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INTRODUCTION

There are several methods of determining the local time of the one-dimensional Brownian motion. One of these methods, proposed by P. Levy, is based on the analysis of the oscillations of the paths near the origin. By this method one should consider the number of times N_t^{ε} that the Brownian path comes to 0 after visits outside the interval $(-\varepsilon, +\varepsilon)$ untill time t. It was checked by K. Ito and H.P.Mc Kean (see [I-K]) that

$$\lim_{\varepsilon \to 0} \varepsilon N_t^{\varepsilon} = \ell_t \ , \quad \text{for each } t, \text{ a.s.},$$

where ℓ is the local time of Brownian motion.

The aim of this paper is to extend this result to the case of multidimensional Brownian motion. So let us consider the Brownian motion in R^d as a standard process X and observe its behaviour near a hyperplane. To fix the notation we consider that the hyperplane is $K=\{x \in R^d : x^d=0\}$. Then the Brownian oscillations near K depend only on the component X^d. Let us denote by $N_t^{\varepsilon}(\omega)$ the number of times that the path $X(\omega)$ hits K after visits outside the neighbourhood $D_\varepsilon=\{x\in R^d: |x^d|<\varepsilon\}$ before time t. Then by the one dimensional result we immediately get

$$\lim_{\varepsilon\to 0} \varepsilon N_t^\varepsilon = A_t(\omega) \quad , \tag{0.1}$$

where $A=(A_t)$ is a continuous additive functional uniformly distributed on K. In fact A is nothing else but the local time at 0 of the component x^d, considered as additive functional with respect to the multidimensional process X.

What will happen if instead of a hyperplane we consider a hypersurface? Then of course, we can treat this problem locally. If x_o is an arbitrary point of the surface, we may find an open neighbourhood of x_o , D and a diffeomorphism $F:D\to B$, which maps D onto an open neighbourhood of 0, B such that $F(x_o)=0$ and maps the piece of surface which is in D onto $K\cap B$. So we reduce the problem of a hypersurface to a hyperplane, except that, transported by F, the Brownian motion becomes a diffusion process having as infinitesimal generator an elliptic operator. This elliptic operator is obtained from $1/2\Delta$ by the change of variable $x\to F(x)$ and it is of the form

$$L=\sum_{i,j=1}^{d} a^{ij}D_{ij}+\sum_{i=1}^{d} b^i D_i \ .$$

Surprisingly we find in Theorem 3.1 that for the hyperplane K we can treat any diffusion generated by an elliptic operator of this form almost as easy as brownian motion, provided that the drift coefficient b^d vanish ($b^d\equiv 0$). However if L is obtained from $1/2\Delta$ by a diffeomorphism chosen as above, then the coefficient b^d depends essentially on the curvature of the surface and it is non-null in general. In Theorem 3.3 we treat the hyperplaen K with a general diffusion, generated by an elliptic operator of the above form (with b^d non-null). The proof is more complicated. The difficulty is related to the L^2 estimate. Then in Theorem 7.2 we treat the case of a compact hypersurface with a general diffusion. As a particular corollary we have the following result, which was our initial aim in doing this work.

THEOREM. Let X be the Brownian motion in R^d with $d \geq 3$ and let K be a compact hypersurface. For each $\varepsilon > 0$ we denote by V^ε the neighbourhood of radius ε of K and by $N_t^\varepsilon(\omega)$ the number of times that the path $X(\omega)$ hits K after visits in $R^d \setminus V^\varepsilon$ before time t. Then there exists a continuous additive functional A such that

$$(0.2)\quad \lim_{\varepsilon \to 0} \sup_t |\varepsilon N_t^\varepsilon - A_t| = 0, \quad \text{a.s.},$$

$$(0.3)\quad E^x(\sup_t |\varepsilon N_t^\varepsilon - A_t|^2)^{1/2} \leq C \varepsilon^{1/4} (\ln 1/\varepsilon)^{3/4}, \quad x \in R^d, \varepsilon \in (0, \varepsilon_o),$$

where C and ε_o are constants. The functional A is uniformly distributed on K, in the sense that its potential is represented with the "surface area" μ by the following formula

$$(0.4)\quad E^x(A_\infty) = \int_K g(x,y)\mu(dy), \quad x \in R^d,$$

where $g(x,y) = k|x-y|^{2-d}$ is the Green function of R^d.

A different approach of the oscillations of Brownian motion (or of diffusions) near a hypersuraface was proposed by N. Portenko (see [P] and [P-Y]). We use the approach of V. Bally

[B_1] and our work is more related to the papers [B2] and [B-S].

Our main tool in the proof is the Green function. Section 1 of this paper is a long introductory section where we gather together several properties of Green function and of Green potentials. In general these are more or less known. In the second section we give a rigurous definition of the functionals numbering the displacements of the path and then study their basic properties. The third section is devoted to the proof of Theorem 3.3 mentioned above. Then Sections 4, 5 and 6 present the lemmas needed in the proof of Theorem 3.3.

The main result of Section 4 is Lemma 4.2. In Section 5 Lemma 5.9 is central, the others are needed to prove it. Lemma 6.1 of Section 6 is used in the proof of Theorem 3.3 while Lemma 6.2 is used in the proof of Theorem 7.2. The last section is devoted to the proof of Theorem 7.2 which is the main result of this paper.

NOTATION

The ball of center x and radius r is denoted by $B(x,r) = \{y \in R^d : |x-y| < r\}$. We call strip a set of the form $\{y \in R^d : \sum_{i=1}^{d} y^i x^i \in (a,b)\}$, where $x \in R^d$, $a,b \in R$, $a<b$ are fixed. Sometimes the last component of R^d is put in a special position, and then we write a point $x \in R^d$ as $x = (x', x^d)$, where $x' \in R^{d-1}$. If D is a subset of R^d and $f: D \to R$, then we denote by supp f the closure in D of the set $\{x \in D: f(x) \neq 0\}$. The set supp f is called the support of f. If D is an open set we say that μ is a measure in D if it is a Radon measure in D. The space of all bounded Borel measurable functions in D is denoted by $\mathcal{B}_b(D)$

and the family of all nonnegative Borel functions in D is denoted by $\mathcal{B}_+(D)$. The space of continuous functions with compact support in D is denoted by $\mathcal{C}_c(D)$. If f(x,t) is a function defined in and open subset of $R^d xR$, we denote its partial derivatives as follows

$$D_i f=\frac{\partial f}{\partial x^i}\ ,\quad D_{ij} f=\frac{\partial^2 f}{\partial x^i \partial x^j}\ ,\quad D^t f=\frac{\partial f}{\partial t}\ .$$

If f(x,y) is defined in an open subset of $R^d xR^d$, we use the notation

$$D_i^x f=\frac{\partial f}{\partial x^i}\ ,\quad D_i^y f=\frac{\partial f}{\partial y^i}\ ,\quad D_{ij}^x f=\frac{\partial^2 f}{\partial x^i \partial x^j}\ ,\quad D_{ij}^y f=\frac{\partial^2 f}{\partial x^i \partial x^j}\ .$$

In the hope that a reader with little knowledge may understand how we use some results of partial differential equations, we recall here the notation of Hölder spaces of functions to help him. Let D be any set in R^d and $\alpha\in(0,1)$. If A is an arbitrary subset of D and $f:D\to R$ we put

$$[f]_0^A=\sup\{\,|f(x)|\ :x\in A\}\ ,$$

$$[f]_\alpha^A=\sup\{\ \frac{|f(x)-f(y)|}{|x-y|^\alpha}:x,y\in A,\ x\neq y\}\ .$$

If D is open and f possess firs order derivatives, we put

$$[f]_1^A=\max_{i=1}^{d}\,[D_i f]_0^A\ ,$$

$$[f]_{1+\alpha}^A=\max_{i=1}^{d}\,[D_i f]_\alpha^A\ .$$

Similarly, if f possess second order derivatives

$$[f]_2^A = \max_{i,j=1}^{d} [D_{ij} f]_o^A$$

$$[f]_{2+\alpha}^A = \max_{i,j=1}^{d} [D_{ij} f]^A \quad .$$

We also write

$$\|f\|_o^A = [f]_o^A, \quad \|f\|_1^A = \|f\|_o^A + [f]_1^A, \quad \|f\|_2^A = \|f\|_1^A + [f]_2^A \quad ,$$

$$\|f\|_{i+\alpha}^A = \|f\|_i^A + [f]_{i+\alpha}^A \,, \quad i=0,1,2 \quad .$$

If A=D we simply write

$$[f]_i = [f]_i^D \,, \quad [f]_{i+\alpha} = [f]_{i+\alpha}^D \,, \quad \|f\|_i = \|f\|_i^D \quad ,$$

$$\|f\|_{i+\alpha} = \|f\|_{i+\alpha}^D \,, \quad i=0,1,2 \;.$$

Moreover we write $\|f\| = \|f\|_o = [f]_o$. Now let us suppose that A is a subset of D. Then for each $x \in D$ we set

$$d_x = \inf \{ |x-y| : y \in A \} \,, \quad \text{if } A \neq \emptyset \quad ,$$

$$d_x = 1, \quad \text{if } A = \emptyset \;,$$

$$d_{xy} = \inf(d_x, d_y) \,,$$

and define another series of seminorms and norms

$${}^A[f]_1 = \sup \{ d_x |D_i f(x)| : x \in D, \; i=1\ldots d \} \,,$$

$${}^A[f]_2 = \sup \{ d_x^2 |D_{ij} f(x)| : x \in D, \; i,j=1\ldots d \} \,,$$

$${}^A[f]_\alpha = \sup \{ d_{xy}^\alpha \frac{|f(x)-f(y)|}{|x-y|^\alpha} : x,y \in D \} \,,$$

$${}^{A}[f]_{1+\alpha}=\sup\{\, d_{xy}^{1+\alpha}\frac{|D_i f(x)-D_i f(y)|}{|x-y|^{\alpha}}: x,y\in D,\ i=1\ldots d\}\ ,$$

$${}^{A}[f]_{2+\alpha}=\sup\{\, d_{xy}^{2+\alpha}\frac{|D_{ij} f(x)-D_{ij} f(y)|}{|x-y|^{\alpha}}: x,y\in D,\ i,j=1\ldots d\}\ ,$$

$${}^{A}\|f\|_{0}=\|f\|_{0}\ ,$$

$${}^{A}\|f\|_{i+1}={}^{A}\|f\|_{i}+{}^{A}[f]_{i+1}\ ,\quad i=0,1\ ,$$

$${}^{A}\|f\|_{i+\alpha}={}^{A}\|f\|_{i}+{}^{A}[f]_{i+\alpha}\ ,\quad i=0,1,2\ .$$

Of course, if $A=\emptyset$, we have ${}^{\emptyset}\|f\|_{i}=\|f\|_{i}$, ${}^{\emptyset}\|f\|_{i+\alpha}=\|f\|_{i+\alpha}$ $i=0,1,2$. We also define the spaces of functions

$$\mathcal{C}^{0}(D)=\{f:D\to R:\ f \text{ is continuous and } \|f\|_{0}<\infty\},$$

$$\mathcal{C}^{1}(D)=\{f\in\mathcal{C}^{0}(D):D_i f\in\mathcal{C}^{0}(D),\ i=1\ldots d\}\ ,$$

$$\mathcal{C}^{2}(D)=\{f\in\mathcal{C}^{1}(D):D_{ij} f\in\mathcal{C}^{0}(D),\ i,j=1\ldots d\}\ ,$$

$$\mathcal{C}^{i+\alpha}(D)=\{f\in\mathcal{C}^{i}(D):\ \|f\|_{i+\alpha}<\infty\},\ i=0,1,2\ .$$

A function $f\in\mathcal{C}^{\alpha}(D)$ admits a unique extention $\overline{f}:\overline{D}\to R$ such that $[\overline{f}]_{\alpha}<\infty$. We use the notation $Bf=\overline{f}_{|\partial D}$; and so we have $Bf\in\mathcal{C}^{\alpha}(\partial D)$. If the boundary ∂D is a hypersurface of class $\mathcal{C}^{2+\alpha}$, we can define the Hölder semi-norms and classes $\mathcal{C}^{i}(\partial D)$, $\mathcal{C}^{i+\alpha}(\partial D)$ $i=0,1,2$, in an obvious manner. We observe that $Bf\in\mathcal{C}^{i+\alpha}(\partial D)$, provided $f\in\mathcal{C}^{i+\alpha}(D)$, for $i=0,1,2$.

The letter C is designed for constants. Though we will have different constants, in various places we will use the same symbol C.

Concerning the Markov processes we use the terminology and notation of the book of R.M. Blumenthal and R.K. Getoor [B-G]. For example, if X is a standard process with state space an open subset $E \subset R^d$, then for each $f \in \mathcal{B}_+(E)$ we may write

$$E^x(f(X_T)) = E^x(f(X_T), T<\infty) ,$$

because f is automatically extended to E_Δ with $f(\Delta)=0$.

1. THE DIFFUSION GENERATED BY AN ELLIPTIC DIFFERENTIAL OPERATOR

The method we employ in this paper is essentially related to the Green function. The most general class of diffusions, which we know to possess satisfactory properties of the Green functions, is the class generated by elliptic operators with Hölder coefficients. The construction of these diffusions and some of the properties we need may be founded in the book of E.B. Dynkin [Dy]. However for the reader's easy we briefly retake here these aspects. We also add several properties of the Green function that will be of special interest to our approach. We begin by listing a series of results from the theory of partial differential equations, which are at the ground of this matter.

Let a^{ij}, $i,j=1,\dots d$, $b^i, i=1\dots d, c$, be functions of Hölder class $\mathcal{C}^{\alpha}(R^d)$, $d\geq 2$. The index α will be fixed throught the paper and will be supposed to satisfy $0<\alpha<1$. It will be reserved to indicate only Hölder continuity. The coefficients a^{ij} are supposed to be symmetric (i.e. $a_{ij}=a_{ji}$, $i,j=1,\dots d$) and to satisfy the uniform ellipticity condition

$$(1.1)\quad \sum_{i,j=1}^{d} a^{ij}(x)\xi^i\xi^j\geq C|\xi|^2,\ x,\xi\in R^d ,$$

where C is a strictly positive constant. We define a second order differential operator as follows

$$(1.2)\quad L=\sum_{i,j=1}^{d} a^{ij}D_{ij}+\sum_{i=1}^{d} b^i D_i+c \quad .$$

Whenever we will say in this paper that an operator L is of the form (1.2), its coefficients will implicitely be assumed as above. For the next six theorems we consider a fixed operator L of the form (1.2).

