$$

whence, by 2°,

$$H+\{0,y_1-y_2\} \ni (1-\lambda)[y_1+\lambda(y_2-y_1)]+\lambda[y_1+(1-\lambda)(y_1-y_2)]=y_1.$$

Hence, either $H \ni y_1$, or $H+(y_1-y_2) \ni y_1$, in which case $H-y_2 \ni 0$, so $y_2 \in H$, in contradiction with our assumption that $y_1, y_2 \in R^n \setminus H$. This proves that $R^n \setminus H$ is convex, and hence H is a hemi-space.

$1^\circ \Rightarrow 4^\circ$. Let H be represented as in 15° of theorem 2.1 and let $C \subseteq R^n$. Then, by (2.36) and (0.6), we have

$$H \,\underline{*}\, C = \bigcap_{c \in C}(H-c) = \bigcap_{c \in C}\{y-c \in R^n \mid y \in H\} = \bigcap_{c \in C}\{y-c \in R^n \mid v(y)\sigma z\} =$$

$$=\{y' \in R^n \mid v(y')\sigma z-v(c) \quad (c \in C)\}. \tag{3.6}$$

Define

$$z'=z - \sup_L v(C), \tag{3.7}$$

$$\sigma' \text{ is } \begin{cases} <_L , & \text{if } \sigma \text{ is } <_L \text{ and } \sup_L v(C) \in v(C) \\ \leqslant_L & \text{otherwise.} \end{cases} \tag{3.8}$$

Then, similarly to the above proof of the implication $1^\circ \Rightarrow 3^\circ$, we obtain

$$H \,\underline{*}\, C=\{y' \in R^n \mid v(y')\sigma'z'\}, \tag{3.9}$$

and hence, by theorem 1.1, implication $2^\circ \Rightarrow 1^\circ$, $H \,\underline{*}\, C$ is a hemi-space.

Finally, the implication $4^\circ \Rightarrow 1^\circ$ follows from $H=H \,\underline{*}\, \{0\}$. ∎

Definition 3.1. For any hemi-space H in R^n, represented (uniquely) as in 18^0 of theorem 2.1, the set

$$M=M(H)=\{y\in R^n | u(y)=x\}\neq\emptyset \tag{3.10}$$

will be called the linear manifold associated to H.

Remark 3.1. a) By remark 2.3 d), we see that H is a semi-space or the complement of a semi-space if and only if M(H) consists of one point (namely, $M(H)=\{u^{-1}(x)\}$).

b) From the uniqueness of (r,τ), x and u in 18^0 of theorem 2.1 it follows that for H represented as in 19^0 (or, in particular, 20^0) of theorem 2.1, we have, by (2.38),

$$u(y)=(\Psi_1(y),\dots,\Psi_r(y))^T = \sum_{j=1}^{r} \Psi_j(y)e_j \qquad (y\in R^n), \tag{3.11}$$

$$x=(\Psi_1(y_0),\dots,\Psi_r(y_0))^T = \sum_{j=1}^{r} \Psi_j(y_0)e_j , \tag{3.12}$$

whence, by (3.10),

$$M(H)=\{y\in R^n | \Psi_j(y)=\Psi_j(y_0)\ (j=1,\dots,r)\}. \tag{3.13}$$

Also, from remark 2.2 d) it follows that the linear manifold associated to the hemi-space $H=H_0$ of (2.40) is

$$M(H_0)=\{y=(\eta_j)_1^n | \eta_j=0 \quad (j=1,\dots,r)\}. \tag{3.14}$$

c) For any hemi-space H in R^n, we have

$$M(R^n\setminus H)=M(H). \tag{3.15}$$

Indeed, by (2.37), $R^n\setminus H=\{y\in R^n | (-u)(y)\tau'(-x)\}$, where τ' is the unique element of $\{<_L, \leq_L\}\setminus\{\tau\}$, whence $M(R^n\setminus H)=\{y\in R^n | (-u)(y)=-x\}=M(H)$.

d) For any hemi-space H, with associated linear manifold M=M(H), the sets $H\cup M$, $H\setminus M$ and H-M are hemi-spaces, namely, representing H as in 18^0 of theorem 2.1, we have

$$H\cup M=\{y\in R^n | u(y) \leq_L x\},\ H\setminus M=\{y\in R^n | u(y) <_L x\}, \tag{3.16}$$

$$H-M=\{y\in R^n | u(y)\tau(H)0\}, \tag{3.17}$$

with $\tau(H)$ of definition 2.1. Indeed, (3.16) is obvious. Furthermore, if $y=y'-y''$, where $u(y')\tau(H)x$, $u(y'')=x$, then $u(y)=u(y')-u(y'')=u(y')-x\tau(H)0$; conversely, if $y\in R^n$, $u(y)\tau(H)0$, then, taking any $y_0\in M$, we have $y=(y+y_0)-y_0$, where $u(y+y_0)=u(y)+x\tau(H)x$, i.e., $y+y_0\in H$, and thus $y\in H-M$.

When τ of (2.37) is $\leq_L$, propositions 2.1 and 2.3, respectively, when τ is $<_L$, propositions 2.2 and 2.4, give geometric characterizations of M, which do not depend on u or x of the representation (2.37) of the hemi-space H. Let us give now a further geometric characterization of M, not depending on τ either. To this end, we shall need

Lemma 3.1. For H and M as in (2.37) and (3.10), where $u=(m_1,\dots,m_r)^T \in \mathcal{L}(R^n, R^r)$, $x=(\xi_1,\dots,\xi_r)\in R^r$, and $(r,\tau)\in\{0,1,\dots,n\}\times\{<_L, \leqslant_L\}$, define

$$A(y)=\{d\in R^n \mid y+\lambda d\in H,\ y-\lambda d\notin H\ (\lambda>0)\} \qquad (y\in R^n). \tag{3.18}$$

Then, we have

$$A(y)=\{d\in R^n \mid (m_1,\dots,m_{p(y)})^T d <_L 0\} \qquad (y\in R^n), \tag{3.19}$$

where

$$p(y)=\max\left[\{i\leqslant r \mid m_j^T y=\xi_j \ (j=1,\dots,i)\}\cup\{0\}\right]. \tag{3.20}$$

Hence, in particular,

$$A(y)=\{d\in R^n \mid u(d) <_L 0\} \qquad (y\in M). \tag{3.21}$$

Proof. Let $y\in R^n$ and $d\in A(y)$. Then, by (3.18) and (2.37), we have

$$u(y+\lambda d)\tau x\tau' u(y-\lambda d) \qquad (\lambda>0), \tag{3.22}$$

where τ' is the only element in $\{<_L, \leqslant_L\}\setminus\{\tau\}$.

Now, if $p(y)=0$, then $m_1^T y\neq\xi_1$, whence either $u(y\pm\lambda d) <_L x$ for sufficiently small $\lambda>0$ (if $m_1^T y<\xi_1$) or $u(y\pm\lambda d) >_L x$ for sufficiently small $\lambda>0$ (if $m_1^T y>\xi_1$), which contradicts (3.22); thus, $1\leqslant p(y)\leqslant r$. Then, by the first inequality in (3.22) we have, in particular,

$$(m_1^T y+\lambda m_1^T d,\dots,m_{p(y)}^T y+\lambda m_{p(y)}^T d)^T \leqslant_L (\xi_1,\dots,\xi_{p(y)})^T \qquad (\lambda>0),$$

whence, by (3.20),

$$(m_1,\dots,m_{p(y)})^T d \leqslant_L 0. \tag{3.23}$$

If $1\leqslant p(y)=r$ and

$$(m_1,\dots,m_{p(y)})^T d=0, \tag{3.24}$$

then, by (3.20), we obtain

$$u(y+\lambda d)=u(y)+\lambda u(d)=x+\lambda.0=x \qquad (\lambda\in R),$$

in contradiction with (3.22). If (3.24) holds with $1\leqslant p(y)<r$, note that, by (3.20), we have $m_{p(y)+1}^T y\neq\xi_{p(y)+1}$. Assume that $m_{p(y)+1}^T y<\xi_{p(y)+1}$, and let

$$0<\lambda_o=\left|\frac{m_{p(y)+1}^T y-\xi_{p(y)+1}}{2m_{p(y)+1}^T d}\right| \quad (=+\infty, \text{ if } m_{p(y)+1}^T d=0). \tag{3.25}$$

Then,

$$\begin{aligned} m_{p(y)+1}^T(y\pm\lambda_o d)&=m_{p(y)+1}^T y\pm\lambda_o m_{p(y)+1}^T d< \\ &<m_{p(y)+1}^T y+(\xi_{p(y)+1}-m_{p(y)+1}^T y)=\xi_{p(y)+1}. \end{aligned} \tag{3.26}$$

Hence, by (3.20) and (3.24), we obtain $u(y\pm\lambda_o d) <_L x$, whence, by (2.37), $y\pm\lambda_o d\in H$, which, by (3.18), contradicts the assumption $d\in A(y)$. On the other hand, if $m^T_{p(y)+1}y > \xi_{p(y)+1}$, then, for λ_o of (3.25), we obtain, similarly, that $u(y\pm\lambda_o d) >_L x$, whence, by (2.37), $y\pm\lambda_o d\notin H$, which, again, contradicts $d\in A(y)$. By (3.23) and since $d\in A(y)$ was arbitrary, this proves the inclusion $\subseteq$ in (3.19).

Conversely, assume now that $y,d\in R^n$ and $(m_1,\dots,m_{p(y)})^T d <_L 0$. If $p(y)=0$, then this assumption means that $0.d <_L 0$, which is impossible; thus, $1\leqslant p(y)\leqslant r$. Then, by (3.20), we have

$$(m_1,\dots,m_{p(y)})^T(y+\lambda d)=(\xi_1,\dots,\xi_{p(y)})^T+\lambda(m_1,\dots,m_{p(y)})^T d$$
$$<_L (\xi_1,\dots,\xi_{p(y)})^T \qquad (\lambda>0), \tag{3.27}$$

whence $u(y+\lambda d) <_L x$ $(\lambda>0)$, and hence, by (2.37), $y+\lambda d\in H$ $(\lambda>0)$. Similarly, we also obtain $u(y-\lambda d) >_L x$ $(\lambda>0)$, whence, by (2.37), $y-\lambda d\notin H$ $(\lambda>0)$. Hence, by (3.18), $d\in A(y)$.

Finally, if $y\in M$, then, by (3.10) and (3.20), we have $p(y)=r$, so $(m_1,\dots,m_{p(y)})^T=u$, which, by (3.19), yields (3.21). ■

From lemma 3.1 we see that, among the sets $A(y)$ $(y\in R^n)$ there are at most $r+1$ distinct ones, and that all sets $A(y)$ $(y\in M)$ coincide. Moreover, from lemma 3.1 we obtain the following characterization of M:

<u>Proposition 3.1</u>. <u>For</u> H <u>and</u> M <u>as in</u> (2.37) <u>and</u> (3.10), <u>where</u> $u\in\mathcal{L}(R^n,R^r)$, $x\in R^r$ <u>and</u> $(r,\tau)\in\{0,1,\dots n\}\times\{<_L, \leqslant_L\}$, <u>we have</u>

$$M=\{y\in R^n | A(y) \text{ is the largest set of the form } (3.18)\}. \quad ■ \tag{3.28}$$

Let us give now, using the representations (2.37) and (3.10), some results on hemi-spaces of types $<_L$ and $\leqslant_L$ respectively, related to theorem 1.4.

<u>Theorem 3.2. A set</u> $H\subseteq R^n$ <u>is a hemi-space of type</u> $<_L$ <u>if and only if there exists a linear manifold</u> M <u>in</u> R^n <u>such that</u> H <u>is a maximal convex set not intersecting</u> M. <u>In this case</u>, $M=M(H)$, <u>the linear manifold associated to</u> H.

<u>Proof</u>. Let H be a hemi-space of type $<_L$ and $M=M(H)$. Then

$$H=\{y\in R^n | u(y) <_L x\},\ M=M(H)=\{y\in R^n | u(y)=x\}, \tag{3.29}$$

where $u\in\mathcal{L}(R^n, R^r)$ anx $x\in R^r$ are as in 18^o of theorem 2.1. Then, clearly, $H\cap M=\emptyset$. Assume now, a contrario, that there exists a convex set $C\subset R^n$ such that $H\subset C$, $H\neq C$, and $C\cap M=\emptyset$, and let $y\in C\setminus H$. Then, by $y\notin H$, we have $u(y) \geqslant_L x$, whence, by $y\in C$ and $C\cap M=\emptyset$, we obtain $u(y) >_L x$. Take any $y_o\in M$ and let $y_1=2y_o-y$. Then

$$u(y_1)=2u(y_o)-u(y) <_L 2x-x=x,$$

whence $y_1\in H\subset C$. Hence, since C is convex and $y\in C$, we obtain $y_o=\frac{1}{2}(y_1+y)\in C$, so $y_o\in C\cap M$, in contradiction with $C\cap M\neq\emptyset$. This proves that H is maximal.

Conversely, let H be a maximal convex set not intersecting a linear manifold M. Then, by theorem 1.4, H is a hemi-space, and hence

$$H = \{y \in R^n \mid u(y) \tau x\}, \tag{3.30}$$

where $u \in \mathcal{L}(R^n, R^r)$, $x \in R^r$ and $\tau \in \{<_L, \leqslant_L\}$ are as in 18^0 of theorem 2.1. If $\inf_L u(M) <_L x$, then there exists $y \in M$ such that $u(y) <_L x$, which, by (3.30), contradicts the assumption $H \cap M = \emptyset$. On the other hand, if we had $\inf_L u(M) >_L x$, then

$$\tilde{H} = \{y \in R^n \mid u(y) <_L \inf_L u(M)\} \tag{3.31}$$

would be a convex set (in fact, a hemi-space) not intersecting M and such that $H \subset \tilde{H}$, $H \neq \tilde{H}$, in contradiction with the maximality of H. Hence, we must have

$$\inf_L u(M) = x. \tag{3.32}$$

We claim that

$$u(M) = \{x\}. \tag{3.33}$$

Indeed, by (3.32), it is enough to show that u(M) is a singleton, i.e., that u is constant on M. If $u = (m_1, \dots, m_r)^T$ is not constant on M, let

$$i_0 = \min\{i \leqslant r \mid m_i^T \text{ is not constant on } M\},$$

and let $y', y'' \in M$ be such that $m_{i_0}^T y' > m_{i_0}^T y''$. Then

$$\lambda m_{i_0}^T y' + (1-\lambda) m_{i_0}^T y'' = \lambda (m_{i_0}^T y' - m_{i_0}^T y'') + m_{i_0}^T y'' \rightarrow -\infty \quad \text{as } \lambda \rightarrow -\infty,$$

whence, since $\lambda y' + (1-\lambda) y'' \in M$ for all $\lambda \in R$ (because M is a linear manifold), and since $u(\lambda y' + (1-\lambda) y'') = (\lambda m_i^T y' + (1-\lambda) m_i^T y'')_1^r$ $(\lambda \in R)$, from (3.32) and lemma 1.1 we obtain, denoting $x = (\xi_i)_1^r$, that

$$\xi_{i_0} = \inf \{m_{i_0}^T y \mid y \in M,\ m_i^T y = \xi_i \quad (i = 1, \dots, i_0 - 1)\} = -\infty,$$

in contradiction with $x \in R^r$. This proves the claim (3.33), whence

$$M \subseteq \{y \in R^n \mid u(y) = x\}. \tag{3.34}$$

From $\tau \in \{<_L, \leqslant_L\}$, (3.30) and (3.34) it follows that τ is $<_L$, whence

$$H = \{y \in R^n \mid u(y) <_L x\}. \tag{3.35}$$

Now let $y_0 \in R^n$ be such that $u(y_0) = x$. Then, by (3.35), $H \cup \{y_0\}$ is a convex set, $H \subset H \cup \{y_0\}$ and $H \neq H \cup \{y_0\}$. Hence, the maximality of H implies that $(H \cup \{y_0\}) \cap M \neq \emptyset$, which, by $H \cap M = \emptyset$, is equivalent to $y_0 \in M$. Thus, we have the opposite inclusion to (3.34), whence the equality

$$M = \{y \in R^n \mid u(y) = x\}, \tag{3.36}$$

and hence, by definition 3.1, M=M(H), the linear manifold associated to M. ∎

<u>Corollary 3.1.</u> <u>A set $H \subseteq R^n$</u> <u>is a hemi-space of type</u> $\leqslant_L$ <u>if and only if there exists</u>

a linear manifold M in R^n such that $R^n \setminus H$ is a maximal convex set not intersecting M. In this case, $M=M(H)$.

Proof. This follows from theorem 3.2, using remarks 2.3 b) and 3.1 c). ∎

Remark 3.2. P.C.Hammer [4] has called a set H in a linear space E a demispace at a linear manifold $M \subset E$, $M \neq E$, provided H is a maximal convex set not intersecting M, and he has called $E \setminus H$ a codemispace at M. Furthermore, be has mentioned, without proof, that every demispace H at M is a hemi-space and then every point of M is a vertex of the cone H ([4], theorem 3, part 1) and, in $E=R^n$, exactly one of each pair of non-empty complementary convex sets H and $R^n \setminus H$ is a demispace at some linear manifold M of dimension j, where $0 \leq j \leq n-1$ ([4], theorem 3, part 2). Also, M.Lassak ([9], theorem 1, part 6) has shown that a set $H \subseteq R^n$ is a hemi-space if and only if one of the sets H, $R^n \setminus H$ is a maximal convex set not intersecting a linear manifold M and, in this case, M is the set of all vertices of the cone H; for some related results, see also [10]. Theorem 3.2 above shows, in addition to the above results, that the "demispaces" in R^n are exactly those hemi-spaces which admit the representation (3.35), i.e., the hemi-spaces of type $<_L$.