THEOREM 1.1. There exists a continuous function $p(t,x,y)$ defined on $(0,\infty)xR^d xR^d$ with the following properties

(1.3.0) $p>0$ and possesses derivatives of second order with respect to x and of first order with respect to t; the derivatives $D_i^x p$, $D_{ij}^x p$, $i,j\leq d$, $D^t p$, are continuous functions on $(0,\infty)xR^d xR^d$.

(1.3.i) for each fixed $y\in R^d$ the function $p(.,.,y)$ satisfies the equation $(L^x - D^t)p=0$.

(1.3.ii) $\lim_{t\to 0} \int p(t,x,y)f(y)dy=f(x)$, $x\in R^d$, $f\in \mathcal{C}^o(R^d)$.

(1.3.iii) $p(t,x,y)\leq C_1 t^{-d/2}\exp(-C_2 |x-y|^2/t)$

$$|D^t p(t,x,y)|\leq C_1 t^{-\frac{d+2}{2}}\exp(-C_2|x-y|^2/t)$$

$$|D_i^x p(t,x,y)|\leq C_1 t^{-\frac{d+1}{2}}\exp(-C_2|x-y|^2/t), \quad i\leq d$$

$$|D_{ij}^x p(t,x,y)|\leq C_1 t^{-\frac{d+2}{2}}\exp(-C_2|x-y|^2/t), \quad i,j\leq d$$

(1.3.iv) $|p(t,x,y -p_y(t,x,y)|\leq C_1 t^{-\frac{d-\alpha}{2}}\exp(-C_2|x-y|^2/t)$,

where

$$p_z(t,x,y)=(4\pi t)^{-d/2}|\det a^{ij}(z)|^{-1/2} \cdot$$

$$\cdot \exp(-\sum_{ij}\bar{a}^{ij}(z)(x^i-y^i)(x^j-y^j)/4t);$$

$\bar{a}^{ij}(z)$ are the elements of the matrix $(\bar{a}^{ij}(z))$ which is the inverse of $(a^{ij}(z))$.

In the above estimates C_1 and C_2 are strictly positive constants. The function p with the above properties is unique.

THEOREM 1.2. Let $T>0$, $h\in\mathcal{C}^o(R^d)$, $f\in\mathcal{C}^\alpha(R^d x(0,T))$ and put

$$u(x,t)=\int_{R^d} p(t,x,y)h(y)dy-\int_o^t\int_{R^d} p(t-s,x,y)f(y,s)dyds,$$

for $x\in R^d$ and $t\in(0,T]$. Then u is bounded, it can be extended as a continuous function on $R^d x[0,T]$ and its derivatives $D^t u$, $D_i u$, $D_{ij} u$ are continuous in $R^d x(0,T]$. Moreover u satisfies the following relations

(1.4) $(L-D^t)u(x,t)=f(x,t)$, $x\in R^d$, $t\in (0,T]$,

(1.5) $u(x,0)=h(x)$, $x\in R^d$.

If another function v is bounded and continuous on $R^d x[0,T]$, its derivatives $D^t v$, $D_i v$, $D_{ij} v$ are continuous in $R^d x(0,T]$ and satisfies relations (1.4) and (1.5), then this function coincides with u.

THEOREM 1.3 (Maximum principle). Suppose that the coefficient c of L satisfies condition $c\leq 0$. Let D be an open set in R^d which is included in a strip and let $u:\bar{D}\to R$ be a bounded continuous function such that the derivatives $D_i u$, $D_{ij} u$, $i,j\leq d$ are continuous in D. If $u\leq 0$ on ∂D and $Lu\geq 0$ in D, then $u\leq 0$ in D.

THEOREM 1.4 (Schauder estimates). Let D be a bounded domain in R^d whose boundary is a hypersurface of class $\mathcal{C}^{2+\alpha}$, let A be a closed subset of ∂D and set $E=\partial D\setminus A$. Then there exists a constant $C>0$ such that

$$^{A}\|u\|_{2+\alpha} \leqslant C(^{A}\|Lu\|_{\alpha}+\|u\|_{0}+\|Bu\|^{E}_{2+\alpha})$$

for any $u\in\mathcal{C}^{2+\alpha}(D)$. The constant C depends only on the following objects D,A, $\|a^{ij}\|_{\alpha}$, $\|b^{i}\|_{\alpha}$, $i,j\leqslant d$, $\|c\|_{\alpha}$ and the ellipticity constant appearing in condition (1.1).

THEOREM 1.5. Let D be a bounded domain in R^d whose boundary is a hypersurface of class $\mathcal{C}^{2+\alpha}$. Assume that $c\leqslant 0$. If $f\in\mathcal{C}^{\alpha}(D)$ and $h\in\mathcal{C}^{2+\alpha}(D)$, then there exists a unique function $u\in\mathcal{C}^{2+\alpha}(D)$ such that $Bu=h$ and $Lu=f$ in D.

THEOREM 1.6. Suppose that $a^{ij}\in\mathcal{C}^{2+\alpha}(R^d)$, $b^i\in\mathcal{C}^{1+\alpha}(R^d)$, $i,j\leqslant d$ and set $b^{*i}=-b^i+2\sum_{j=1}^{d} D_j a^{ij}$, $c^*=c-\sum_{i=1}^{d} D_i b^i+\sum_{i,j=1}^{d} D_{ij}a^{ij}$.

Then the operator

$$L^*=\sum_{i,j=1}^{d} a^{ij}D_{ij}+\sum_{i=1}^{d} b^{*i}D_i+c^*$$

is of the form (1.2). Let $p(t,x,y)$ and $p^*(t,x,y)$ be the functions associated by Theorem 1.1 to L and L^*. Then $p(t,x,y)=p^*(t,y,x)$.

Theorem 1.1 was proved by W. Pogorzelski and Theorems 1.4 and 1.5 are do to J. Schauder. For a systematic treatment of all the above results the reader is refered to the books [F] and [L-S-U]. The function u which satisfies conditions (1.4) and (1.5) is called solution of the Cauchy problem. The function u from Theorem 1.5 is called solution of the Dirichlet-Poisson problem. (If $f \equiv 0$, we say Dirichlet problem. If $h \equiv 0$, we call it Poisson problem). An elegant and self-contained proof of Theorems 1.4 and 1.5 may also be found in the work [B-M]. It is not difficult to see that the proofs given there work also in the case where D is a strip. We will use this fact with no other comment.

L- DIFFUSIONS

From now on in this section we will assume that L is a fixed operator of the form (1.2) with $c \equiv 0$. We denote by $p_t(x,y)=p(t,x,y)$ the function given in Theorem 1.1

LEMMA 1.7.

(a) $\int p_t(x,y)dy=1, \quad x \in R^d, \quad t>0$.

(b) $p_{t+s}(x,y)=\int p_t(x,z)p_s(z,y)dz, \quad x,y \in R^d, \ t>0, \ s>0.$

(c) $\limsup_{t \to 0} \sup_{x \in R^d} t^{-1} \int_{|x-y| \geq r} p_t(x,y)dy=0, \quad r>0$.

(d) $\lim_{t \to 0} \int p_t(x,y)f(x)dx=f(y), \quad y \in R^d, \quad f \in \mathcal{C}^o(R^d)$.

Proof

For (a) and (b) it suffices to use the unicity assertion from Theorem 1.2. To deduce (c) one applies (1.3. iii) and (d) follows from (1.3. iv).

From (a) and (b) in the preceding lemma it follows that $p_t(x,y)$ is the density (with respect to Lebesgue measure) of a

Markov semigroup. We denote by (P_t) this semigroup.

LEMMA 1.8.

$$P_t f(x) - f(x) - \int_0^t P_s(Lf)(x)ds = 0, \quad x \in R^d, \quad t \geq 0, \quad f \in \mathcal{C}^2(R^d) .$$

Proof

First we take $f \in \mathcal{C}^{2+\alpha}(R^d)$. Then the relation follows from the unicity assertion of Theorem 1.2. If $f \in \mathcal{C}^2(R^d)$, then we can take a sequence (f_n) of functions in $\mathcal{C}^\infty(R^d)$ which approximate f in $\mathcal{C}^2(R^d)$. The relation holds for each f_n and, as $n \to \infty$, we get it for f.

The theory of stochastic differential equations suggests us the following definition (see Proposition 2.1 of page 155 in [Y-W]).

DEFINITION 1.9. Let E be an open set in R^d and $X = (\Omega, \mathcal{M}, \mathcal{M}_t, X_t, \theta_t, P^x)$ a standard process with continuous paths with state space E. We say that X is an L-diffusion in E provided that the process

(1.7) $f(X_t) - \int_0^t Lf(X_s)ds, \quad t \in [0, \infty)$

is a martingale with respect to each measure P^x, $x \in E$, for any $f \in \mathcal{C}_c(E) \cap \mathcal{C}^2(E)$.

One can obviously see that the restriction of an L-diffusion in E to an open set $D \subset E$ is an L-diffusion in D.

Now we reformulate a result of Dynkin (see [Dy]).

THEOREM 1.10. There exists an L-diffusion in R^d whose transition function is given by the semigroup (P_t). Moreover for any

$f \in \mathcal{C}^2(R^d)$ the process (1.7) is a martingale with respect to each measure P^x.

Proof

The first estimate in (1.3. iii) shows that the semigroup (P_t) maps $\mathcal{C}_o(R^d)$ into itself and from (1.3. ii) it follows that it is continuous. Therefore there exists a Hunt process whose transition function is given by this semigroup. Because of Lemma 1.7 (c) the process has continuous paths. Now let $f \in \mathcal{C}^2(R^d)$. To show that the process (1.7) is a martingale we have to prove the following equality

$$(1.8) \qquad E^x(f(X_t) - \int_u^t Lf(X_s)ds / \mathcal{M}_u) = f(X_u)$$

for $u \leq t$. Thus we apply the Markov property and the left side becomes

$$E^{X_u}(f(X_{t-u}) - \int_o^{t-u} Lf(X_s)ds) = P_{t-u}f(X_u) - \int_o^{t-u} P_s Lf(X_u)ds \quad .$$

By Lemma 1.8 the last expression equals $f(X_u)$, proving the equality (1.8). The proof is complete.

The following proposition gives the probabilistic interpretation of the Dirichlet-Poisson problem.

PROPOSITION 1.11. Let E be an open set in R^d and X and L-diffusion in E. Moreover let D be a bounded domain with boundary of class $\mathcal{C}^{2+\alpha}$ such that $\bar{D} \subset E$ and $f \in \mathcal{C}^\alpha(D)$, $h \in \mathcal{C}^{2+\alpha}(\partial D)$. The the function

$$(1.9) \qquad u(x) = E^x(h(X_T)) - E^x(\int_o^T f(X_t)dt), \quad x \in D,$$

with $T=T_{R^d\setminus D}$, is the solution of the Dirichlet-Poisson problem, i.e. $u\in\mathcal{C}^{2+\alpha}(D)$, $Lu=f$ in D and $Bu=h$. Moreover for each $x\in D$ one has $E^x(T<\infty)=1$.

The statement is still true if $E=R^d$ and D is a strip.

Proof

Let us denote by v the solution of the Dirichlet-Poisson problem: $v\in\mathcal{C}^{2+\alpha}(D)$, $Lu=f$ in D and $Bv=h$. Since the boundary is of class $\mathcal{C}^{2+\alpha}$, we can extend v as a function of class $\mathcal{C}^{2+\alpha}$, in R^d. Let w be such an extension which moreover has compact support and supp $w\subset E$. We know that

$$w(X_t)-\int_o^t Lw(X_s)ds$$

is a martingale. Stopping this martingale at T and taking expectation we get

$$(*)\qquad v(x)=E^x(v(X_{t\wedge T}))-E^x\left(\int_o^{t\wedge T} Lv(X_s)\,ds\right) \quad .$$

Now let us assume that $h\equiv 1$ and $f\equiv 0$. Then $v\equiv 1$ and for $x\in D$, the above relation gives

$$1=v(x)=E^x(t\wedge T<\zeta)$$

Since t is arbitrary in this relation, we deduce $T\leq\zeta$, P^x-a.s. On the other hand supposing $h\equiv 0$ and $f\equiv 1$ we get

$$v(x)=E^x(v(X_{t\wedge T}))-E^x\left(\int_o^{t\wedge T} 1(X_s)ds\right) \quad .$$

For $s<\zeta$ we have $1(X_s)=1$. (The usual convention from the theory of standard processes is that any function $\psi:E\rightarrow R$ is extended on E_Δ such that $\psi(\Delta)=0$. Therefore we have $1(X_s)=0$ if $s\geq\zeta$.)

Because v is bounded the above relation shows that $E^x(T)<\infty$. In particular T is finite, and hence $\lim_{t\to\infty} v(X_{t\wedge T})=v(X_T)$. Letting $t\to\infty$ in relation (*) we deduce that v coincides with function u given by (1.9). The proof is complete.

REMARK 1.12. Let E be an open set in R^d and X a standard process with continuous paths with state space E. Suppose that for any bounded domain D with boundary of class $\mathscr{C}^{2+\alpha}$ such that $\bar{D}\subset E$ and for any $h\in\mathscr{C}^{2+\alpha}(\partial D)$, $f\in\mathscr{C}^{\alpha}(D)$ the solution u of the Dirichlet-Poisson problem satisfy relation (1.9). Then X is an L-diffusion in E.

We will not use this characterisation of L-diffusions and therefore we do not insist on the proof.

PROPOSITION 1.13. Let E be an open set in R^d and X, $\bar{X}$ two L-diffusions in E. Then the transition functions of X and $\bar{X}$ coincide.

Proof

Let us consider a bounded domain D with boundary of class $\mathscr{C}^{2+\alpha}$ such that $\bar{D}\subset E$. We denote by (Q_t) (resp. $(\bar{Q}_t)$) the transition function of the process X (resp. $\bar{X}$) restricted to D:

$$Q_t(x,A)=E^x(1_A(X_t),\ t<T_{E\setminus D}),\quad x\in D,\ A\in\mathcal{B}(D).$$

A similar formula expresses $\bar{Q}_t(x,A)$. We will show first that these transition functions coincide.