Using theorem 3.2, we obtain, for hemi-spaces H of type $<_L$, the following characterization of the associated linear manifold $M(H)$:

Theorem 3.3. If H is a hemi-space of type $<_L$, then

$$M(H)=\{y \in R^n \setminus H \mid H \cup \{y\} \text{ is convex}\}. \tag{3.37}$$

Proof. Assume that we have (3.29), with u and x as in 18° of theorem 2.1. Then, clearly, $M \subseteq R^n \setminus H$. Furthermore, if $y \in M$, $y' \in H$ and $0<\lambda<1$, then, by (3.29),

$$u((1-\lambda)y+\lambda y')=(1-\lambda)u(y)+\lambda u(y') <_L (1-\lambda)x+\lambda x=x,$$

whence $(1-\lambda)y+\lambda y' \in H$, which proves that $H \cup \{y\}$ is convex (since H is convex). Thus, we have the inclusion $\subseteq$ in (3.37).

Conversely, let $y \in R^n \setminus H$ be such that $H \cup \{y\}$ is convex. If $y \notin M$, then $(H \cup \{y\}) \cap M = \emptyset$, so H is not a maximal convex set satisfying $H \cap M = \emptyset$, which contradicts theorem 3.2. Thus, we must have $y \in M$, which proves the inclusion $\supseteq$ in (3.37) and hence the equality (3.37). ∎

Corollary 3.2. If H is a hemi-space of type $\leq_L$, then

$$M(H)=\{y \in H \mid (R^n \setminus H) \cup \{y\} \text{ is convex}\}. \tag{3.38}$$

Proof. This follows from theorem 3.3, using remarks 2.3 b) and 3.1 c). ∎

We can use the set M to express $H \overset{*}{-} H$, as follows:

Proposition 3.2. Let H be a hemi-space, with associated linear manifold M. Then, we have

$$H \overset{*}{-} H = (H \cup M) - M. \tag{3.39}$$

Proof. Let H be represented (uniquely) as in 18° of theorem 2.1. Then, by (3.16) and (3.17) (and since $\tau(H \cup M)$ is $\leq_L$), we have

$$(H \cup M) - M = \{y \in R^n \mid u(y) \leq_L 0\}, \tag{3.40}$$

whence, by (2.37), we obtain

$$\bar{y} + H = \{\bar{y} + y \in R^n \mid u(y) \tau x\} = \{y' \in R^n \mid u(y') \tau x + u(\bar{y})\} \subseteq$$
$$\subseteq \{y' \in R^n \mid u(y') \tau x\} = H \qquad (\bar{y} \in (H \cup M) - M), \tag{3.41}$$

which proves that

$$(H \cup M) - M \subseteq H \overset{*}{-} H.$$

To prove the opposite inclusion, assume, a contrario, that there exists an element $\bar{y} \in (H \overset{*}{-} H) \setminus ((H \cup M) - M)$, whence, by (3.40), $u(\bar{y}) >_L 0$. Let

$$y = u^*(x - \tfrac{1}{2} u(\bar{y})). \tag{3.42}$$

Then, since $uu^* = I$ (by 18° of theorem 2.1), we obtain

$$u(y) = uu^*(x - \tfrac{1}{2} u(\bar{y})) = x - \tfrac{1}{2} u(\bar{y}) <_L x, \tag{3.43}$$

whence, by (2.37), $y \in H$. But, by (3.43) and (3.40), we have

$$u(\bar{y} + y) = u(\bar{y}) + x - \tfrac{1}{2} u(\bar{y}) = x + \tfrac{1}{2} u(\bar{y}) >_L x,$$

which, by (2.37), implies that $\bar{y} + y \notin H$, contradicting $\bar{y} \in H \overset{*}{-} H$. ∎

We shall denote by $D(M)$ the direction (i.e., the translate to the origin) of the linear manifold M associated to a hemi-space $H \subseteq R^n$; thus,

$$D(M) = M - M = M - y_o \qquad (y_o \in M). \tag{3.44}$$

Corollary 3.3. For H and M as above, we have

$$H \overset{*}{-} H = (H - M) \cup D(M). \tag{3.45}$$

Proof. This follows from (3.39) and the obvious equalities

$$(H \cup M) - M = (H - M) \cup (M - M) = (H - M) \cup D(M). \quad ∎ \tag{3.46}$$

Lemma 3.2. Let $H \subseteq R^n$ be a hemi-space, with associated linear manifold $M = M(H)$. Then

$$H - M = H - y_o \qquad (y_o \in M). \tag{3.47}$$

Proof. Let $y_o \in M$. Then, by (2.37), (3.10) and (3.17), we have

$$H - y_o = \{y - y_o \in R^n \mid u(y) \tau x\} = \{y - y_o \in R^n \mid u(y) \tau u(y_o)\} = \{y' \in R^n \mid u(y') \tau 0\} = H - M. \quad ∎$$

Corollary 3.4. For H and M as above, we have

$$(H \overset{*}{-} H) \setminus D(M) = (H - M) \setminus D(M) = (H \setminus M) - M. \tag{3.48}$$

Proof. From (3.45) and (3.47) we obtain, for any $y_o \in M$,

$$(H \circledast H)\setminus D(M)=[(H-M)\cup D(M)]\setminus D(M)=(H-M)\setminus D(M)=$$

$$=(H-M)\setminus(M-M)=(H-y_o)\setminus(M-y_o)=(H\setminus M)-y_o=(H\setminus M)-M. \quad \blacksquare \qquad (3.49)$$

Remark 3.3. If H is represented as in 18° of theorem 2.1, then, by the above, we have

$$(H \circledast H)\setminus D(M)=\{y\in R^n \mid u(y) <_L 0\}. \qquad (3.50)$$

Corollary 3.5. *For H and M as in lemma 3.2, H-M is the unique translate of H whose associated linear manifold is a linear subspace of R^n (namely, D(M)).*

Proof. Let $y_o \in M$. Then the linear manifold associated to $H-M=H-y_o$ is $M-y_o=D(M)$.

Conversely, if $H-y_1$ is a translate of H, whose associated linear manifold $M-y_1$ is a linear subspace of R^n, then $y_1 \in M$, whence, by (3.47), $H-y_1=H-M$. $\blacksquare$

Remark 3.4. From (3.39), (3.50) and corollary 3.5, it follows that, for H and M as above, $H \circledast H$ (respectively, $(H \circledast H)\setminus D(M)$) is the unique translate of $H\cup M$ (respectively, of $H\setminus M$), whose associated linear manifold is a linear subspace (namely, D(M), in both cases).

Corollary 3.6. *Let H and M be as in lemma 3.2. For a set $H'\subseteq R^n$, the following statements are equivalent:*

1°. $H'=H \circledast H$.

2°. H' *is a hemi-space with* $M(H')=D(M)$, *and* H' *is a translate of* $H\cup M$.

3°. H' *is a hemi-space of type* $\leqslant_L$, *with* $M(H')=D(M)$, *and* $H'\setminus M(H')$ *is a translate of* $H\setminus M$.

Proof. The equivalence $1°\Leftrightarrow 2°$ and the implication $1°\Rightarrow 3°$ follow from remark 3.4 and (3.39), (3.40) (by which $H \circledast H$ is of type $\leqslant_L$).

$3°\Rightarrow 1°$. If 3° holds, then, since H' is of type $\leqslant_L$, we have $M(H')\subseteq H'$, so $H'\setminus M(H')$ is a hemi-space of type $<_L$, with $M(H'\setminus M(H'))=M(H')=D(M)$ (by 3°). Hence, by the last property in 3° and by remark 3.4, we have

$$H'\setminus M(H')=(H \circledast H)\setminus D(M),$$

whence, by $D(M)=M(H')\subseteq H'$ and $D(M)\subseteq H \circledast H$ (see (3.45)), we obtain

$$H'=(H'\setminus M(H'))\cup M(H')=((H \circledast H)\setminus D(M))\cup D(M)=H \circledast H. \quad \blacksquare$$

Let us also give

Proposition 3.3. *If $H\subseteq R^n$ is a hemi-space, with associated linear manifold M, then*

$$D(M)=\{d\in R^n \mid y+\lambda d\in H \quad (y\in H,\ \lambda\in R)\}=$$

$$=\{d\in R^n \mid y+\lambda d\in R^n\setminus H \quad (y\in R^n\setminus H,\ \lambda\in R)\}. \qquad (3.51)$$

Proof. If H is (uniquely) represented as in 18^0 of theorem 2.1, then, by the definition (3.10) of M, we have $D(M)=\text{Ker } u$.