This will follow once we have proved that the resolvents $(U_\lambda)_{\lambda\geq 0}$ and $(\bar{U}_\lambda)_{\lambda\geq 0}$ given by

$$U_\lambda=\int e^{-\lambda t}Q_t dt,\quad \bar{U}_\lambda=\int e^{-\lambda t}\bar{Q}_t dt\quad ,$$

coincide. In order to prove this we take $f \in \mathcal{C}^{\alpha}(D)$. Then by relation (1.9) we know that $U_o f$ and $\bar{U}_o f$ both belong to $\mathcal{C}^{2+\alpha}(D)$ and coincide with the solution of the Poisson problem. Chosing $f \equiv 1$ we observe that $U_o 1 = \bar{U}_o 1$ is a bounded function and so $U_o = \bar{U}_o$ may be viewed as a bounded operator on $\mathcal{B}_b(D)$. On the other hand the resolvent equation leads to the following relation

$$U_\beta = \sum_{n=0} (\lambda-\beta)^n U_\lambda^{n+1} ,$$

which holds provided that $|\lambda-\beta| \; \|U_\lambda\| < 1$. Of course this relation holds also with respect to $(\bar{U}_\beta)$. Thus from $U_o = \bar{U}_o$ we get $U_\beta = \bar{U}_\beta$ for $\beta < \| U_o \|^{-1}$. Let us put $\Lambda = \{\lambda \in [0,\infty): U_\lambda = \bar{U}_\lambda$. Reasoning as above we deduce that Λ is an open set. The continuity of the resolvents shows that Λ is closed, and therefore $\Lambda = [0,\infty)$. This implies $Q_t = \bar{Q}_t$.

Now let us choose a sequence (D_n) of bounded domains with boundaries of class $\mathcal{C}^{2+\alpha}$ such that $\bar{D}_n \subset D_{n+1}$, n=1,2,... and $E = \bigcup_n D_n$. The transition functions of the restrictions of X and $\bar{X}$ to D_n coincide for each n and, in the limit, we have that the transition functions of X and $\bar{X}$ coincide, completing the proof.

LEMMA 1.14. Let E be an open set in R^d and X an L-diffusion in E. Then a.s. the left limit at the life-time ζ, $X_{\zeta-}(\omega)$ exists in R^d and $X_{\zeta-}(\omega) \in \partial E$ provided that $\zeta(\omega) < \infty$. If $E = R^d$, then $\zeta \equiv \infty$, a.s.

Proof

First let us consider an L-diffusion X in R^d. Since $P_t 1 = 1$ we have $E^x(\zeta > t) = 1$ for each $x \in R^d$, which shows that $\zeta \equiv \infty$ a.s.

Now let us suppose that E is an open subset of R^d and denote by X' the restriction of X to E.

If $X=(\Omega,\mathcal{M},\mathcal{M}_t,X_t,\theta_t,P^x)$ is our L-diffusion in R^d, then $X'=(\Omega,\mathcal{M},\mathcal{M}_t,X_t',\theta_t',P^x)$, where X_t' and θ_t' are defined by

$$X_t'(\omega)=\begin{cases} X_t(\omega), & \text{if } t<T_{R^d\setminus E}(\omega), \\ \Delta, & \text{if } t\geq T_{R^d\setminus E}(\omega), \end{cases}$$

$$\theta_t'(\omega)=\begin{cases} \theta_t(\omega), & \text{if } t<T_{R^d\setminus E}(\omega), \\ \omega_\Delta, & \text{if } t\geq T_{R^d\setminus E}(\omega). \end{cases}$$

The life-time ζ' of X' coincides with $T_{R^d\setminus E}$. Thus X' has the property asserted by the lemma. Further we define the canonical function space type process with state space E. First for a function $\omega:[0,t)\to E_\Delta=E\cup\{\Delta\}$, we put $\zeta(\omega)=\inf\{t\in[0,\infty):\omega(t)=\Delta\}$. Then we set

$$W_E=\{\omega:[0,\infty)\to E_\Delta\}$$

$$\Omega_E=\{\omega\in W_E:\ \text{if } \zeta(\omega)<\infty, \text{ then } \omega(\zeta-) \text{ exists in } R^d \text{ and } \omega(\zeta-)\in\partial E\},$$

$$Y_t(\omega)=\omega(t),\quad t\in[0,\infty),\ \omega\in W_E,$$

$$\mathcal{F}=\sigma(Y_t;\ t\in[0,\infty)).$$

Now we define the map $\pi:\Omega_E\to W_E$ as follows

$$\pi(\omega)(t)=X_t'(\omega).$$

Because a.s. $X'_{\zeta-} \in \partial E$ if $\zeta < \infty$, it follows that $P^x \circ \Pi^{-1}$ is carried by Ω_E. If $\bar{X} = (\bar{\Omega}, \bar{\mathcal{M}}, \bar{\mathcal{M}}_t, \bar{X}_t, \bar{\theta}_t, \bar{P}^x)$ is another L-diffusion in $E^{\cdot}$, then we define $\bar{\Pi}: \bar{\Omega} \to W_E$,

$$\bar{\Pi}(\omega)(t) = \bar{X}_t(\omega) \quad .$$

The measures $\bar{P}^x \circ \bar{\Pi}^{-1}$ and $P^x \circ \Pi^{-1}$ should coincide, because, on account of Lemma 1.13, they are determined by the same transition function. Thus $\bar{P}^x \circ \bar{\Pi}^{-1}$ is also carried by Ω_E. The proof is complete.

THE GREEN FUNCTION

In this sub-section X will be a fixed L-diffusion in R^d. If E is an open set in R^d such that $\partial E \neq \emptyset$ and $h \in \mathcal{B}_b(\partial E)$, then we introduce the notation

$$H^E h(x) = E^x(h(X(T_{R^d \setminus E}))), \quad x \in E \ .$$

If h would be defined on R^d, then of course $H^E h(x) = P_{R^d \setminus E} h(x)$ for $x \in E$.

LEMMA 1.15. Let E be an open set such that $\partial E \neq \emptyset$. If $h \in \mathcal{B}_b(\partial E)$ and B is a bounded open set such that $\bar{B} \subset E$, then $H^E h \in \mathcal{C}^{2+\alpha}(B)$ and $LH^E h = 0$ in E. If E is a strip or a bounded domain with boundary of class $\mathcal{C}^{2+\alpha}$, then one has

$$(1.10) \qquad {}^{\partial E}\| H^E h \|_{2+\alpha} \leq C \| h \| , \quad h \in \mathcal{B}_b(\partial E) \quad .$$

Moreover if h is continuous, then one has

$$(1.11) \qquad \lim_{\substack{y\to x\\ y\in E}} H^E h(y)=h(x), \quad x\in\partial E .$$

Proof

If B is as in the statement we can chose an open set D which is the union of a finite number of bounded domains with boundary of class $\mathcal{C}^{2+\alpha}$ such that $\bar{B}\subset D$ and $\bar{D}\subset E$. Putting $u=H^E h$, from the strong Markov property we get $u(x)=H^D u(x)$, $x\in D$. This shows that we may suppose from the begining that E is a bounded domain with boundary of class $\mathcal{C}^{2+\alpha}$. If $h\in\mathcal{C}^{2+\alpha}(E)$, then from Proposition 1.11 we know that $H^E h$ is the solution of a Dirichlet problem. On the other hand we have

$$\|H^E h\| \leq \|h\| \ , \quad \text{for each } h\in\mathcal{B}_b(\partial E) \ .$$

Therefore if $h\in\mathcal{C}(\partial E)$ we approximate uniformly h by a sequence (h_n) of functions in $\mathcal{C}^\infty(\partial E)$. The preceding inequality shows that $H^E h_n$ uniformly approximate $H^E h$ and relation (1.11), which holds for the functions h_n is obtained also for h. Relation (1.10) follows by a monotone class argument.

DEFINITION 1.16. Let E be an open set in R^d and u a function that possesses continuous second order derivatives. We say that u is L-harmonic in E if $Lu=0$ in E.

LEMMA 1.17. Let $u\in\mathcal{B}(E)$ be such that it is bounded on each compact subset of E.

(a) If $(D_i)_{i\in I}$ is an open covering of E such that $\bar{D}_i\subset E$ and

$$H^{D_i}u(x)=u(x), \quad x\in D_i \quad ,$$

for each $i \in I$, then u is L-harmonic in E.

(b) If u is L-harmonic in E and if D is a bounded open set such that $\overline{D} \subset E$, then $u \in \mathcal{C}^{2+\alpha}(D)$ and one has

(1.12) $\qquad H^D u(x) = u(x), \quad x \in D.$

Proof

(a) From the preceding lemma we know that u is L-harmonic in each of the sets D_i , therefore it is L-harmonic in E.

(b) We may conisder that D is included in a bounded domain B with boundary of class $\mathcal{C}^{2+\alpha}$ such that $\overline{D} \subset B$ and $\overline{B} \subset E$. Then the function

$$v(x) = u(x) - H^B u(x), \quad x \in B,$$

is L-harmonic in B and vanishes on the boundary. The maximum principle implies $v \equiv 0$, which proves (1.12) with B. Then the strong Markov property leads to (1.12) with D. From (1.10) we get $u \in \mathcal{C}^{2+\alpha}(D)$.

From now on we assume that the dimension of the space satisfies $d \geq 3$. Then the Green function associated to L on R^d has the expression

$$g(x,y) = \int_0^\infty p(t,x,y)\,dt, \quad x,y \in R^d .$$

The first estimation in (1.3 iii) shows that $g(x,y) < \infty$ if $x \neq y$. In the case d=2 this is not true. Here is the reason we are assuming $d \geq 3$ in the rest of this paper. The next lemma gathers together some basic properties of Green's function.

LEMMA 1.18.

(a) There exist $C>0$ and $r>0$ such that

(1.13) $$g(x,y)\leq C|x-y|^{2-d}\ ,\quad x\neq y\ ,$$

(1.14) $$C^{-1}|x-y|^{2-d}\leq g(x,y)\ ,\quad x\neq y,\quad |x-y|<r\ .$$

(b) One has $g(x,x)=\infty$ for each $x\in R^d$ and g is continuous on $R^d x R^d$.

(c) For each $y\in R^d$ the function $g(\cdot,y)$ is L-harmonic in $R^d\setminus\{y\}$. Moreover there exists a constant $C>0$ such that

(1.15) $$|D_i^x g(x,y)|\leq C|x-y|^{1-d}\ ,$$

$$|D_{ij}^x g(x,y)|\leq C|x-y|^{-d}\ ,\quad x\neq y\ .$$

(d) For each $y\in R^d$, $g(\cdot,y)$ is excessive.

Proof

(a) Relation (1.13) follows from the first inequality in (1.3 iii). To prove (1.14) we use (1.3 iv) and first obtain

$$\left|g(x,y)-\int_0^\infty p_y(t,x,y)\,dt\right|\leq C|x-y|^{2+\alpha-d}\ .$$

An examination of the expression of $p_z(t,x,y)$ shows that

$$c_1 t^{-d/2}\exp(-c_2|x-y|^2/t)\leq p_z(t,x,y)\ ,$$

which leads to

$$c|x-y|^{2-d}\leq\int_0^\infty p_z(t,x,y)\,dt\ .$$

From these estimates one easily deduces the existence of $r>0$ such that (1.14) holds.

(b) As in the preceding situation we use (1.3 iv) to obtain an estimate of the form

$$C_1 t^{-d/2} - Ct^{-\frac{d-\alpha}{2}} \leqslant p(t,x,x),$$

which shows that $g(x,x)=\infty$. The continuity of g on the diagonal follows from estimate (1.14). The continuity outside the diagonal is a consequence of estimate (1.3 iii) and of the fact that $p(t,x,y)$ is continuous.

(c) The estimates from (1.3 iii) show that we may interchange derivation with the integral so that

$$D_i^x g(x,y) = \int_o^\infty D_i^x p(t,x,y)dt, \quad x \neq y \ ,$$

$$D_{ij}^x g(x,y) = \int_o^\infty D_{ij}^x p(t,x,y)dt, \quad x \neq y \quad ,$$

and these immediately lead to (1.15). On the other hand for $x \neq y$ we have

$$L^x g(x,y) = \int_o^\infty L^x p(t,x,y)dt = \int_o^\infty \frac{\partial}{\partial t} p(t,x,y)dt \quad .$$

Since $\lim_{t\to 0} p(t,x,y) = \lim_{t\to\infty} p(t,x,y) = 0$ we observe that the last term is nul, proving that $g(\cdot,y)$ is L-harmonic.

(d) We have

$$P_t g(\cdot,y)(x) = \int_t^\infty p(s,x,y)ds \quad ,$$

which allows (us) to deduce that $g(\cdot,y)$ is excessive. The lemma is proved.

We will denote by V the potential kernel of the process X. It is expressed by

(1.16) $$Vf(x)=E^x\left(\int_0^\infty f(X_t)\,dt\right)=$$

$$=\int_{R^d} g(x,y)f(y)\,dy,\quad f\in\mathcal{B}_+(R^d),\qquad x\in R^d.$$

LEMMA 1.19.

(a) If $f\in\mathcal{B}_b(R^d)$ has compact support, then Vf is a bounded continuous function and is L-harmonic in $R^d\setminus \text{supp } f$.

(b) If $f\in\mathcal{C}^\alpha(R^d)$ has compact support, then Vf has continuous second order derivatives and $LVf=-f$.

<u>Proof</u>

(a) From the estimate (1.13) we deduce that the functions $\{g(x,\cdot),\ x\in R^d\}$ are uniformly integrable on any compact K. Taking K=supp f we deduce that, Vf is bounded and continuous. Then in order to prove L-harmonicity we are going to use Lemma 1.17. So let B be any open ball such that $\overline{B}\cap K=\emptyset$ and put $T=T_{R^d\setminus B}$. For $x\in B$ we have

$$H^B Vf(x)=E^x(Vf(X_T))=\int_K E^x(g(X_T,y))f(y)\,dy=$$

$$=\int_K H^B g(\cdot,y)(x)f(y)\,dy=\int_K g(x,y)f(y)\,dy$$

The last equality holds because $g(\cdot,y)$ is L-harmonic in $R^n\setminus K$ if $y\in K$. Thus Vf satisfies the condition from Lemma 1.17 (a) and hence it is L-harmonic in $R^d\setminus K$.