Now, if $d\in\text{Ker } u$, we have

$$u(y+\lambda d)=u(y)\,\tau\, x \qquad (y\in H,\ \lambda\in R),$$

whence, by (2.37), we obtain

$$y+\lambda d\in H \qquad (y\in H,\ \lambda\in R), \qquad (3.52)$$

which proves the inclusion $\subseteq$ in the first part of (3.51). Conversely, if $d\in R^n$ satisfies (3.52), then, by (3.10), we have

$$u(y+\lambda d)\tau x=u(y) \qquad (y\in M,\ \lambda\in R),$$

whence, by $\tau\in\{<_L, \leqslant_L\}$, we obtain

$$\lambda u(d)=u(\lambda d)\leqslant_L 0 \qquad (\lambda\in R),$$

which implies $u(d)=0$, i.e., $d\in\text{Ker } u$. Finally, the second representation of D(M) in (3.51) follows from the first one, replacing H by $R^n\setminus H$ and using remark 3.1 c). ∎

Our main result in this Section is the following "decomposition theorem":

Theorem 3.4. A set $H\subseteq R^n$ is a hemi-space if and only if there exist a linear manifold M and a set $S\subseteq (D(M))^{\perp}$ (where D(M) is the direction of M) which is either a semi-space at 0 in $D(M)^{\perp}$ or the complement of a semi-space at 0 in $D(M)^{\perp}$, such that

$$H = M + S. \qquad (3.53)$$

Moreover, in this case, M and S are unique, namely, M is the linear manifold associated to H and S is the projection of H-M onto $D(M)^{\perp}$ along D(M), whence

$$S = (H - M)\cap D(M)^{\perp}. \qquad (3.54)$$

Proof. Assume that we have (3.53), with M a linear manifold and S either a semi-space at 0, or the complement of a semi-space at 0, in $D(M)^{\perp}$, and let

$$r = \text{codim } D(M). \qquad (3.55)$$

Then, by [15], lemma 1.1, there exists a linear isomorphism w of $D(M)^{\perp}$ onto R^r, such that

$$S =\{y\in D(M)^{\perp} \mid w(y)\,\tau\, 0\}, \qquad (3.56)$$

where τ is $<_L$, if S is a semi-space, and τ is $\leqslant_L$, if S is the complement of a semi-space. Define $u\in\mathcal{L}(R^n, R^r)$ by

$$u(y)=\begin{cases} w(y) & \text{if } y\in D(M)^{\perp} \\ 0 & \text{if } y\in D(M), \end{cases} \qquad (3.57)$$

and let $y_o \in M$ be arbitrary. Then, by (3.44) and (3.57), we have $u(y-y_o)=0$ $(y \in M)$. Conversely, if $y \in R^n$ satisfies $u(y-y_o)=0$, then, by (3.57), (3.44), and since w is an isomorphism, we have $y-y_o \in M-y_o$, so $y \in M$. Thus,

$$M=\{y \in R^n \mid u(y)=u(y_o)\}. \tag{3.58}$$

Furthermore, let $y \in R^n$. Then, there exist unique $y_1 \in D(M)$ and $y_2 \in D(M)^{\perp}$, such that $y-y_o=y_1+y_2$. By $y_1 \in D(M)$ and (3.57), we have $u(y_1)=0$. Also, by (3.53) and $S \subseteq D(M)^{\perp}$, we have $y-y_o \in H-y_o=(M-y_o)+S=D(M)+S$ if and only if $y_2 \in S$, that is, by (3.56), $w(y_2)\tau 0$, which, by (3.57) and $y_2 \in S \subseteq D(M)^{\perp}$, is equivalent to $u(y_2)\tau 0$. Hence, since $u(y)=u(y_o)+u(y_1)+u(y_2)$, we obtain

$$H=\{y \in R^n \mid u(y)\,\tau\, u(y_o)\}, \tag{3.59}$$

and thus H is a hemi-space. Moreover, from propositions 2.1-2.4 it follows that M is the linear manifold associated to H. Finally, by (3.53) and $S \subseteq D(M)^{\perp}$, we have

$$H-M=(M-M)+S=D(M)+S \subseteq D(M) \oplus D(M)^{\perp}=R^n, \tag{3.60}$$

and hence S is the projection of $H-y_o$ onto $D(M)^{\perp}$ along $D(M)$. Hence, since $0 \in D(M)$, we obtain

$$S \subseteq (D(M)+S)\cap(D(M))^{\perp}=(H-M)\cap D(M)^{\perp}. \tag{3.61}$$

On the other hand, if $y \in (H-M)\cap D(M)^{\perp}=(D(M)+S)\cap D(M)^{\perp}$, say, $y=y_1+y_2 \in D(M)^{\perp}$, where $y_1 \in D(M)$, $y_2 \in S \subseteq D(M)^{\perp}$, then $y_1+(y_2-y)=0$, where $y_2-y \in D(M)^{\perp}$, and hence, by $R^n=D(M) \oplus D(M)^{\perp}$, we obtain $y_1=0$ and $y=y_2 \in S$. Thus, $(H-M)\cap D(M)^{\perp} \subseteq S$, which, together with (3.61), proves (3.54).

Conversely, if H is a hemi-space, represented (uniquely) as in 18^o of theorem 2.1, define M by (3.10) and S by

$$S=\{y \in D(M)^{\perp} \mid u(y)\tau 0\}. \tag{3.62}$$

For any $y_o \in M$ we have $u(y_o)=x$, whence (3.58), (3.59). Given $y \in R^n$, there exist unique $y_1 \in D(M)=M-y_o$ and $y_2 \in D(M)^{\perp}$, such that $y-y_o=y_1+y_2$. Then, since $y_1 \in M-y_o$, we have, by (3.10), $u(y_1)=0$. Hence, by (3.58), (3.59) and (3.62), we have $y \in H$ if and only if $y_2 \in S$. Thus, if $y \in H$, then, since $y_o+y_1 \in M$, we have $y=(y_o+y_1)+y_2 \in M+S$. Conversely, if $y=y_o'+y_2$, where $y_o' \in M$, $y_2 \in S$, then $y_o'=y_o+(y_o'-y_o)$, where $y_o'-y_o \in M-y_o=D(M)$, whence, by the above (with $y_1=y_o'-y_o$), we obtain $y \in H$. This proves (3.53).

Finally, since

$$D(M)^{\perp}=(\mathrm{Ker}\ u)^{\perp}=u^*(R^r) \tag{3.63}$$

(identifying $(R^r)^*$ with R^r) and since $uu^*=I$, we have $u(D(M)^{\perp})=uu^*(R^r)=R^r$, whence, by (3.10),

$$\dim u(D(M)^{\perp})=r=\operatorname{codim} D(M)=\dim D(M)^{\perp},$$

and hence $u|_{D(M)^{\perp}}$ is an isomorphism of $D(M)^{\perp}$ onto R^r. Consequently, by [15], lemma 1.1, S is either a semi-space at 0 in $D(M)^{\perp}$, or the complement of a semi-space at 0 in $D(M)^{\perp}$. ■

Remark 3.5. a) As shown by the above proof, S is a semi-space (respectively, the complement of a semi-space) at 0, in $D(M)^{\perp}$, if and only if τ of 18° of theorem 2.1 is $<_L$ (respectively, $\leq_L$). Therefore, by theorem 2.2, the equivalence class $(r(H), \tau(H))$ of a hemi-space H is uniquely determined by $\dim D(M)$ $(=n-r(H))$ and the type of S, or, equivalently, by $\dim S(=r(H))$ and the type of S.

b) In the situation of theorem 3.4, we have

$$H = M \oplus S, \tag{3.64}$$

i.e., each $y\in H$ admits an unique decomposition $y=m+s$, with $m\in M$, $s\in S$; indeed, the relations $m_1,m_2\in M$, $s_1,s_2\in S$, $m_1+s_1=m_2+s_2$ imply $m_1-m_2=s_2-s_1\in D(M)\cap S\subseteq D(M)\cap D(M)^{\perp}=\{0\}$.

c) Theorem 3.4 remains valid if we replace $D(M)^{\perp}$ by any linear subspace E of R^n such that $D(M) \oplus E=R^n$. Indeed, we have used $E=D(M)^{\perp}$ only in (3.63), but that part can be replaced by the following argument: by (3.57) (with $D(M)^{\perp}$ replaced by E), we have $u(E)=$ $=w(E)=R^r$, whence, by (3.10),

$$\dim u(E)=r=\operatorname{codim} D(M)=\dim E,$$

and hence $u|_E$ is an isomorphism of E onto R^r. ■

Theorem 3.5. Let H_1 and H_2 be two hemi-spaces, with (unique) decompositions $H_1=M_1+S_1$, respectively, $H_2=M_2+S_2$ (of theorem 3.4). The folllowing statements are equivalent:

1°. $S_1=S_2$.

2°. H_2 is a translate of H_1.

Proof. 1°⇒2°. If 1° holds, then, since S_i is either a semi-space at 0 in $D(M_i)^{\perp}$, or the complement of a semi-space at 0 in $D(M_i)^{\perp}$ $(i=1,2)$, we have

$$D(M_1)^{\perp}=S_1\cup\{-S_1\}\cup\{0\}=S_2\cup\{-S_2\}\cup\{0\}= D(M_2)^{\perp}, \tag{3.65}$$

whence

$$D(M_1)=D(M_1)^{\perp\perp}=D(M_2)^{\perp\perp}=D(M_2). \tag{3.66}$$

Therefore, there exists $\bar{y}\in R^n$ such that $\bar{y} + M_1=M_2$, whence, again by 1°, we obtain

$$H_2 =M_2 + S_2 = \bar{y} + M_1 + S_1 = \bar{y} + H_1.$$

2°⇒1°. If $H_2=\bar{y}+H_1=\bar{y}+(M_1+S_1)=(\bar{y}+M_1)+S_1$ for some $\bar{y}\in R^n$, then, since $\bar{y}+M_1$ is a linear manifold and $S_1\subseteq D(M_1)^{\perp}=D(\bar{y}+M_1)^{\perp}$, from $H_2=M_2+S_2$ and the uniqueness part of theorem 3.4 we obtain $S_1=S_2$ (and $\bar{y} + M_1=M_2$). ■

Theorem 3.6. Let H_1 and H_2 be two hemi-spaces, with associated linear manifolds

M_1 and M_2 respectively. The following statements are equivalent:

1°. $D(M_1)=D(M_2)$.

2°-3°. There exists $v\in\mathcal{U}(R^n)$, respectively $v\in\mathcal{O}(R^n)$, such that

$$v(D(M_1))=D(M_1), \tag{3.67}$$

$$v(H_1-M_1)=H-M_2 \text{ for some } H\in\{H_2,\ R^n\setminus H_2\}. \tag{3.68}$$

4°-5°. There exists $v\in\mathcal{U}(R^n)$, respectively $v\in\mathcal{O}(R^n)$, satisfying (3.68) and

$$v(y)=y \qquad (y\in D(M_1)). \tag{3.69}$$

Proof. $1^\circ \Rightarrow 5^\circ$. Assume 1°, and let (with $\tau(H_i)$ of definition 2.1)

$$H=\begin{cases} H_2, & \text{if } \tau(H_1)=\tau(H_2) \\ R^n\setminus H_2, & \text{if } \tau(H_1)\neq\tau(H_2), \end{cases} \tag{3.70}$$

that is, the unique element of the doubleton $\{H_2,\ R^n\setminus H_2\}$, satisfying

$$\tau(H)=\tau(H_1). \tag{3.71}$$

Let us define $v\in\mathcal{L}(R^n)$ by

$$v(y)=\begin{cases} y, & \text{if } y\in D(M_1) \\ u_2^*u_1(y), & \text{if } y\in D(M_1)^\perp, \end{cases} \tag{3.72}$$

where $u_1,u_2\in\mathcal{L}(R^n,\ R^r)$ (with $r=\operatorname{codim} D(M_1)=\operatorname{codim} D(M_2)$, by 1°) are the operators such that $u_iu_i^*=I$ $(i=1,2)$, corresponding to H_1 and H respectively, by 18° of theorem 2.1. Then, there holds (3.69), and, by (3.63) (for u_1, u_2) and 1°, we have

$$u_2^*u_1(D(M_1)^\perp)=u_2^*u_1u_1^*(R^r)=u_2^*(R^r)=D(M_2)^\perp=D(M_1)^\perp. \tag{3.73}$$

From (3.72) and (3.73), it follows that

$$v^*(\varphi)(y)=\varphi(v(y))=\begin{cases} \varphi(y) & \text{if } y,\varphi\in D(M_1) \\ \varphi(u_2^*u_1(y))=0=\varphi(y) & \text{if } y\in D(M_1)^\perp,\ \varphi\in D(M_1) \\ \varphi(y)=0=\varphi(u_2^*u_1(y)) & \text{if } y\in D(M_1),\ \varphi\in D(M_1)^\perp \\ \varphi(u_2^*u_1(y))=u_1^*u_2(\varphi)(y) & \text{if } y,\varphi\in D(M_1)^\perp, \end{cases}$$

whence

$$v^*(\varphi)=\begin{cases} \varphi & \text{if } \varphi\in D(M_1) \\ u_1^*u_2(\varphi) & \text{if } \varphi\in D(M_1)^\perp. \end{cases} \tag{3.74}$$

By (3.74) and (3.69), we have

$$v^*v(y)=v(y)=y \qquad (y\in D(M_1)). \tag{3.75}$$

On the other hand, if $y\in D(M_1)^\perp=u_1^*(R^r)$, then there exists $x\in R^r$ such that $y=u_1^*(x)$, whence, by (3.72)-(3.74) and $u_iu_i^*=I$, we obtain

$$v^*v(y)=u_1^*u_2u_2^*u_1(u_1^*(x))=u_1^*(u_2u_2^*u_1u_1^*(x))=u_1^*(x)=y,$$

which, together with (3.75), proves that $v^*v=I$, i.e. $v\in\mathcal{O}(R^n)$.

Furthermore, by (3.69) and 1^o, we have

$$u_2v(D(M_1))=u_2(D(M_1))=u_2(D(M_2))=u_2(\text{Ker } u_2)=\{0\}= =u_1(\text{Ker } u_1)=u_1(D(M_1)), \quad (3.76)$$

whence

$$u_2v(y)=0=u_1(y) \qquad (y\in D(M_1)); \quad (3.77)$$

on the other hand, by (3.72) and $u_2u_2^*=I$, we have

$$u_2v(y)=u_2u_2^*u_1(y)=u_1(y) \qquad (y\in D(M_1)^\perp), \quad (3.78)$$

which, together with (3.77), yields

$$u_2v = u_1. \quad (3.79)$$

Hence, by (3.17)(for H_1, M_1, u_1 and H,M_2, u_2) and (3.71), we obtain

$$v(H_1-M_1)=\{v(y)|u_1(y)\tau(H_1)0\}=\{v(y)|u_1(y)\tau(H)0\}= =\{y'\in R^n|u_1v^{-1}(y')\tau(H)0\}=\{y'\in R^n|u_2vv^{-1}(y')\tau(H)0\}= =\{y'\in R^n|u_2(y')\tau(H)0\}=H-M_2.$$

The implications $5^o\Rightarrow4^o\Rightarrow2^o$ and $5^o\Rightarrow3^o\Rightarrow2^o$ are obvious.

$2^o\Rightarrow1^o$. The linear manifolds associated to $v(H_1-M_1)$ and $H-M_2$, where $H\in\{H_2, R^n\setminus H_2\}$, are $v(D(M_1))$ and $D(M_2)$, respectively. Hence, if 2^o holds, then $D(M_1)=v(D(M_1))=D(M_2)$. ∎

Remark 3.6. a) Formula (3.68) means that either $v(H_1-M_1)=H_2-M_2$ or $v(H_1-M_1)= =(R^n\setminus H_2)-M_2$. Note that, since (by remark 3.1 d),c)) $\tau(H_2-M_2)=\tau(H_2)\neq\tau(R^n\setminus H_2)=\tau((R^n\setminus H_2)- -M_2)$, from theorem 2.2 it follows that there cannot exist v_1, $v_2\in\mathcal{U}(R^n)$ satisfying $v_1(H_1-M_1)=H_2-M_2$ and $v_2(H_1-M_1)=(R^n\setminus H_2)-M_2$.

b) If v is as in 3^o of theorem 3.6, then

$$v(S_1)=S_2, \quad (3.80)$$

where $H_1=M_1+S_1$ and $H=M_2+S_2$ are the decompositions given by theorem 3.4. Indeed, by (3.60), (3.67) and 1^o, we have

$$v(H_1-M_1)=v(D(M_1)+S_1)=v(D(M_1))+v(S_1)=D(M_1)+v(S_1), \quad (3.81)$$

$$H-M_2=D(M_2)+S_2=D(M_1)+S_2, \quad (3.82)$$

where $v(S_1)\subseteq v(D(M_1)^\perp)=D(M_1)^\perp$ (by (3.72), (3.73)) and $S_2\subseteq D(M_2)^\perp=D(M_1)^\perp$, whence, by (3.68) and the uniqueness part of theorem 3.4, we obtain (3.80). ∎

c) In 3^o of theorem 3.6, $v|_{D(M_1)^\perp}$, and hence, in 5^o, v, are uniquely determined. Indeeed, if $v_i\in\mathcal{O}(R^n)$, $v_i(H_1-M_1)=H-M_2$ $(i=1,2)$, with $H\in\{H_2, R^n\setminus H_2\}$, then $v_2^{-1}v_1(H_1-M_1)=$

$=v_2^{-1}(H-M_2)=H_1-M_1$, whence, by corollary 4.3 and theorem 4.3 below, $v_2^{-1}v_1(y)=y$ $(y\in D(M_1)^{\perp})$, i.e., $v_1(y)=v_2(y)$ $(y\in D(M_1)^{\perp})$. Moreover, if 1^o holds, then $v|_{D(M_1)^{\perp}}$ above is the unique orthogonal transformation of $D(M_1)^{\perp}$ onto itself, satisfying (3.80); indeed, (3.80)-(3.82) imply $v(H_1-M_1)=H-M_2$, whence, by the above, $v|_{D(M_1)^{\perp}}$ is uniquely determined. Similarly, using (the proof of) theorem 4.4 below and theorem 3.4, one can prove that if $v_1, v_2\in\mathcal{U}(R^n)$ satisfy (3.67), then the following statements are equivalent:

1^o. $v_1(H_1-M_1)=v_2(H_1-M_1)$.