(b) We write $Vf(x)=u(x,t)+v(x,t)$, with $t>0$ and u, v given by

$$u(x,t)=\int_o^t P_s f(x)ds \quad ,$$

$$v(x,t)=\int_t^\infty P_s f(x)ds=P_t Vf(x) \quad .$$

By Theorem 1.2 we know that u and v possess continuous second order derivatives with respect to x, first order with respect to t and

$$(L-D^t)u(x,t)=-f(x) \quad ,$$

$$(L-D^t)v(x,t)=0 \quad .$$

As one can easily see $D^t u(x,t)=P_t f(x)$ and $D^t v(x,t)=-P_t f(x)$. These relations lead to $L(u+v)=-f$, which completes the proof.

LEMMA 1.20. Let D be a strip or a bounded domain with boundary of class $\mathcal{C}^{2+\alpha}$. Then any point $x\in\partial D$ is regular for $R^d\setminus D$.

Proof

Let $f\in\mathcal{B}_b(R^d)$ be such that supp f is compact. Then the function u defined by

$$u(x)=\begin{cases} Vf(x), & \text{if } x\ R^d\setminus D, \\ H^D Vf(x), & \text{if } x\in D, \end{cases}$$

is continuous, on account of relation (1.11). The function $v=P_{R^d\setminus D}Vf$ is excessive and u=v in $R^d\setminus\partial D$. Since ∂D is negligible with respect to Lebesgue measure we have $P_t u=P_t v$ for each $t>0$. By (1.3 ii) we have $\lim_{t\to 0} P_t u=u$ and since v is excessive we also have $\lim_{t\to 0} P_t v=v$. Therefore we get u=v. For $x\in\partial D$ we may write

$$E^x(\int_0^\infty f(X_t)dt)=Vf(x)=P_T Vf(x)=E^x(\int_T^\infty f(X_t)dt),$$

with $T=R^d\setminus D$, which shows that

$$E^x(\int_0^T f(X_t)dt)=0 .$$

The function f is arbitrary and so we may choose it so that $f>0$ in a neighbourhood of x. Then the preceding relation implies $P^x(T>0)=0$, completing the proof.

Now we introduce some notation related to an arbitrary open set $E\subset R^d$. The potential kernel of an L-diffusion in E coincides with the potential kernel of the restriction of X to E. We denote it by V^E. It is given by

(1.17) $$V^E f(x)=E^x(\int_0^T f(X_t)dt), \quad f\in\mathcal{B}_+(E), \quad x\in E,$$

with $T=T_{R^d\setminus E}$. If $f\in\mathcal{B}_b(R^d)$ has compact support, then

(1.18) $$V^E f(x)=Vf(x)-P_{R^d\setminus E}Vf(x), \quad x\in E .$$

We denote by g^E the Green function in E. It is defined by

(1.19) $$g^E(x,y)=g(x,y)-h(x,y), \quad x,y\in E ,$$

where h is given by

$$h(x,y)=E^x(g(X_T,y)) .$$

From relation (1.18) we immediately get

(1.20) $$V^E f(x)=\int_E g^E(x,y)f(y)dy , \quad f\in\mathcal{B}_+(E), \quad x\in E .$$

The main properties of Green's function in E are listed in the following lemma. The proof is omitted because is easy.

LEMMA 1.21

(a) The function h is finite and continuous in $E \times E$. For each fixed $y \in E$, the function $h(\cdot, y)$ is L-harmonic in E.

(b) The function g^E is continuous in $E \times E$ and for each $y \in E$, $g(\cdot, y)$ is excessive in E and L-harmonic in $E \setminus \{y\}$. Moreover one has

$$g^E(x,y) \leq g(x,y), \quad x,y \in E$$

(c) For each compact set $K \subset E$, there exists $r > 0$ and $C > 0$ such that

$$C^{-1} |x-y|^{2-d} \leq g^E(x,y), \quad x \in E, \quad y \in K, \quad |x-y| < r .$$

(d) Assume that E is a strip or a bounded domain with boundary of class $\mathcal{C}^{2+\alpha}$. Then for each fixed $y \in E$, the function $g^E(\cdot, y)$ can be extended to $\overline{E}$ as a continuous function vanishing on ∂E.

Green potentials

There are several approaches to Green's function and Green potentials in the axiomatic potential theory as well as in probabilistic potential theory. Chapter VI of the book [B-G] and the book [M] are most familiar to probabilists. We will prefer the approach presented in [M] because the hypotheses K-W (of Kunita-Watanabe) are very easy to check in our case.

Lemma 1.22

Let E be an open set in R^d. Then the semigroup of an L-diffussion in E satisfies the hypotheses K-W with Lebesgue measure as duality measure. For each $x \in E$ the function $g^E(x, \ )$ is coexcessive and the dual potential kernel is given by

$$\hat{U}(dx,y) = g^E(x,y)dx, \ y \in E.$$

Proof.

In the case $E = R^d$ we can write explicitely the dual semigroup

$$\hat{P}_t(dx,y) = \rho(t,x,y)dx.$$

The hypotheses K-W can be easily checked on account of Lemma 1.7, estimation (1.13) and using the continuity of $\rho(t,x,y)$,

Now let E be an arbitrary open set. We will work with a process X which is an L-diffusion in R^d. Its restriction to E, which is an L-diffusion in E, has the following resolvent

$$V^E_\lambda f(x) = E^x\left(\int_0^T e^{-\lambda t} f(X_t)dt\right). \ f \in \mathcal{B}_+(E), x \in E,$$

with $T = T_{R^d \setminus E}$. Denoting by (V_λ) the resolvent of X we can write

$$V^E_\lambda f(x) = V_\lambda f(x) - E^x(e^{-\lambda T} V_\lambda f(X_T)).$$

Then we introduce the notation

$$g_\lambda(x,y) = \int_0^\infty e^{-\lambda t}\rho(t,x,y)dt, \lambda \geq 0, \ x,y \in R^d,$$

$$h_\lambda(x,y) = E^x(e^{-\lambda T} g_\lambda(X_T,y)),$$

$$g^E_\lambda(x,y) = g_\lambda(x,y) - h_\lambda(x,y), \quad \lambda \geq 0, x,y \in E$$

Thus we deduce

$$V_\lambda^E f(x)=\int_E g_\lambda^E(x,y)f(y)dy,\ x\in E,\ f\in \mathcal{B}_+(E).$$

Now we assert that g_λ^E is continuous on $E\times E$ for each $\lambda\geq o$.

Of course for $\lambda=o$ we have $g_o^E=g^E$ and we know it is continuous.

For $\lambda>o$ we should repeat the reasoning. First, the relation

$$e^{-\lambda t}P_t f-f-\int_o^t e^{-\lambda s}P_s(L-\lambda)f\,ds,\ f\in \mathcal{C}^2(R^d)$$

is analogous to that in Lemma 1.8. The operator $L-\lambda$ should be used instead of L. The martingale from (1.7) should be replased by

$$\exp(-\lambda t)f(X_t)-\int_o^t \exp(-\lambda s)(L-\lambda)f(X_s)ds.$$

Formula (1.9) is generalised so that

$$u_\lambda(x)=E^x(\exp(-\lambda R)h(X_R)-E^x\left(\int_o^R \exp(-\lambda t)f(X_t)dt\right),x\in D,$$

with $R=T_{R^d\setminus D}$, is the solution of the Dirichlet-Poisson problem with respect to $L-\lambda$, i.e. $(L-\lambda)u_\lambda=f$ in D and $u_\lambda=h$ on ∂D.

We conclude that $h_\lambda(x,y)$ is continuous, which leads to the continuity (of g_λ^E).

Further to verify the hypotheses K-W everything is obvious except the following relation

$$\lim_{\lambda\to\infty}\lambda\int_E f(x)\ g_\lambda^E(x,y)\ dx=f(y),f\in\mathcal{C}_c(E),\ y\in E.$$

This relation will be proved once we have established the following

$$(*)\qquad \lim_{\lambda\to\infty}\lambda\int_E f(x)\ h_\lambda(x,y)\ dx=o.$$

From the first estimation in (1.3.iii) we have

$$\lambda g_\lambda(x,y)\leq C_1\int_o^\infty \lambda e^{-\lambda t}\ t^{-d/2}e^{-C_2r^2/t}\,dt,$$

provided that $|x-y|\geq r$. Putting $M(r)=\sup_t\{t^{-d}\exp(-C_2r^2/t)$ we get $\lambda g_\lambda(x,y)\leq C_1M(r)$. Then for a fixed $y\in E$ we put $r=d(y,\partial E)$ and obtain

$\lambda h_\lambda(x,y)\leq C_1\ M(r)E^x(e^{-\lambda T})$. From this relation we easily get relation (*), completing the verification of the hypotheses K-W.

Now from the resolvent equation we have

$$g^E(x,y)=g_\lambda^E(x,y)+\lambda\int_E g^E(x,z)g_\lambda(z,y)dz,$$

From estimate (1.3.iii) we deduce $\lim_{\lambda \to \infty} g_\lambda(x,y) = 0$, and hence $\lim_{\lambda \to \infty} g^E_\lambda(x,y) = 0$. Therefore letting $\lambda \to \infty$ in the preceding relation we deduce that $g^E(x, \;)$ is co-excessive, completing the proof.

Remark.

The dual semigroup $(\hat{P}_t)$ is not sub-Markov in general. More precisely under the hypotheses of Theorem 1.6 the semigroup $(\hat{P}_t)$ is sub-Markov if and only if the coeficient c^* appearing there satisfies $c^* \leq 0$. Therefore we could not use directly the results of Chapter VI in [B-G].

If μ is a Radon measure in E we introduce the notation

$$G^E_\mu(x) = \int_E g^E(x,y)\,\mu(dy),\ x \in E.$$

This function is called the Green potential in E of measure μ. Since $g^E(\cdot,y)$ is excessive in E for each $y \in E$, one easily deduces that G^E_μ is excessive in E for each μ. As a particular case of Theorem T9 from page 66 in [M] we have the following result.

Theorem 1.23

Let E be an open set in R^d and u an excessive function in E. Suppose that there exists $f \in \mathcal{B}_+(E)$ such that $V^E f$ is finite and $u \leq V^E f$. Then there exists a unique Radon measure μ such that $u = G^E_\mu$. Let M be a closed set in E such that $P_M u = u$, where P_M is considered with respect to an L-diffusion in E. Then $\operatorname{supp}\mu \subset M$.

The next lemma gather together some properties of Green potentials.

Lemma 1.24

Let X be an L-diffusion in an open set $E \subset R^d$.

(a) If D is an open set such that $D \subset E$ and μ is a Radon measure such that $\operatorname{supp}\mu \subset D$, then $P_D\, G^E_\mu = G^E_\mu$ and

(2.21) $G^E_\mu(x) = G^D_\mu(x) + P_{E \setminus D}\, G^E_\mu(x)$, $x \in D$.

(b) Let μ be such that G^E_μ is bounded. Then G^E_μ is L-harmonic in $E \setminus \operatorname{supp}\mu$. If G^E_μ is L-harmonic in an open set $D \subset E$, then $\operatorname{supp}\mu \cap D = \emptyset$.

Proof.

(a) The first assertion follows from Theorem T6 of page 65 in [M2]. Relation (1.21) is a consequence of the following equality

$$g^E(x,y) = g^D(x,y)+E^x(g^E(X_T,y)),\ x \in D,\ y \in D,\ T=T_{E\setminus D},$$

which results directly from (1.19).

(b) The first assertion follows from Lemma 1.15. Now let us suppose that $\operatorname{supp}\mu \cap D \neq \emptyset$. Then we take $x_o \in \operatorname{supp}\mu \cap D$ and choose $0<\delta < d(x_o, CD)$. Then the set $B=B(x_o,\delta)$ satisfies $\bar{B} \subset D$. From Lemma 1.21. (c) we get $C > 0$ and $r \in (0,\delta)$ such that

$$C^{-1}|x_o-y|^{2-d} \leq g^B(x_o,y),\ y \in B(x_o,r).$$

Since $x_o \in \operatorname{supp}\mu$, it follows that the measure $\nu = 1_{B(x_o,r)}\cdot\mu$ is non-null. Thus from the preceding estimate we have

$$G^B_\nu(x_o) \geq C^{-1}r^{2-d}\nu(1) > 0.$$

and using relation (1.21) we obtain

$$(*)\quad G^E_\nu(x_o)-P_{E\setminus B}\,G^E_\nu(x_o) > 0.$$

Let us put $\tau = \mu - \nu$. Since G^E_μ is L-harmonic in D, from Lemma 1.17. (b) we get

$$G^E_\tau + G^E_\nu = G^E_\mu = P_{E\setminus B}G^E_\mu = P_{E\setminus B}\,G^E_\tau + P_{E\setminus B}G^E_\nu.$$

On the other hand the functions G^E_τ and G^E_ν are excessive, and so

$$G^E_\tau \geq P_{E\setminus B}G^E_\tau,$$

$$G^E_\nu \geq P_{E\setminus B}G^E_\nu.$$

These inequalities and the preceding equality shows that $G^E_\nu = P_{E\setminus B}G^E_\nu$, which is in contradiction with the relation $(*)$. Therefore our supposition fails, and hence $\operatorname{supp}\mu \cap D=\emptyset$. The proof is complete.

Lemma 1.25

Let E be an open set in R^d and μ, μ_n, $n \in N$ Radon measures in E such that

(i) $\lim_n \mu_n(f) = \mu(f)$, $f \in \mathcal{C}_c(E)$,

(ii) $\lim_{r \to o} \limsup_n \| G^E_{\mu_n(a,r)} \| = 0$, $a \in E$,

(iii) $\lim_{r \to o} \| G^E_{\mu(a,r)} \| = 0$, $a \in E$,

where $\mu_n(a,r) = 1_{B(a,r)} \cdot \mu_n$, $\mu(a,r) = 1_{B(a,r)} \cdot \mu$.