2^o. There exists $\ell\in\mathcal{U}(D(M_1)^{\perp})$ such that ℓ preserves the lexicographical order in the basis $\{e_j'\}_1^r$ of $D(M_1)^{\perp}$ and $\pi_{D(M_1)^{\perp}}v_1|_{D(M_1)^{\perp}}=\pi_{D(M_1)^{\perp}}v_2|_{D(M_1)^{\perp}}\ell$.

Theorem 3.7. Let H_1 and H_2 be two hemi-spaces, with associated linear manifolds M_1 and M_2, respectively. The following statements are equivalent:

1^o. $M_1=M_2$.

2^o-3^o. There exist $v\in\mathcal{U}(R^n)$, respectively $v\in\mathcal{O}(R^n)$, and $y_1\in M_1$, such that we have (3.67) and

$$v(H_1-y_1)=H-y_1 \quad \text{for some } H\in\{H_2, R^n\setminus H_2\}. \tag{3.83}$$

4^o-5^o. There exist $v\in\mathcal{U}(R^n)$, respectively, $v\in\mathcal{O}(R^n)$, and $y_1\in M_1$, satisfying (3.69) and (3.83).

6^o-7^o. There exists $v\in\mathcal{U}(R^n)$, respectively, $v\in\mathcal{O}(R^n)$, such that we have (3.67) and (3.83) for all $y_1\in M_1$.

8^o-9^o. There exists $v\in\mathcal{U}(R^n)$, respectively, $v\in\mathcal{O}(R^n)$, such that we have (3.69) and (3.83) for all $y_1\in M_1$.

Proof. $1^o\Rightarrow 9^o$. Assume 1^o, and let $y_1\in M_1=M_2$. Then, $D(M_1)=D(M_2)$, whence, by theorem 3.6, implication $1^o\Rightarrow 5^o$, there exists $v\in\mathcal{O}(R^n)$ satisfying (3.69) and (3.68). Hence, by (3.47),

$$v(H_1-y_1)=v(H_1-M_1)=H-M_2=H-y_1, \text{ for some } H\in\{H_2, R^n\setminus H_2\}.$$

Since $M_1\neq\emptyset$, the implications $9^o\Rightarrow 8^o\Rightarrow 6^o\Rightarrow 2^o$, $9^o\Rightarrow 7^o\Rightarrow 3^o\Rightarrow 2^o$ and $9^o\Rightarrow 5^o\Rightarrow 4^o\Rightarrow 2^o$ are obvious.

$2^o\Rightarrow 1^o$. The linear manifolds associated to $v(H_1-y_1)$ and $H-y_1$, where $y_1\in M_1$, are $v(M_1-y_1)=v(D(M_1))$ and M_2-y_1, respectively. Hence, if 2^o holds, then $D(M_1)=v(D(M_1))=$ $=M_2-y_1$, whence $M_2=y_1+D(M_1)=M_1$. ■

In theorem 3.7, one cannot replace H_1-y_1 and $H-y_1$ by H_1-M_1 and $H-M_1$ respectively (where $H\in\{H_2, R^n\setminus H_2\}$), as shown by

Example 3.1. Let

$$H_1=\{y=(\eta_1, \eta_2)^T\in R^2 \mid \eta_1\leq 0\}, \quad H_2=\{y\in R^2 \mid y \leq_L 0\}. \tag{3.84}$$

Then, $M_1=\{y=(\eta_1, \eta_2)^T\in R^2 \mid \eta_1=0\}$, whence $H_1-M_1=H_1=H_2-M_1$, and hence $v=I\in\mathcal{O}(R^n)$ satisfies (3.69) and $v(H_1-M_1)=H_2-M_1$, but $M_2=\{0\}\neq M_1$.

§4. Affine transformations preserving a hemi-space

Let us first consider translations $y\to\bar{y}+y$ $(y\in R^n)$, where $\bar{y}\in R^n$.

Theorem 4.1. Let H be a hemi-space, with associated linear manifold M. For an element $\bar{y}\in R^n$, the following statements are equivalent:

1°. $\bar{y} + H = H$.

2°. $\bar{y} + M = M$.

3°. $\bar{y} + M\subseteq M$.

4°. $\bar{y}\in D(M)$.

Proof. $1^\circ\Rightarrow 2^\circ$. This follows from the above proof of theorem 3.5, implication $2^\circ \Rightarrow 1^\circ$ (applied to $H_1=H_2=H$).

$2^\circ\Rightarrow 1^\circ$. If 2° holds and (3.53) is the decomposition of H, given by theorem 3.4, then

$$\bar{y} + H=\bar{y} +(M+S)=(\bar{y}+M)+S=M+S=H.$$

Finally, the equivalences $2^\circ\Leftrightarrow 3^\circ\Leftrightarrow 4^\circ$ hold for any linear manifold M. Indeed, the implications $2^\circ\Rightarrow 3^\circ\Rightarrow 4^\circ$ are obvious. Finally, if 4° holds, then $\bar{y}\in M-m$ $(m\in M)$, whence $\bar{y}+M\subseteq M$. Also, since D(M) is a linear subspace, 4° implies $-\bar{y}\in D(M)=M-m$ $(m\in M)$, whence $m\in\bar{y}+M$ $(m\in M)$, that is, $M\subseteq \bar{y}+M$; thus, $4^\circ\Rightarrow 2^\circ$. ∎

Remark 4.1. a) It is also easy to deduce the implication $4^\circ\Rightarrow 1^\circ$ above, from proposition 3.3. Indeed, if 4° holds, then, by the first part of (3.51), with $d=\bar{y}$ and $\lambda=1$, we have $y + \bar{y}\in H$ $(y\in H)$, so $\bar{y} + H\subseteq H$; similarly, by the first part of (3.51), with $d=\bar{y}$ and $\lambda=-1$, we have $y-\bar{y}\in H$ $(y\in H)$, so $H\subseteq \bar{y}+H$, whence $\bar{y}+H=H$. ∎

b) One can also write the above equivalences as formulas for D(M); e.g., one can write the equivalence $1^\circ\Leftrightarrow 4^\circ$ in the form

$$D(M)=\{d\in R^n \mid d+H=H\}. \tag{4.1}$$

In particular, for demispaces, i.e. (see remark 3.2), for hemi-spaces of type $<_L$, formula (4.1) has been given, without proof, by P.C.Hammer ([4], theorem 3, part 1). Note that this also yields (4.1) in the general case, since if H is a hemi-space of type $\leq_L$, then $R^n\setminus H$ is of type $<_L$, whence, by (3.15) and (4.1) for $R^n\setminus H$, we obtain

$$D(M)=\{d\in R^n \mid d+(R^n\setminus H)=R^n\setminus H\}=\{d\in R^n \mid d+H=H\}.$$

Corollary 4.1. Let $H\subsetneq R^n$ be a hemi-space, with associated linear manifold M, and let $\bar{y}\in R^n$. Then, $\bar{y}+H$ is the unique translate of H, whose associated linear manifold is $\bar{y}+M$.

Proof. Clearly, we have to prove only the uniqueness statement. If $\bar{y}+H$, $\bar{y}_1+H$ are translates of H, both with the same associated linear manifold $\bar{y}+M$, then $\bar{y}+M=\bar{y}_1+M$, whence $\bar{y}-\bar{y}_1\in M-M=D(M)$, and hence, by theorem 4.1, implication $4^\circ\Rightarrow 1^\circ$, $\bar{y}-\bar{y}_1+H=H$; thus, $\bar{y}+H=\bar{y}_1+H$. ∎

Remark 4.2. In other words, corollary 4.1 means that any translate of a hemi-space is uniquely determined by its associated linear manifold.

Corollary 4.2. Let $H\subsetneq R^n$ be a hemi-space, with associated linear manifold M. For a set $C\subseteq R^n$, the following statements are equivalent:

1°. $H \overset{*}{-} C=H-C$.

2°. $C-C\subseteq D(M)$.

Proof. Since $H\overset{*}{-}C=\bigcap_{c\in C}(H-c)$ and $H-C=\bigcup_{c\in C}(H-c)$, 1° is equivalent to $H-c_1=H-c_2$ $(c_1,c_2\in C)$, i.e., to $c_2-c_1+H=H$ $(c_1,c_2\in C)$, which, by theorem 4.1, equivalence $1^\circ\Leftrightarrow 4^\circ$, is equivalent to 2°. ∎

Let us consider now affine transformations of the form

$$y\to \bar{y}+v(y) \qquad (y\in R^n), \tag{4.2}$$

where $\bar{y}\in R^n$ and $v\in\mathcal{U}(R^n)$, or $v\in\mathcal{O}(R^n)$; note that such transformations (with v replaced by v^{-1}), have been used in 21° of theorem 2.1 and in theorem 2.2. We shall give some necessary and sufficient conditions in order that (4.2) should preserve a hemi-space H.

Theorem 4.2. Let $H\subseteq R^n$ be a hemi-space, with associated linear manifold M. For $\bar{y}\in R^n$ and $v\in\mathcal{U}(R^n)$, the following statements are equivalent:

1°. $\bar{y}+v(H)=H$.

2°. $\bar{y}+v(M)=M$ and $v(H-M)=H-M$.