(a) Then for each $f \in \mathcal{C}_c(E)$, $f \geqslant 0$ one has

$$\lim_n \| G^E_{f \cdot \mu_n} - G^E_{f \cdot \mu} \| = 0$$

(b) Moreover if there exists $f \in \mathcal{B}_+(E)$ such that the function $v = V^E f$ is finite and G^E_μ, $G^E_{\mu_n} \leqslant v$, $n \in N$, then for any compact $K \subset E$ one has

$$\lim_n \| G^E_{\mu_n} - G^E_\mu \|^K = 0$$

Proof.

(a) Let $f_o \in \mathcal{C}_c(E)$, $f \geqslant 0$ be fixed. We choose a compact (set) M such that supp $f \subset M$. Conditions (ii) and (iii) together with a compacity argument show that for each fixed $\varepsilon > 0$ there exists $r > 0$ such that

$$\| G^E_{\mu(a,r)} \| < \varepsilon \text{ and } \| G^E_{\mu_n(a,r)} \| < \varepsilon$$

for each $a \in M$ provided n is large enough. Further we choose $\psi \in \mathcal{C}(E \times E)$ such that $0 \leqslant \psi \leqslant 1$, $\psi(x,y) = 0$ if $|x-y| \leqslant r/2$, and $\psi(x,y) = 1$ if $|x-y| \geqslant r$. Then the functions in the family $\{\psi(x,\cdot) g^E(x,\cdot) f(\cdot) : x \in M\}$ are equal uniform continuous. Therefore the measures μ_n converge to μ uniform with respect to this family of functions. Thus we have

$$(*) \left| \int \psi(x,y) g^E(x,y) f(y) \mu_n(dy) - \int \psi(x,y) g^E(x,y) f(y) \mu_n(dy) \right| < \varepsilon,$$

for each $x \in M$, provided n is large enough. On the other hand sinc $0 \leqslant 1-\psi(x,\cdot) \leqslant 1_B(x,r)$, we have

$$0 \leqslant \int g^E(x,y)f(y)\mu_n(dy) - \int \psi(x,y)g^E(x,y)f(y)\mu_n(dy) \leqslant$$

$$\leqslant \|f\|\, G^E_{\mu_n(x,r)}(x) \leqslant \varepsilon \|f\|,$$

for each $x \in M$ and n large. A similar estimate holds with μ instea of μ_n. These estimates and (*) lead to

$$\left|\int g^E(x,y)f(y)\mu_n(dy) - \int g^E(x,y)f(y)\mu(dy)\right| \leqslant \varepsilon(2\|f\|+1),$$

for each $x \in M$ and n large. With our notation this can be written as

$$\left| G^E_{f\cdot\mu_n}(x) - G^E_{f\cdot\mu}(x)\right| \leqslant \varepsilon(2\|f\|+1),\ x \in M.$$

By Lemma 1.24 (a) we have $P_M G^E_{f\cdot\mu} = G^E_{f\cdot\mu}$ and $P_M G^E_{f\cdot\mu_n} = G^E_{f\cdot\mu_n}$, because M has been choosen so that supp $f \subset M$. Thus the preceding estimate leads to

$$\| G^E_{f\cdot\mu_n} - G^E_{f\cdot\mu}\| \leqslant \varepsilon(2\|f\|+1),$$

which completes the proof of (a).

(b) We choose a sequence of bounded open sets such that $\overline{B}_n \subset B_{n+1}$, and $E = \bigcup_n B_n$. Working with a process X which is an L-diffusion in E we can write

$$P_{E\setminus B_n} v(x) = E^x\left(\int_{T_n}^{\infty} f(X_t)\,dt\right),\ x \in E,$$

with $T_n = T_{E\setminus B_n}$. Since $\lim_n T_n \geqslant \zeta$, we get

$$\lim_n P_{E\setminus B_n} v(x) = 0,\ x \in E.$$

Since $P_{E\setminus B_n} v$ is L-harmonic in B_n, it follows that the preceding limit relation holds uniformly on each compact set. Now let K be a fixed compact set and $\varepsilon > 0$. We choose k so that $B_k \supset K$ and

$$\| B_{E\setminus B_k} v\|^K < \varepsilon .$$

Then we choose $f \in \mathcal{C}_c(E)$ such that $0 \leqslant f \leqslant 1$ in E and f=1 on a neighbourhood of $\overline{B}_k$. If we denote by h=1-f, we can write supp $h \cap \overline{B}_k=\emptyset$. Applying Lemma 1.24 (a) we get

$$G^E_{h\cdot\mu} = P_{E\setminus B_k} G^E_{h\cdot\mu} \leq P_{E\setminus B_k} G^E_{\mu} \leq P_{E\setminus B_k} v ,$$

and similarly $G^E_{h\cdot\mu_n} \leq P_{E\setminus B_k} v$. It follows

$$\|G^E_{h\cdot\mu}\|^K < \varepsilon \ , \ \|G^E_{h\cdot\mu_n}\|^K < \varepsilon.$$

Combining these estimates with assertion (a) we obtain the dessired conclusion, completing the proof.

Lemma 1.26

Let E be an open set and μ a Radon measure in E such that G^E_μ is bounded. Suppose that $a \in E$ is such that

$$(1.22) \quad \lim_{r\to 0} \| G^E_{\mu_r} \| = 0,$$

where $\mu_r = 1_{B(a,r)}\cdot\mu$. Then the function G^E_μ is continuous at a.

Proof.

We put $\mu'_r=\mu-\mu_r$. The function $G^E_{\mu'_r}$ is L-harmonic in B(a,r), in particular continuous. Since

$$\| G^E_\mu - G^E_{\mu'_r} \| = \| G^E_{\mu_r} \| ,$$

relation (1.22) implies the continuity of G^E_μ.

Remark

Relation (1.22) characterises the continuity, i.e. if G^E_μ is continuous in a, then relation (1.22) holds.

Lemma 1.27

Assume that the coeficients of L satisfies the conditions $a^{ij} \in \mathcal{C}^{2+\alpha}(R^d)$ and $b^i \in \mathcal{C}^{1+\alpha}(R^d)$, $i,j \leq d$ so that the adjoint operator L^* has coefficients of class $\mathcal{C}^\alpha(R^d)$. Then for each $x \in R^d$ the function g(x,·) possesses continuous second order derivatives in $R^d \setminus \{x\}$. Moreover there exists $C>0$ such that

$$|D_i^y\ g(x,y)| \leqslant C|x-y|^{1-d} \quad ,x,y \in R^d ,x\neq y$$

$$|D_{ij}^y\ g(x,y)| \leqslant C|x-y|^{-d} \quad ,x,y \in R^d,x\neq y.$$

Proof.

It follows from Theorem 1.6 and Lemma 1.18.(C).

Lemma 1.28

Assume the hypotheses of the preceding lemma.Let D be a bounded domain with boundary of class $\mathcal{C}^{2+\alpha}$. If ν is a Radon measure such that G_ν^D is bounded and $h\in \mathcal{C}^2(D)\cap\mathcal{C}_c(D)$, then one has

$$\int_D h(x)\ \nu(dx)=-\int_D G_\nu^D(x)\ L^* h(x)dx.$$

Proof.

If $f\in\mathcal{C}^\infty(D)\cap\mathcal{C}_c(D)$, relation (1.9) ensures us that

$$f=-LV^Df$$

Then by the Green's formula we have

$$\int_D hf = -\int_D hLV^Df = -\int_D (L^*h)\ V^Df$$

We put

$$v(y) = \int_D L^*h(x)g^D(x,y)dx,\ y\in D.$$

The properties of g^D allow us to deduce that v is a continuous function. The preceding identity may be written as

$$\int hf = -\int vf.$$

Since f is arbitrary, it follows h = -v. Integrating this equality with $\nu(dy)$ we get the relation in the statement.

2. Functionals which Number the Crossings

In this section L will be an operator of the form(1.2)with $c \equiv$ and E an open set in R^d.X will be an L-diffusion in E and K a clos subset of E.For each $\varepsilon > 0$ we put

$$V^{\varepsilon} = \{x \in E : d(x,K) < \varepsilon\} .$$

Then we define the following stopping times

$$T = T_K, \quad S = T_{E \setminus V^{\varepsilon}},$$

$$T_1 = S + T \circ \theta_S,$$

$$T_{n+1} = T_n + T_1 \circ \theta_{T_n} \quad , \ n = 1,2\ldots$$

Of course these stopping times depend on ε .For $t \in [0,\infty)$ we put

$$(2.1) \quad A_t^{\varepsilon} = \varepsilon \sum_{n=1}^{\infty} 1_{\{T_n \leq t\}}.$$

For each $\omega \in \Omega$, the number of distinct displacemens of the path from the exterior of V^{ε} to K until the time t equals the quantity $\varepsilon^{-1} A_t^{\varepsilon}(\omega)$.The process $A^{\varepsilon} = (A_t^{\varepsilon})$ is adapted, nondecreasing, right-continuous and of pure jumps.The jumps accur only at the times T_n,n =1,2..., and are of constant length .Now let us look at the moments

$$T_n \leq T_n + S \circ \theta_{Tn} \leq T_{n+1}.$$

The following statements hold almost surely:

if $T_n(\omega) < \infty$,then $X_{Tn}(\omega) \in K$ and if $(T_n + S \circ \theta_{Tn})(\omega) < \infty$, then $X(T_n + S \circ \theta_{Tn})(\omega) \in E \setminus V^{\varepsilon}$.On account of Lemma 1.14 we deduce that one should have $\lim_n T_n = \infty$ a.s.Therefore A_t^{ε} is finite a.s. for each $t < \infty$ .We set $A_{\infty}^{\varepsilon} = \lim_{t \to \infty} A_t^{\varepsilon}$.One obviously has $T_1 > 0$a.s.,and hence $A_0^{\varepsilon} = 0$a.s.We note that

$$\int_0^{\infty} 1_{E \setminus K}(X_t)\, dA_t^{\varepsilon} = 0 \quad , \text{ a.s.}$$

The process A^{ε} looks very much like an additive functional supported by K. This feature is stressed by Lemma 2.3 from below.

Lemma 2.1

Let $t \in [0,\infty)$ and put $T_0 = 0$ and, for $n = 0,1,2\ldots$, set

$$\Lambda_{nt} = \{T_n \leq t \leq t + S\circ\theta_t < T_{n+1} < \infty\},$$

$$\Lambda'_{nt} = \{T_n < t < T_{n+1} < t + S\circ\theta_t\},$$

$$\Lambda''_{nt} = \{T_n \leq t,\ T_{n+1} = \infty\},$$

$$\Lambda_t = \bigcup_{n=0}^{\infty} \Lambda_{nt},\ \Lambda'_t = \bigcup_{n=0} \Lambda'_{nt},\ \Lambda''_t = \bigcup_{n=0} \Lambda''_{nt}.$$

Then the sets $\Lambda_{nt}, \Lambda'_{nt}, \Lambda''_{nt}$, $n = 0,1,2\ldots$ are mutually disjoint and $\Omega = \Lambda_t \cup \Lambda'_t \cup \Lambda''_t$ almost surely.

Proof.

Since $\lim_n T_n = \infty$ a.s., we have

$$\Omega = \bigcup_{n=0}^{\infty} \{T_n \leq t < T_{n+1}\} \text{ a.s.}$$

We assert that for each n

$$(*)\ \{T_n \leq t < T_{n+1}\} = \Lambda_{nt} \cup \Lambda'_{nt} \cup \Lambda''_{nt}.$$

Since the inclusion „$\supset$" is obvious we have to prove only the inclusion „$\subset$". Thus let us consider ω and n such that $T_n(\omega) \leq t < T_{n+1}(\omega)$.

If $T_{n+1}(\omega) = \infty$, then $\omega \in \Lambda''_{nt}$. If $T_{n+1}(\omega) < \infty$, then $X_{T_{n+1}}(\omega) \in K$. If besides we suppose that $t = T_n(\omega)$, then $t + S\circ\theta_t(\omega) \leq T_{n+1}(\omega) < \infty$ and so $X(t + S\circ\theta_t)(\omega) \in E\setminus V^{\varepsilon}$. This implies $t + S\circ\theta_t(\omega) < T_{n+1}(\omega)$, and hence $\omega \in \Lambda_{nt}$.

Now let us suppose that $T_n(\omega) < t < T_{n+1}(\omega) < \infty$. Suppose besides (that) $S\circ\theta_t(\omega) = \infty$. Then clearly $\omega \in \Lambda'_{nt}$. If $S\circ\theta_t(\omega) < \infty$, then $X(t + S\circ\theta_t)(\omega) = X_{S\circ\theta_t}(\omega) \in E\setminus V^{\varepsilon}$. Therefore $t + S\circ\theta_t(\omega) \neq T_{n+1}(\omega)$ and we conclude that either $\omega \in \Lambda_{nt}$ or $\omega \in \Lambda'_{nt}$, which completes the proof of $(*)$. The reminder proof is obvious.

Lemma 2.2

Assume the notation of the preceding lemma. If $\omega \in \Lambda_{nt}$, then one has

$$(2.2)\quad T_{n+k}(\omega) = t + T_k\, \theta_t(\omega),\ k=1,2\ldots$$

If $\omega \in \Lambda'_{nt}$, then one has

$$(1.3)\quad T_{n+k+1}(\omega) = t + T_k \circ \theta_t(\omega),\ k=1,2\ldots$$

If $\omega \in \Lambda''_{nt}$, then $T_1 \circ \theta_t(\omega) = \infty$, in particular relation (2.2) holds.

Proof.

First we assert that the inequality $T_n(\omega) \leq t$ implies

$$(2.4)\quad T_{n+1}(\omega) \leq t + T_1 \circ \theta_t(\omega).$$

To check this we use the following general relation

$$(2.5)\quad u + T_M \circ \theta_u = \inf\{ s > u : X_s \in M\},\ M \in \mathcal{B}(E), u \in [0,\infty).$$

This relation gives first

$$T_n(\omega) + \theta_{T_n}(\omega) \leq t + S \circ \theta_t(\omega),$$

and then using it again we get inequality (2.4).