Proof. $1^\circ\Rightarrow 2^\circ$. For any $y\in R^n$ and $v\in\mathcal{U}(R^n)$, the linear manifold associated to the hemi-space $\bar{y}+v(H)$ is $\bar{y}+v(M)$ (e.g., by propositions 2.1-2.4). Hence, if 1° holds, then $\bar{y}+v(M)=M$. Thus, fixing any $y_o\in M$, we have $\bar{y}+v(y_o)\in M$, whence

$$\bar{y}+v(y_o)-y_o\in M-y_o=D(M), \tag{4.3}$$

and hence, by theorem 4.1, implication $4^\circ\Rightarrow 1^\circ$,

$$\bar{y}+v(y_o)-y_o+H = H. \qquad (4.4)$$

Hence, by lemma 3.2 and 1^o, we obtain

$$v(H-M)=v(H-y_o)=v(H)-v(y_o)=-\bar{y}+H-v(y_o)=$$
$$=-\bar{y}+(\bar{y}+v(y_o)-y_o+H)-v(y_o)=H-y_o=H-M.$$

$2^o\Rightarrow 1^o$. Assume 2^o and let $y_o\in M$. Then, by the first part of 2^o, we have $\bar{y}+v(y_o)\in M$, whence, as in the above proof of the implication $1^o\Rightarrow 2^o$, we obtain (4.4). But, by lemma 3.2 and the second part of 2^o, we have

$$v(H)-v(y_o)=v(H-y_o)=v(H-M)=H-M=H-y_o, \qquad (4.5)$$

whence, by (4.4), we obtain

$$\bar{y}+v(H)=\bar{y}+v(y_o)-y_o+H=H.$$ ■

Theorem 4.3. Let $H\subseteq R^n$ be a hemi-space, with associated linear manifold M. For $\bar{y}\in R^n$ and $v\in\mathcal{O}(R^n)$, the following statements are equivalent:

1^o. $\bar{y}+v(H)=H$.

2^o. $\bar{y} + H=H=v(H)$.

3^o. $\bar{y}\in D(M)$ *and* $v(H)=H$.

Proof. $1^o\Rightarrow 2^o$. Assume 1^o and let H be represented (uniquely) as in 18^o of theorem 2.1. Then, by (2.37), we have

$$\bar{y}+v(H)=\{\bar{y}+v(y)\in R^n|u(y)\tau x\}=\{y'\in R^n|uv^{-1}(y'-\bar{y})\tau x\}=$$
$$=\{y'\in R^n|uv^{-1}(y')\tau x+uv^{-1}(\bar{y})\}. \qquad (4.6)$$

Since $v^{-1}\in\mathcal{O}(R^n)$, we have $v^{-1}(v^{-1})^*=I$, whence, by $uu^*=I$, we obtain

$$(uv^{-1})(uv^{-1})^*=uv^{-1}(v^{-1})^*u^*=uu^*=I. \qquad (4.7)$$

Hence, by 1^o, (4.6), (2.37) and the uniqueness of u in 18^o of theorem 2.1, we obtain

$$u = uv^{-1}. \qquad (4.8)$$

Consequently, by (2.37),

$$v(H)=\{v(y)\in R^n|u(y)\tau x\}=\{y'\in R^n|uv^{-1}(y')\tau x\}=$$
$$=\{y'\in R^n|u(y')\tau x\}= H,$$

and hence, by 1^o,

$$\bar{y} + H=\bar{y} + v(H) = H.$$

$2^o\Rightarrow 1^o$. If 2^o holds, then $\bar{y}+v(H)=\bar{y}+H=H$.

Finally, the equivalence $2^o\Leftrightarrow 3^o$ is an immediate consequence of theorem 4.1, equivalence $1^o\Leftrightarrow 4^o$. ■

In connection with 2^o of theorem 4.3, let us give now some necessary and sufficient conditions in order that an operator $v\in\mathcal{U}(R^n)$, respectively, $v\in\mathcal{O}(R^n)$, should preserve a hemi-space H.

Theorem 4.4. Let $H \subseteq R^n$ be a hemi-space, with associated linear manifold M. For a linear isomorphism $v \in \mathcal{U}(R^n)$, the following statements are equivalent:

1°. $v(H)=H$.

2°. $v(M)=M$ and $\pi_{D(M)^\perp} v|_{D(M)^\perp}$ preserves the lexicographical order in the orthonormal basis $\{e_j'\}_1^r$ of $D(M)^\perp$, given in 20° of theorem 2.1, where $\pi_{D(M)^\perp}$ denotes the orthogonal projection operator from R^n onto $D(M)^\perp$.

Proof. Note first that, by remark 3.1 b), we have, for $\{e_j'\}_1^n$ and $\{\Psi_j\}_1^n$ as in 20° of theorem 2.1,

$$D(M)=\{y \in R^n | \Psi_j(y)=0 \ (j=1,\dots,r)\}=\mathrm{lin}\{e_j'\}_{r+1}^n , \tag{4.9}$$

$$D(M)^\perp=\{y \in R^n | \Psi_j(y)=0 \ (j=r+1,\dots,n)\}=\mathrm{lin}\{e_j'\}_1^r , \tag{4.10}$$

so $\{e_j'\}_1^r$ is indeed an orthonormal basis of $D(M)^\perp$.

Assume now that H is also represented in the form 18° of theorem 2.1. Then, by (2.37), we have

$$v(H)=\{v(y)| y \in R^n , u(y)\tau x\} = \\ =\{y' \in R^n | uv^{-1}(y')\tau x\}, \tag{4.11}$$

where, since $v \in \mathcal{U}(R^n)$, $v(H)$ is a hemi-space, too, with associated linear manifold

$$M(v(H))=\{y' \in R^n | uv^{-1}(y')=x\} = \{v(y)| y \in R^n, u(y)=x\}=v(M). \tag{4.12}$$

Hence, by remark 2.2 e), we have 1° if and only if there exists a (unique) lexicographical order preserving $\ell \in \mathcal{U}(R^r)$ such that

$$\ell uv^{-1}=u, \tag{4.13}$$

$$\ell(x)=x. \tag{4.14}$$

Note also that, if 1° holds, then, by (4.12), we have $v(M)=M(v(H))=M(H)=M$. Thus, for the proof of the equivalence 1°⇔2°, we may (and it will be convenient to) assume that

$$v(M) = M. \tag{4.15}$$

We claim that

$$\ell=uvu^* \tag{4.16}$$

is the unique operator in $\mathcal{U}(R^r)$, satisfying (4.13). Indeed, the proof of the fact that ℓ of (4.16) satisfies (4.13) and $\ell \in \mathcal{U}(R^r)$, is similar to that of the corresponding statement in remark 2.2 e), using the decomposition $R^n=D(M)^\perp \oplus D(M)$ and the equalities

$$u^*u=\pi_{(\mathrm{Ker}\ u)^\perp}=\pi_{D(M)^\perp} . \tag{4.17}$$

In order to see (4.17), note that, by $uu^*=I$, we have

$$u^*u(u^*(x'))=u^*(uu^*(x'))=u^*(x') \quad (u^*(x')\in u^*(R^r)=(\mathrm{Ker}\ u)^{\perp});$$

also, clearly, $u^*u|_{D(M)}=u^*u|_{\mathrm{Ker}\ u}=0$, which proves (4.17). Now, to see that ℓ of (4.16) satisfies (4.13), or, equivalently,

$$\ell u=uv, \tag{4.18}$$

note that by (4.17) we have $\ell u|_{D(M)^{\perp}}=uvu^*u|_{D(M)^{\perp}}=uv|_{D(M)^{\perp}}$. On the other hand, by (3.44) and (4.15), we have

$$v(D(M))=v(M-M)=v(M)-v(M)=M-M=D(M); \tag{4.19}$$

but, (4.19) implies $v(y)\in D(M)=\mathrm{Ker}\ u$ $(y\in D(M))$, whence $\ell u(y)=0=uv(y)$ $(y\in D(M))$, and thus, finally, (4.18). Let us also show that $\ell\in\mathcal{U}(R^r)$, or, equivalently, $\mathrm{Ker}\ \ell=\{0\}$. If $x'\in R^r$, $\ell(x')=0$, then there exists $y\in D(M)^{\perp}$ such that $u(y)=x'$, whence, by (4.18), $u(v(y))=\ell(u(y))=$ $=\ell(x')=0$. Therefore, $v(y)\in\mathrm{Ker}\ u=D(M)$, whence, by (4.19), $y\in v^{-1}(D(M))=v^{-1}(v(D(M)))=D(M)=$ $=\mathrm{Ker}\ u$. Thus, $x'=u(y)=0$ (moreover, $y\in D(M)^{\perp}\cap D(M)=\{0\}$). Finally, if ℓ_1, $\ell_2\in\mathcal{U}(R^r)$, $\ell_1u=$ $=uv=\ell_2u$, then $\ell_1=\ell_1uu^*=\ell_2uu^*=\ell_2$. This proves the above claim on (4.16). Furthermore, taking any $y_o\in M$, by (4.15), (4.12), (4.13) and (3.10) we obtain $\ell(x)=\ell(uv^{-1}(y_o))=u(y_o)=x$, and thus ℓ satisfies also (4.14). Consequently, we have 1° if and only if $v(M)=M$ and uvu^* preserves the lexicographical order.