Now let us suppose that $\omega \in \Lambda''_{nt}$. From inequality (2.4) we obtain $T_1 \circ \theta_t(\omega) = \infty$, which implies relation (2.2). Then let us suppose that $\omega \in \Lambda_{nt}$. Since in this case $X_{T_{n+1}}(\omega) \in K$, from relation (2.5) and the inequality $t + S \circ \theta_t(\omega) < T_{n+1}(\omega)$ we deduce

$$t + S \circ \theta_t(\omega) + T \circ \theta(t + S \circ \theta_t)(\omega) \leq T_{n+1}(\omega).$$

The left side of this inequality is $t + T_1 \circ \theta_t(\omega)$. Therefore we get $t + T_1 \circ \theta_t(\omega) = T_{n+1}(\omega)$, because of (2.4). Relation (2.2) follows by induction. Further let us consider the case $\omega \in \Lambda'_{nt}$. Using again relation (2.5), from the inequality $t < T_{n+1}(\omega)$ we obtain $t + T_1 \circ \theta_t(\omega) \leq T_{n+2}(\omega)$.

Similarly from the inequality $T_{n+1}(\omega) < t + S\circ\theta_t(\omega)$ we obtain $T_{n+1}(\omega) + S\circ\theta_{T_{n+1}}(\omega) \leq t + S\circ\theta_t(\omega)$ and also $T_{n+2}(\omega) \leq t+T_1\circ\theta_t(\omega)$. Relation(2.3)follows by induction.The lemma is proved.

Lemma 2.3

Assume the notation of Lemma 2.1.If t 0,) and s 0, ,then

(2.6) $A_{t+s}(\ ) = A_t(\ ) + A_s\ \theta_t(\ )$,

provided that $\quad_t \quad_t$ and

(2.7) $A_{t+s}(\ ) = A_t(\ ) + A_s\ \theta_t(\ ) +\ $,

provided that $\quad_t$.

Proof.

Relation (2.6) follows from (2.2) and relation (2.7) from (2.3).The proof is complete.

Let I be y Borel subset of K and put

(2.8) $u_I(x) = E^x(\ _0 \quad (X_t)dA_t),\ x \quad E.$

Obviously we have $u_K(x) = E^x(A\ )$.We will simply write $u = u_K$.

Lemma 2.4

For each I (K) the function u_1 is excessive.One has $u_I \quad P_I u_I +$ and $P_D u_I = u_I$provided that D is an open set such that V D.

Proof.

We treat the case I = K only,because the general case is similar.From the relations (2.6)and(2.7) we obtain

$$E^x(u(X_t)) = E^x(A \quad \theta_t) \quad E^x(A\ ) = u(x),\ x \quad E.$$

Now we are going to prove that $\lim_{t\ 0} E^x(u(X_t)) = u(x)$ for each x E. First from relation (2.5) we observe that s t implies s + θ_s t + θ_t.Therefore the sets $_t$ = t + θ_t T_1 ,t 0,) satisfy the relation $_t \quad _s$ provided that s t.Then we remark that $\lim_{t\ 0}$ t + θ_t = ,and hence

$$_{t\ 0}\ _t = S \quad T_1 .$$

Since $\{S < T_1\} = \{S < \infty\}$ we deduce that

$$\bigcup_{t>0} \Omega_t \cup \{S = \infty\} = \Omega .$$

By Lemma 2.3 we have

$$A^{\varepsilon}_{\infty}(\omega) = A^{\varepsilon}_{t}(\omega) + A^{\varepsilon}_{\infty} \circ \theta_t(\omega),$$

provided that $\omega \in \Omega_t \cup \{S = \infty\}$. Since $\lim_{t \to 0} A^{\varepsilon}_{t} = 0$ we obtain

$$A^{\varepsilon}_{\infty} \circ \theta_t \cdot 1_{\Omega_t \cup \{S = \infty\}} \nearrow A^{\varepsilon}_{\infty} \text{ as } t \to 0 .$$

Taking the expectation we get

$$\lim_{t \to 0} E^x(A^{\varepsilon}_{\infty} \circ \theta_t ; \Omega_t \cup \{S = \infty\}) = E^x(A^{\varepsilon}_{\infty}),$$

which in turn implies $\lim_{t \to 0} E^x(A^{\varepsilon}_{\infty} \circ \theta_t) = \lim_{t \to 0} E^x(u(X_t)) = u(x)$. Thus u is excessive. Now let us check that $u \leq P_K u + \varepsilon$. This follows once we have proved that

$$(*) \quad A^{\varepsilon}_{\infty} \leq \varepsilon + A^{\varepsilon}_{\infty} \circ \theta_{T_K} .$$

We have $T_K \leq T_1$. If $T_K(\omega) < T_1(\omega)$, then $A^{\varepsilon}_{T_K}(\omega) = 0$ though if $T_K(\omega) = T_1(\omega)$, then $\omega \in \Lambda_{T_K(\omega)} \cup \Lambda'_{T_K(\omega)}$. Therefore we get relation (*) applying Lemma 2.3.

Let D be an open set such that $\overline{V^{\varepsilon}} \subset D$. If $x \in D$, the relation $P_D u(x) = u(x)$ is obvious. If $x \in E \setminus D$, then $X(T_D) \in \partial D, P^x$-a.s., and hence $S \circ \theta_{T_D} = 0, P^x$-a.s. Since $A^{\varepsilon}(T_D) = 0$, relation (2.6) gives us $A^{\varepsilon}_{\infty} = A^{\varepsilon}_{\infty} \circ \theta_{T_D}$ P^x-a,s. Therefore $P_D u(x) = u(x)$, completing the proof. The aim of this paper is to prove that in the case when K is a hypersurface the functionals A^{ε} converge (as $\varepsilon \to 0$) to a continuous additive functional supported by K. The basic estimate in proving the convergence is the following lemma which is a version of a result of Bàlly (see Lemma 1 of [B1]). We omit the proof because it is similar to the original.

<u>Lemma 2.5</u>

Let $A = (A_t)$ be an increasing process which is right continuous, adapted and such that $A_0 = 0$. Assume that $A_\infty = \lim_{t\to\infty} A_t$ is P^x-integrable for each $x \in E$ and set $u(x) = E^x(A_\infty)$. Moreover suppose that there are two constants Δ and Γ such that

$$A_t - A_{t-} \leq \Delta, \quad t > 0,$$

$$|A_\infty - A_t - A_\infty \circ \theta_t| \leq \Gamma, \quad t > 0.$$

Let A' be a CAF and $u'(x) = E^x(A'_\infty)$ its potential. Then one has

$$E^x(\sup_t |A_t - A'_t|^2)^{1/2} \leq \Gamma + \|u - u'\| +$$

$$+ 2\sqrt{2}(u(x) + u'(x))^{1/2}(\Gamma + \Delta + \|u - u'\|)^{1/2}.$$

3. The Case of a Hyperplane

In this section L is an operator of the form (1.2) in $R^d (d \geqslant 3)$ with $c \equiv 0$. For $\beta > 0$ we put $D_\beta = \{x \in R^d : |x^d| < \beta\}$ and $K = \{x \in R^d : x^d = 0\}$. We retake the frame of the preceding section with respect to the set $E = D_1$. So X is an L-diffusion in D_1. The set V^ε coincides with D_ε. For $\varepsilon \in (0.1)$ we preserve the notation T, S, T_n and A^ε will be defined by (2.1). The Lebesgue measure in K will be denoted by μ. (Since K is isomorphic to R^{d-1} we may conside this measure). We denote by λ the measure $2a^{dd}\cdot\mu$, where a^{dd} is the coeficient appearing in the expression of L. The integral of a function $f \in \mathcal{B}_+(R^d)$ with respect to λ can be written as follows

$$\int f(x)\ \lambda(dx) = \int_{R^{d-1}} f(x',0)\, 2a^{dd}(x',0)\, dx'.$$

If $x \in R^d$ we write $x = (x', x^d)$, with $x' \in R^{d-1}$, $x^d \in R$. Now we state the result of convergence of A^ε in the case when the coeficient b^d appearing in the expression of L is null.

Theorem 3.1

Assume that $b^d \equiv 0$. Then there exists a CAF, A, such that

(3.1) $\lim\limits_{\varepsilon \to 0} \sup\limits_t |A_t^\varepsilon - A_t| = 0$, a.s.

(3.2) $E^x(\sup\limits_t |A_t^\varepsilon - A_t|^2)^{1/2} \leq C\sqrt{\varepsilon}$, $x \in D_1$.

The functional A is determined by the following relation

(3.3) $E^x(A_\infty) = \int_K g^1(x,y)\ \lambda(dy)$, $x \in D_1$,

where $g^1 = g^{D_1}$ is the Green function associated with L in D_1.

Proof.

First we are going to prove the following relation

(*) $E^x(T < \infty) = 1 - |x^d|$, $x \in D_1$.

We observe that (the) function appearing in the right side of this equality is L-harmonic in $D_1 \setminus K$. (Here we use the fact that $b^d \equiv 0$.) This function is the solution of the Dirichlet problem in $D_1 \setminus K$

with the following boundary conditions: 1 on K and 0 on ∂D_1. Since our process X may be viewed as the restriction to D_1 of an L-diffusion in R^d, we apply formula (1.9) to obtain relation (*) with $x \in D_1 \setminus K$. Further, a point $x \in K$ is regular for the set $\{x \in R^d : x^d \geqslant 0\}$ and for the set $\{x \in R^d : x^d \leqslant 0\}$, on account of Lemma 1.20. A little analisys of the trajectories shows that x should be regular for K, which implies relation (*). Similarly we have

$$E^x(S < \infty) = 1, \; x \in D_1.$$

Further we are going to prove the following relation

$$(3.3') \quad E^x(T_1 < \infty) = (1-\varepsilon) \wedge (1-|x^d|), \; x \in D_1.$$

By the strong Markov property we have

$$E^x(T_1 < \infty) = E^x(E^{X_S}(T < \infty); S < \infty).$$

If $x \in \overline{D_\varepsilon}$, then $X_S \in \partial D_\varepsilon$, P^x-a.s., and hence

$$E^{X_S}(T < \infty) = 1-\varepsilon, \; P^x\text{-a.s.}$$

If $x \in D_1 \setminus \overline{D_\varepsilon}$, then $X_S = x$, P^x-a.s., and hence

$$E^x(T_1 < \infty) = E^x(T < \infty) = 1-|x^d|,$$

which proves (3.3'). Computing further we get

$$E^x(T_{n+1} < \infty) = E^x(E^{X_{T_n}}(T_1 < \infty); T_n < \infty) =$$
$$= (1-\varepsilon)E^x(T_n < \infty) = (1-\varepsilon)^n E^x(T_1 < \infty),$$

because almost surely $T_n(\omega) < \infty$ implies $X_{T_n}(\omega) \in K$.

Now we can obtain an explicite expression for $u(x) = E^x(A^\varepsilon_\infty)$

$$u(x) = \varepsilon \sum_{n=1}^{\infty} E^x(T_n < \infty) = (1-\varepsilon) \wedge (1-|x^d|).$$

Let us put $u'(x) = 1-|x^d|$. By Lemma 3.2 from below we know that u' is a regular potential. Hence, there exists a CAF A such that $u'(x) = E^x(A_\infty)$. The formula proved in Lemma 3.2 gives relation (3.3)

Estimation (3.2) follows from Lemma 2.5 and the following easily observed relations

$$\|u-u'\| = \varepsilon \;,$$

$$A_t^{\varepsilon} - A_{t-}^{\varepsilon} \leq \varepsilon \;,\; t > 0$$

$$0 \leq A_{\infty}^{\varepsilon} - A_t^{\varepsilon} - A_{\infty}^{\varepsilon} \circ \theta_t \leq \varepsilon \;, t > 0.$$

The last estimate is obtained from Lemma 2.3.It remains to prove relation (3.1).Then we set $\varepsilon_k = k^{-2}$ and $A^k = A^{\varepsilon_k}$.

A Borel-Cantelli argument together with estimate (3.2) leads to

$$(**) \quad \lim_k \sup_t |A_t^k - A_t| = 0, \text{a.s.}$$

On the other hand, putting $T_n^{\varepsilon} = T_n$ and looking at the definition we observe that $T_n^{\varepsilon} \leq T_n^{\delta}$ provided $\varepsilon \leq \delta$ .Therefore we have

$$\delta^{-1} A_t^{\delta} \leq \varepsilon^{-1} A_t^{\varepsilon}.$$

Now for ε such that $\varepsilon_{k+1} \leq \varepsilon \leq \varepsilon_k$ we obtain

$$k^2 \varepsilon^{-1} A_t^k \leq A_t^{\varepsilon} \leq (k+1)^2 \varepsilon^{-1} A_t^{k+1} \;,$$

and further

$$|A_t^{\varepsilon} - A_t| \leq \left(\frac{k}{k+1}\right)^2 |A_t^k - A_t| + \left(\frac{k+1}{k}\right)^2 |A_t^{k+1} - A_t| + \frac{6}{k} A_t.$$

This estimate together with (**) leads to relation (3.1),which completes the proof of the theorem.

Lemma 3.2

Under the condition $b^d \equiv 0$, the function $u'(x) = 1 - x^d$ is a regular potential in D_1.It can be represented as a Green potential of measure λ :

$$u'(x) = \int_K g^1(x,y)\, \lambda(dy), \; x \in D_1 \;.$$

Proof.