Finally, let us show that $\pi_{D(M)^{\perp}}v|_{D(M)^{\perp}}$ preserves the lexicographical order in the basis $\{e'_j\}_j^r$ if and only if uvu^* preserves the lexicographical order. By the biorthogonality of $\{e'_j, \Psi_j\}_1^n$ we obtain, taking $y=e'_i$ $(i=1,\dots,r)$ in (3.11), that

$$u(e'_i)=\sum_{j=1}^{r}\Psi_j(e'_i)e_j=e_i \qquad (i=1,\dots,r). \tag{4.20}$$

On the other hand, by (4.10), (4.9), and the definition of $\{e'_j, \Psi_j\}_1^n$, we have

$$\pi_{D(M)^{\perp}}v(v^{-1}(y))=\pi_{D(M)^{\perp}}(y)=\sum_{j=1}^{r}\Psi_j(y)e'_j \qquad (y\in R^n). \tag{4.21}$$

Hence, replacing y by $v(y)$ and using (4.20), (3.11) and (4.13), (4.16), we obtain

$$u(\pi_{D(M)^{\perp}}v(y))=\sum_{j=1}^{r}\Psi_j(v(y))u(e'_j)=\sum_{j=1}^{r}\Psi_j(v(y))e_j=$$
$$=u(v(y))=\ell uv^{-1}(v(y))=\ell u(y)=uvu^*(u(y)) \qquad (y\in R^n). \tag{4.22}$$

Observe now that, by (3.11), $y=\sum_{j=1}^{r}\Psi_j(y)e'_j\in D(M)^{\perp}$ satisfies $y\leqslant_L 0$ in the basis $\{e'_j\}_1^r$ of $D(M)^{\perp}$ if and only if $u(y)\leqslant_L 0$ (in the unit vector basis $\{e_j\}_1^r$ of R^r); hence, for $y\in R^n$, we have $\pi_{D(M)^{\perp}}v(y)\leqslant_L 0$ in $\{e'_j\}_1^r$ if and only if $u(\pi_{D(M)^{\perp}}v(y))\leqslant_L 0$. Consequently, by (4.22) and (4.20), the implication

$$y\in D(M)^{\perp},\ y\leqslant_L 0 \text{ in } \{e'_j\}_j^r\Rightarrow\pi_{D(M)^{\perp}}v(y)\leqslant_L 0 \text{ in } \{e'_j\}_j^r$$
$$(\Leftrightarrow u(\pi_{D(M)^{\perp}}v(y))\leqslant_L 0)$$

holds if and only if we have the implication

$$y\in D(M)^{\perp},\ u(y)\in R^{r}=u(D(M)^{\perp}),\ u(y)\leqslant_{L} 0\Rightarrow uvu^{*}(u(y))\leqslant_{L} 0.$$ ■

Theorem 4.5. Let $H\subseteq R^{n}$ be a hemi-space, with associated linear manifold M. For $v\in\mathcal{O}(R^{n})$, the following statements are equivalent:

1°. $v(H)=H$.

2°. $v(M)=M$ and $v(y)=y$ $(y\in D(M)^{\perp})$.

Proof. $1^{\circ}\Rightarrow 2^{\circ}$. If 1° holds, then, by theorem 4.4, we have $v(M)=M$. Furthermore, if H is represented (uniquely) as in 18° of theorem 2.1, then, by the above proof of theorem 4.3, we have (4.7). Hence, by 1°, (4.11), (2.37) and the uniquess of u in 18° of theorem 2.1, we obtain (4.8), that is,

$$uv = u, \tag{4.23}$$

whence $uvu^{*}=I$. Therefore, by (4.22),

$$u(\pi_{D(M)^{\perp}}v(y))=u(y) \qquad (y\in R^{n}), \tag{4.24}$$

and hence, since $u|_{D(M)^{\perp}}$ is an isomorphism, we obtain

$$\pi_{D(M)^{\perp}}v(y)=y \qquad (y\in D(M)^{\perp}). \tag{4.25}$$

On the other hand, by (3.44) and $v(M)=M$, we have (4.19), whence, by $v\in\mathcal{O}(R^{n})$,

$$v(D(M)^{\perp})=D(M)^{\perp}, \tag{4.26}$$

and hence, since $\pi_{D(M)^{\perp}}(y)=y$ $(y\in D(M)^{\perp})$, we obtain

$$\pi_{D(M)^{\perp}}v(y)=\pi_{D(M)^{\perp}}(v(y))=v(y) \qquad (y\in D(M)^{\perp}). \tag{4.27}$$

Thus, by (4.25) and (4.27), we have $v(y)=y$ $(y\in D(M)^{\perp})$.

$2^{\circ}\Rightarrow 1^{\circ}$. If 2° holds, then $\pi_{D(M)^{\perp}}v|_{D(M)^{\perp}}=\pi_{D(M)^{\perp}}|_{D(M)^{\perp}}=I$, which preserves the lexicographical order in the basis $\{e'_j\}_1^r$ of $D(M)^{\perp}$. Hence, by $v(M)=M$ and theorem 4.4, we obtain $v(H)=H$. ■

Remark 4.3. Theorem 4.5 can be also deduced from theorem 4.4, using a result of [13]. Indeed, by (3.44) and $v(M)=M$ we obtain, as above, (4.27), that is, $\pi_{D(M)^{\perp}}v|_{D(M)^{\perp}}=v|_{D(M)^{\perp}}$. Thus, theorem 4.5 follows from theorem 4.4 and the fact [13] that a linear isometry preserves the lexicographical order in some basis if and only if it is the identity. ■

§5. Appendix

A) Separation of p sets by hemi-spaces.

Let us give now the separation theorem, announced in remark 1.1 c).

Theorem 5.1. For any sets $G_1,\ldots,G_p\subset R^{n}$, the following statements are equivalent:

1°. $\bigcap_{i=1}^{p} \text{co } C_i=\emptyset$.

2°. There exist p hemi-spaces $H_1,\dots,H_p$ in R^n, such that

$$C_i \subseteq H_i \qquad (i=1,\dots,p), \tag{5.1}$$

$$\bigcap_{i=1}^{p} H_i=\emptyset. \tag{5.2}$$

Proof. 1°⇒2°. If p=1, one can take $H_1=\emptyset=C_1$.

Now, let $p\geqslant 2$, and assume that for some $k\in\{0,1,\dots,p-1\}$ there exist k hemi-spaces $H_1,\dots,H_k$ in R^n, such that

$$C_i \subseteq H_i \qquad (i=1,\dots,k), \tag{5.3}$$

$$\left(\bigcap_{i=1}^{k} H_i\right)\cap\left(\bigcap_{i=k+1}^{p} \text{co } C_i\right)=\emptyset; \tag{5.4}$$

note that, for k=0, this assumption is vacuously satisfied (by 1°). Then, by (5.4), the convex sets co C_{k+1} and $\left(\bigcap_{i=1}^{k} H_i\right)\cap\left(\bigcap_{i=k+2}^{p} \text{co } C_i\right)$ are disjoint, and hence, by the separation theorem for two sets (see remark 1.1 b)), there exists a hemi-space H_{k+1} in R^n, such that

$$C_{k+1}\subseteq \text{co } C_{k+1}\subseteq H_{k+1}, \tag{5.5}$$

$$\left(\bigcap_{i=1}^{k+1} H_i\right)\cap\left(\bigcap_{i=k+2}^{p} \text{co } C_i\right)=\emptyset. \tag{5.6}$$

Thus, by induction, we arrive (for k=p-1) at (5.1) and (5.2).

2°⇒1°. If 2° holds, then, by (5.1) and the convexity of H_i we have co $C_i\subseteq H_i$ $(i=1,\dots,p)$, whence, by (5.2), we obtain 1°. ∎

B) On a theorem of V. Klee.

We have the following results, which, in R^n, extend a theorem given by V. Klee for semi-spaces of linear spaces ([6], theorem 2.2):

Theorem 5.2. a) For a subset H and a linear subspace M, of R^n, the following statements are equivalent:

1°. H is a hemi-space of type $<_L$, with M(H)=M.

2°. H is convex and

$$H\cup(-H)=R^n\setminus M. \tag{5.7}$$

b) For a linear subspace V of R^n and a subset H of V, the following statements are equivalent:

1°. H is a hemi-space in V.

2°. $H=\tilde{H}\cap V$, for some hemi-space $\tilde{H}$ in R^n.

c) If Q is one of the open half-spaces bounded by a hyperplane P in R^n, and H is a hemi-space in P, then $H \cup Q$ is a hemi-space in R^n; conversely, every hemi-space H' in R^n, with $\emptyset \neq H' \neq R^n$, is of the form $H \cup Q$, for suitable Q and H as above.

We shall give the proofs elsewhere (in preparation).

Remark 5.1. A result related to theorem 5.2 b) has been announced, without proof, in [10] (see [10], p.196, equivalence $5^o \Leftrightarrow 12^o$).

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