Let U be the potential kernel of Brownian motion in the interval (-1,1).The Green fonction $\bar{g}$ corresponding to U has the expression (see page 258 of [Do])

$$\bar{g}(s,t) = \begin{cases} (1-s)(t+1) & , s \geqslant t, \\ (1-t)(s+1) & , t \geqslant s. \end{cases}$$

We choose a function $f \in \mathscr{C}^\infty(R)$ such that supp $f \subset (-1,1), f \geqslant 0$ and $\int f = 1$. Then we put $f_n(t) = nf(nt)$, $n \in N$. Then the functions

$$U f_n(t) = \int_{-1}^{1} \bar{g}(t,s) f_n(s)ds, \; n \in N,$$

are of class $\mathscr{C}^\infty(-1,1)$ and all vanish at the boundary points. A strightforward computation shows that

$$d^2/dt^2 \, U f_n(t) = -2f_n(t).$$

Further a change of variable in the inegral shows that

$$U f_n(t) = \int_{-1}^{1} \bar{g}(t,u/n) f(u)du.$$

Sice $\bar{g}$ is unifomly continuous on $(-1,1) \times (-1,1)$, we deduce from this formula that $U f_n$ converges uniformly to $\bar{g}(\cdot,0)$. Now we put $u_n(x) = U f_n(x^d)$ for each $x \in D_1$. Then $u_n \in \mathscr{C}^\infty(D_1)$ and $u_n(x',-1) = u_n(x',1) = 0$ for each $x' \in R^{d-1}$. Applying L we obtain

$$Lu_n(x) = -2a^{dd}(x) f_n(x^d).$$

Here we have used the fact that $b^d \equiv 0$. By formula (1.9) we deduce that $u_n = V^{D_1} h_n$, where $h_n(x) = 2a^{dd}(x) f_n(x^d)$, $x \in D_1$. Then we can write

$$u_n(x) = \int_{D_1} g^1(x,y) h_n(y)dy, \; x \in D_1.$$

Since $\bar{g}(s,0) = 1-|s|$, it follows that u_n converges uniformly to u'. The functions u_n are regular potentials, and hence the limit u' is also a regular potential. On the other hand the measures $h_n(y)dy$ converge to the measure $\lambda(dy)$ so that condition (i) of Lemma 1.25 is satisfied. Using estimate (1.13) one can easily see that conditions (ii) and (iii) of that lemma are verified too. Moreover choosing a number $a > 0$ large enough, the potential au_1 will satisfy $u_n \leq au_1$ for each $n \in N$. Thus assertion (b) of Lemma 1.25 implies

$$\lim_{n} u_n(x) = G^{D}_{\lambda} 1 \ (x),$$

completing the proof.

In the next theorem we treat the same problem as in Theorem 3.1 for the general case when b^d is nonnull. If one wants to generalise only relation (3.1), this can be easily done using the Cameron--Martin-Girsanov formula. However relation (3.2) seems to be much more difficult to generalise by this method. In fact we are not able to obtain it but a weaker version (see relation (3.5) from below) using a different method of analytical feature. The proof is based on several lemmas which have their own interest. Since they are also lengthy we present them in the next three sections.

Theorem 3.3

Assume that b^d is non-null. Then there exists a CAF, A, such that

$$(3.4)\ \lim_{\varepsilon \to 0} \sup_{t} |A_t - A_t| = 0, \text{ a.s.}$$

$$(3.5)\ E^x(\sup_{t} |A_t - A_t|^2)^{1/2} \leq C(\varepsilon \ln 1/\varepsilon)^{1/4}, \ x \in D_1, \varepsilon \in (0, \varepsilon_0)$$

where ε_0 is a constant independent of x. The fonctional A is determined by

$$(3.6)\ E^x(A_\infty) = \int_K g^1(x,y)\lambda(dy), x \in D_1.$$

Proof.

First we remark that the statement will hold true once we are able to prove it for a process transformed by a mild random time change. Both relations (3.4) and (3.5) remain unmodified after a random time change. For relation (3.6) we should observe the following facts. Let $\tilde{L} = aL$, where $a \in \mathcal{C}^{\alpha}(R^d)$. Then the Green function $\tilde{g}^1$ associated with $\tilde{L}$ in D_1 is related to g^1 by

$$\tilde{g}^1(x,y) = g^1(x,y)a^{-1}(y)$$

and the measure $\tilde{\lambda}$ corresponding to $\tilde{L}$ is related to λ by $\tilde{\lambda} = a\cdot\lambda$. So relation (3.6) holds for L once it was established for $\tilde{L}$. We use this remark choosing $a = (2a^{dd})^{-1}$, so that the coefficient $\tilde{a}^{dd}$ corresponding to $\tilde{L}$ is $\tilde{a}^{dd} = 1/2$. Thus in the rest of this proof we will assume that $a^{dd} = 1/2$. This assumption will allow us to use Lemma 5.9 at the proper time.

As in the proof of Theorem 3.1, relation (3.4) follows from relation (3.5). Also inequality (3.5) follows from Lemma 2.5 once we are able to estimate $\| u-G_\lambda^{D_1}1 \|$, where $u(x) = E^x(A_\infty^\varepsilon)$, $x \in D_1$. This is difficult point now and the reminder proof is devoted to obtain such an estimate. Under the hypothesis we have just made the measures λ and μ coincide. Besides ε we will use a second parameter β which will be small. We will see that the optimal choice is something like $\beta = \sqrt{\varepsilon}$. However until the proper time we only suppose that $0<\varepsilon<\beta<1$. We set $R = T_{D_1 \setminus D_\beta}$. Since $S\circ\theta_R = 0$ a.s., we deduce from relation (2.6) that

$$A_\infty^\varepsilon = A_R^\varepsilon + A_\infty^\varepsilon\circ\theta_R \quad \text{a.s.}$$

Setting $v(x) = E^x(A_R^\varepsilon)$ we obtain

$$(3.7)\quad u(x) = v(x) + P_{D_1\setminus D_\beta}u(x),\ x \in D_\beta.$$

Now we are going to study function v :

$$\bar{v}(x) = \varepsilon\sum_{n=1}^{\infty} E^x(T_n \leq R),\ x \in D_\beta.$$

We know from Proposition 1.11 that $P^x(R<\infty) = 1$ for each $x \in D_\beta$. On the other hand we have $\{T_n < \infty\} \subset \{T_n < T_{n+1}\}$ a.s., because $T_1 > 0$ a.s. Therefore the following equalities hold P^x-a.s. for each $x \in D_\beta$,

$$\{T_{n+1} \leq R\} = \{T_n + T_1\circ\theta_{T_n} \leq R; T_n < R\} = \{T_1\circ\theta_{T_n} \leq R\circ\theta_{T_n}; T_n < R\}$$

Furhter we have $R\circ\theta_R = 0$ a.s. because $D_1 \setminus D_\beta$ is regular. This leads to

$$E^x(T_1\circ\theta_{T_n} \leq R\circ\theta_{T_n} \,;\, T_n = R) \leq E^x(T_1\circ\theta_{T_n} \leq 0) = 0,$$

which in turn implies

$$E^x(T_{n+1} \leq R) = E^x(T_1\circ\theta_{T_n} \leq R\circ\theta_{T_n} \,; T_n \leq R) =$$

$$= E^x(E^{X_{T_n}}(T_1 \leq R); T_n \leq R).$$

Now we define an operator on $\mathcal{B}_b(D_\beta)$ by

$$Mf(x) = E^x(f(X_{T_1}); T_1 \leq R), \quad x \in D_\beta.$$

This operator is in fact a kernel supported by K. With this notatio we can repeat the above reasoning to obtain

$$E^x(f(X_{T_{n+1}}); T_{n+1} \leq R) = E^x(E^{X_{T_n}}(f(X_{T_1}); T_1 \leq R); T_n \leq R) =$$

$$= E^x(Mf(X_{T_n}); T_n \leq R) = M^{n+1}f(x).$$

We deduce the following expression for v

$$(3.7') \quad v = \varepsilon \sum_{n=1}^{\infty} M^n 1.$$

Further let us look at the operator M. Because $\varepsilon < \beta$ we have $R = S + R\circ\theta_S$ a.s., and therefore

$$Mf(x) = E^x(E^{X_S}(f(X_T); T \leq R)).$$

The probabilistic interpretation of the Dirichlet problem shows that the function

$$h(x) = E^x(f(X_T); T \leq R), \quad x \in D_\beta,$$

is L-harmonic in $D_\beta \setminus K$. Moreover if f is continuous then h can be extended as a continuous function on $\overline{D_\beta}$ such that $h(x) = 0$ for $x \in \partial D_\beta$ and $h(x) = f(x)$ if $x \in K$. Similarly the function

$$l(x) = E^x(h(X_S)), \quad x \in D_\varepsilon,$$

is L-harmonic in D_ε and can be extended as a continuous function on $\overline{D}_\varepsilon$ such that $l(x) = h(x)$ for $x \in \partial D_\varepsilon$ .The last expression of Mf shows that $Mf(x) = h(x)$ if $x \in D_\beta \setminus D_\varepsilon$ and $Mf(x) = l(x)$ if $x \in D_\varepsilon$.In particular Mf is continuous provided f is continuous. Further,from Lemma 4.3 and the maximum principle,we deduce that $M1 \leqslant 1-\gamma\varepsilon\beta^{-1}$,with a constant $\gamma \in (0,1]$.Therefore $\|M\| < 1$ and we may consider the series $\sum_{n=1}^{\infty} M^n$,which defines a bounded operator satisfying

$$(3.8) \quad \Big\|\sum_{n=1}^{\infty} M^n\Big\| \leqslant \beta/\gamma\varepsilon.$$

From relation(3.7′)we deduce that v is a bounded continuous function.Taking $\beta = 1$ we have $u = v$ and so we find that u is also bounded and continuous.From Lemmas 2.4 and 5.12 we deduce that u is a regular potential.

Now we associate to L another operator defined by

$$\overline{L} = \sum_{i,j=1}^{d} a^{ij} D_{ij}.$$

We denote by $\overline{X}$ an L-diffusion process in D_1 and $\overline{u},\overline{v},\overline{M}$ will be objects similar to u,v,M associated with respect to $\overline{X}$.As we have seen in the proof of Theorem 3.1,it is quite easy to work with such a process.For example the expression of $\overline{M}1$ can be explicitely computed and similar to relation (3.3′) we obtain

$$\overline{M}1(x) = (1-|x^d|\beta^{-1}) \wedge (1-\varepsilon\beta^{-1}), x \in D_\beta .$$

This formula leads to

$$\overline{v}(x) = \varepsilon \sum_{n=1}^{\infty} \overline{M}^n 1(x) = (\beta-|x^d|) \wedge (\beta-\varepsilon),$$

$$\|\overline{M}\| = 1-\varepsilon\beta^{-1},$$

$$\Big\|\sum_{n=1}^{\infty} \overline{M}^n\Big\| \leqslant (1-\varepsilon\beta^{-1})\beta/\varepsilon .$$

Further we apply Lemma 4.2 and obtain the following estimate

$$(3.9) \quad |Mf(x)-\overline{M}f(x)| \leqslant C\varepsilon(1 + \ln\beta/\varepsilon)\|f\| , x \in D_\varepsilon ,$$

which holds provided that $\beta<\beta_0$, where $\beta_0 \in (0,1]$ is a constant. Then we are going to prove the following inequality

$$(3.10) \quad |v(x)-\bar{v}(x)| \leq C\beta^2 \ln(\beta/\varepsilon), x \in D_\varepsilon,$$

which holds provided that $\beta<\beta_0$ and $\varepsilon \in (0,\beta)$. To this aim we begin with the following identity which can be checked by a strightforward computation

$$\sum_{n=1}^{\infty} M^n - \sum_{n=1}^{\infty} \bar{M}^n = (\sum_{n=0}^{\infty} M^n)(M-\bar{M})(\sum_{n=0}^{\infty} \bar{M}^n).$$

Applying the above operators to function 1, we get

$$v-\bar{v} = (\sum_{n=0}^{\infty} M^n)(M-\bar{M})(\bar{v}+\varepsilon).$$

Since $\|\bar{v}+\varepsilon\|=\beta$, relation (3.9) implies

$$|(M-\bar{M})(\bar{v}+\varepsilon)(x)| \leq C\varepsilon\beta(1+\ln\beta/\varepsilon), x \in D_\varepsilon.$$

Finally estimate (3.10) follows from this inequality and relation (3.8).

Now we fix some notation. For each β we denote by g^β the Green function associated to L in D_β. The Green potential of measure τ in D_β will be denoted by G_τ^β. Similarly $\bar{g}^\beta$ and $\bar{G}_\tau^\beta$ will indicate analogous objects associated with $\bar{L}$. By Lemma 3.2 we have

$$\bar{G}_\mu^\beta = \beta - |x^d|,$$

which leads to

$$(3.11) \quad \|\bar{v} - \bar{G}_\mu^\beta\| = \varepsilon.$$

At this moment we apply Lemma 5.9 with respect to L and then with respect to $\bar{L}$, obtainig

$$(3.12) \quad \|G_\mu^\beta - \bar{G}_\mu^\beta\| \leq C\beta^2.$$

This inequality together with (3.10) and (3.11) show that

$$|v(x) - G_\mu^\beta(x)| \leq \varepsilon + C\beta^2 \ln\beta/\varepsilon, x \in D_\varepsilon.$$

Since the functions appearing in this inequality are continuous near the boundary ∂D_ε we observe that the inequality holds even for $x \in \partial D_\varepsilon$. On the other hand the expression of $\bar{G}_\mu^\beta$ allows us to deduce $\beta - \varepsilon \leq \bar{G}_\mu^\beta(x)$, for each $x \in \bar{D}_\varepsilon$. Then by (3.12) we obtain

$$\beta/2 \leq G_{\mu}^{\beta}(x), x \in \overline{D}_{\varepsilon},$$

provided that $\varepsilon < \beta/4$ and β is small enough. This inequality and the preceding one imply

$$|v(x) - G_{\mu}^{\beta}(x)| \leq C(\varepsilon\beta^{-1} + \beta \ln(\beta/\varepsilon))G_{\mu}^{\beta}(x), x \in \overline{D}_{\varepsilon}.$$

We intend to apply Lemma 6.1. Therefore we look at the function G_{μ}^{1}. By Lemma 1.26 we deduce that G_{μ}^{1} is a continuous function and from (3.12) with $\beta = 1$, we see it is bounded. Then we deduce that G_{μ}^{1} is a regular potential, on account of Lemma 5.12. Finally we apply Lemma 6.1 and from the above inequality we get

$$|u - G_{\mu}^{1}| \leq C(\varepsilon\beta^{-1} + \beta\ln(\beta/\varepsilon))G_{\mu}^{1}.$$

Since G_{μ}^{1} is bounded this implies

$$\|u - G_{\mu}^{1}\| \leq C(\varepsilon\beta^{-1} + \beta\ln(\beta/\varepsilon)).$$

Now we ovserve that the sharpest estimate we may obtain from this inequality is when β is choosen to be $\beta = \sqrt{\varepsilon}/\sqrt{\ln 1/\varepsilon}$. Namely we get

$$(3.13) \quad \|u - G_{\mu}^{1}\| \leq C\sqrt{\varepsilon \ln 1/\varepsilon}, \text{ for small } \varepsilon.$$

As we have already mentioned the reminder proof is similar to that of Theorem 3.1.

4. Application of the Shauder Estimates

In this section β is a parameter belonging to $(0,1]$ and we put $D_\beta = \{x \in R^d : |x^d| < \beta\}$, $D'_\beta = \{x \in R^d : 0 < x^d < \beta\}$. We suppose that L is an operator of the form (1.2) with $c \equiv 0$. With the coefficients a^{ij} of L we define another operator

$$\bar{L} = \sum_{i,j=1}^{d} a^{ij} D_{ij}.$$

Lemma 4. 1.

For $f \in \mathscr{C}^0(R^{d-1})$ and $\beta \in (0,1]$, we denote by u the $\bar{L}$ - harmonic function in D'_β which is the solution of the following Dirichlet problem

$$u(x',0) = f(x'),\ u(x',\beta) = 0, \quad x' \in R^{d-1}.$$

Then, for $\varepsilon \in (0,\beta)$, we denote by v the $\bar{L}$-harmonic function in D_ε which is the solution of the Dirichlet problem

$$v(x', \varepsilon) = u(x', \varepsilon), \quad v(x', -\varepsilon) = 0, \quad x' \in R^{d-1}.$$

There exists a constant C independent of f, β and ε such that

(4.1) $^A[u]_1 \leq C\|f\|$,

with $A = \{ x \in R^d : x^d = 0\}$ $(\subset \supset D'_\beta)$ and

(4.2) $[v]_1 \leq C\varepsilon^{-1}\|f\|$.

Proof

For any $t > 0$ we define the map $F_t(x) = tx$, $F_t : R^d \to R^d$. This map is a diffeomorphism and, for $\beta \in (0,1]$, we have $F_\beta\ (D_1) = D_\beta$. One can easily see that if $h \in \mathscr{C}^a(D_\beta)$, then $h \circ F_\beta \in \mathscr{C}^a(D_1)$ and the following relations hold

(4.3) $[h \circ F\ \ ]_a = \beta^a [h]_a$,

(4.4) $^A[h\ \ F\ \ ]_a = {}^A[h]_a$.

Let us put $L_\beta = \sum_{i,j=1}^{d} a^{ij} \circ F_\beta\ D_{ij}$ and observe that $\|a^{ij} \circ F_\beta\|_\alpha \leq \|a^{ij}\|_\alpha$, for any $\beta \in (0,1]$. Then

Theorm 1.4 gives us a constant $C > 0$, independent of β, such that

$$(4.5)\quad \|h\|_{2+\alpha} \leq C(\|L_\beta h\|_\alpha + \|h\|_0 + \|Bh\|_{2+\alpha}),$$

$$(4.5')\quad {}^A\|h\|_{2+\alpha} \leq C({}^A\|L_\beta h\|_\alpha + \|h\|_0 + \|Bh\|^E_{2+\alpha}),$$

for any $h \in \mathscr{C}^{2+\alpha}(D_1)$, with $E = \{x \in R^d : x^d = 1\}$. If the function f in the statement is in $\mathscr{C}^{2+\alpha}(R^{d-1})$, then $u \in \mathscr{C}^{2+\alpha}(D'_\beta)$ and, as one can easily see, we have $L_\beta(u \circ F_\beta) = 0$. Then from (4.5') we get

$${}^A\|u \circ F_\beta\|_{2+\alpha} \leq C\|u \circ F_\beta\| = C\|f\|.$$

The same inequality holds for $f \in \mathscr{C}^0(R^{d-1})$, on account of an approximation argument. Using the equality (4.4) we get (4.1). Further, from the preceding inequality we have

$$[u(\cdot,\varepsilon)]_0 + \varepsilon[u(\cdot,\varepsilon)]_1 + \varepsilon^2[u(\cdot,\varepsilon)]_2 + \varepsilon^{2+\alpha}[u(\cdot,\varepsilon)]_{2+\alpha} \leq C\|f\|.$$

Then, relation (4.3) allows us to transform this inequality into

$$\|u \circ F_\varepsilon(\cdot,1)\|_{2+\alpha} \leq C\|f\|.$$

Applying relation (4.5) to $v \circ F_\varepsilon$, we get

$$\|v \circ F_\varepsilon\|_{2+\alpha} \leq C(\|v \circ F\|_0 + \|u \circ F_\varepsilon(\cdot,1)\|_{2+\alpha}) \leq C\|f\|.$$

Then, using relation (4.3), we get the estimate (4.2). The proof is complete.

Lemma 4. 2.

If $f \in \mathscr{C}^0(R^{d-1})$, we denote by u the L-harmonic function in $D_\beta \setminus \{x^d = 0\}$ which is the solution of the boundary problem

$$u(x',0) = f(x'),\ u(x',\beta) = u(x',-\beta) = 0,\ x' \in R^{d-1}.$$

Then, for $\varepsilon \in (0,\beta)$, we denote by v the function which is L-harmonic in D_ε and satisfies

$$v(x',\varepsilon) = u(x',\varepsilon),\ v(x',-\varepsilon) = u(x',-\varepsilon),\ x' \in R^{d-1}.$$

Similarly, we define the function $\bar{u}$ to be $\bar{L}$-harmonic in $D_\beta \setminus \{x^d = 0\}$ satisfying the conditions

$$\bar{u}(x',0) = f(x'),\ \bar{u}(x',\beta) = \bar{u}(x',-\beta) = 0,\ x' \in R^{d-1}$$

and the function $\bar{v}$ to be $\bar{L}$-harmonic in D_ε satisfying

$$\bar{v}(x', \varepsilon) = \bar{u}(x', \varepsilon), \ \bar{v}(x', -\varepsilon) = \bar{u}(x', -\varepsilon), \ x' \in R^{d-1}.$$

There exists a constant $C > 0$ and $\beta_0 \in (0,1]$ such that

(4.6) $\|v - \bar{v}\| \leq C\,\varepsilon(1 + \ln \beta/\varepsilon)\|f\|,$

for any $f \in \mathscr{C}^0(R^{d-1})$ and $0 < \varepsilon < \beta < \beta_0$.

<u>Proof</u>

First we note that $\bar{u}$ satisfies the estimate (4.1). Also $\bar{v}$ may be written as $\bar{v} = v_1 + v_2$, with v_1, v_2 both $\bar{L}$-harmonic in D_ε satisfying the conditions

$$v_1(x', \varepsilon) = \bar{u}(x', \varepsilon), \ v_1(x', -\varepsilon) = 0, \quad x' \in R^{d-1},$$

$$v_2(x', \varepsilon) = 0, \ v_2(x', -\varepsilon) = \bar{u}(x', -\varepsilon), \ x' \in R^{d-1}.$$

Therefore $\bar{v}$ satisfies the estimate (4.2). Our next aim is to prove the following inequality

(4.7) $|u(x) - \bar{u}(x)| \leq C\,|x^d| \ln(\beta/|x^d|)\|f\|.$

From (4.1) we have

$$|x^d|\,|D_i\bar{u}(x)| \leq C\|f\|.$$

and this allows us to obtain

$$|L\bar{u}(x)| = \left|\sum_i b^i D_i \bar{u}(x)\right| \leq C'\|f\|/|x^d|.$$

Let us put $l(x) = x^d \ln(\beta/x^d)$, $x \in D'_\beta$. This function vanishes on the boundary of D'_β. Applying L we get

$$Ll(x) = -a^{dd}(x)/x^d + b^d(x)(\ln(\beta/x^d) - 1).$$

Choosing β_0 to be small enough we will have $Ll(x) \leq -C''/x^d$, for any $x \in D'_\beta$ provided $0 < \beta < \beta_0$. Now let k be a positive constant and note $w = k\|f\|\,l + \bar{u} - u$. This function also vanishes on the boundary of D'_β and applying L we get

$$Lw(x) \leq (C' - kC'')\|f\|/x^d.$$

Therefore we choose k so that $C' < kC''$ and consequently $Lw < 0$ in D'_β. The maximum principle (Theorem 1.3) gives $w \geq 0$, which, written in a different form, is

$$u - \bar{u} \leq k\|f\|l.$$

Repeating the above reasoning with $w = k\|f\|l - \bar{u} + u$ we will get

$$\bar{u} - u \leq k\|f\|l.$$

These estimates together lead to the inequality (4.7), for $x^d \in (0, \beta)$. Similarly one gets the inequality for $x^d \in (-\beta, 0)$.

Further we introduce the function $\bar{\bar{v}}$, which is L-harmonic in D_ε and satisfies the boundary conditions

$$\bar{\bar{v}}(x', \varepsilon) = \bar{u}(x', \varepsilon),\ \bar{\bar{v}}(x', -\varepsilon) = \bar{u}(x', -\varepsilon),\ x' \in R^{d-1}.$$

We are now going to show that

$$(4.8)\quad |\bar{\bar{v}}(x) - \bar{v}(x)| \leq C\varepsilon^{-1}(\varepsilon^2 - (x^d)^2)\|f\|.$$

From (4.2) we have

$$|L\bar{v}| = |\sum_i b^i D_i \bar{v}| \leq C'\varepsilon^{-1}\|f\|.$$

Then we take $l(x) = \varepsilon^2 - (x^d)^2$. If β_0 is small enough, we will get $Ll < -C''$ in D_ε, provided $0 < \varepsilon < \beta < \beta_0$, where C'' is a constant. This time we put $w = k\varepsilon^{-1}\|f\|l + \bar{v} - \bar{\bar{v}}$ and again w vanish on the boundary (of D_ε). Inside D_ε we have

$$Lw \leq (C' - kC'')\varepsilon^{-1}\|f\|.$$

Choosing k such that $C' < kC''$ we get $Lw < 0$. On account of the maximum prnciple we get $w \geq 0$, which can be written as

$$\bar{\bar{v}} - \bar{v} \leq k\varepsilon^{-1}\|f\|l.$$

Similarly we get

$$\bar{v} - \bar{\bar{v}} \leq k\varepsilon^{-1}\|f\|l,$$

completing the proof of the estimate (4.8).

Now from the maximum principle we have

$$\|v - \bar{\bar{v}}\| \leq \sup\{|u(x) - \bar{u}(x)| : x \in \partial D_{\varepsilon}\}.$$

The right side can be estimated from (4.7) obtaining

$$\|v - \bar{\bar{v}}\| \leq C\,\varepsilon \ln(\beta/\varepsilon)\|f\|.$$

This inequality together with (4.8) lead to (4.6). The lemma is proved.

Lemma 4.3.

Let u be the L-harmonic function in $D_{\beta}\setminus\{x^d = 0\}$ which satisfies the boundary conditions

$$u(x',0) = 1,\ u(x',\beta) = u(x',-\beta) = 0,\ x' \in R^{d-1}.$$

Then there exist a constant $\gamma \in (0,1]$ such that

$$u(x) \leq 1 - \gamma\beta^{-1}|x^d|,\ x \in D_{\beta},\ \beta \in (0,1].$$

Proof

We define $f(t) = (e^b - 1)(e^b - \exp bt)$ and $w(x) = f(\beta^{-1}|x^d|)$, with b a constant to be choosen. We see that

$$w(x',0) = 1,\ w(x',\beta) = w(x',-\beta) = 0 \text{ and}$$

$$Lw(x) = -b\beta^{-1}(e^b - 1)\exp(b\beta^{-1}|x^d|)(a^{dd}(x)b\beta^{-1} + b^d(x)).$$

Choosing b large enogh we have $ba^{dd} \geq b^d$, and hence $Lw \leq 0$ in $D_{\beta}\setminus\{x^d = 0\}$ for each $\beta \in (0,1]$. The maximum principle implies $u \leq w$ in D_{β}.

On the other hand the derivative of the function $h(t) = f(t) - 1 + (e^b - 1)^{-1}bt$ satisfies $h'(t) \leq 0$, for $t \in [0,\infty)$. Therefore we have

$$f(t) \leq 1 - (e^b - 1)^{-1}bt.$$

This implies $u(x) \leq w(x) \leq 1 - (e^b - 1)^{-1}b|x^d|$, completing the proof.

5. Potentials Supported by a Hyperplane

In the first part of this section we are assuming that $L = \frac{1}{2}\Delta$ and X is the Brownian motion in $R^d, d \geq 3$. The Green function is known to be

(5.1) $g(x,y) = k|x - y|^{2-d}$,

with k a constant which depends on the dimension d. We put $D_1 = \{ x \in R^d : |x^d| < 1\}$.

Lemma 5.1.

The Green function for D_1 has the following expression

$$g^1(x,y) = \sum_{i \in \mathbb{Z}} (-1)^i g(x,(y',2i + (-1)^i y^d)), \quad x,y \in D_1.$$

Proof

In order to check that the function g^1 given by the above formula is the Green function, one should prove that $g(\cdot,y) - g^1(\cdot,y)$ is harmonic in D_1 and $g^1(x,y) = 0$ for any $x \in \partial D_1$, $y \in D_1$ (see relation (1.19)).

First we write

$$g^1(x,y) = g(x,y) + h'(x,y) + h''(x,y),$$

$$h'(x,y) = \sum_{i=1}^{\infty} (-1)^i g(x,(y',2i + (-1)^i y^d)),$$

$$h''(x,y) = \sum_{i=-1}^{-\infty} (-1)^i g(x,(y',2i + (-1)^i y^d)).$$

The general term which appears in the above series, $g(x,y',2i + (-1)^i y^d)$, decreases to zero provided $i \to \infty$ or $i \to -\infty$ and $x,y \in D_1$ are fixed. Thus the series are convergent because of the factor $(-1)^i$ which makes them alternating. If $i \neq 0$, the function $g(\cdot, (y',2i + (-1)^i y^d))$ is harmonic in D_1, and hence h' and h" are both harmonic in D_1. Further, for $x = (x',1)$, we have

$$g(x,(y',2i + (-1)^i y^d)) = g(x,(y',2(-i + 1) + (-1)^{-i+1} y^d)),$$

which shows that $g^1(x,y) = 0$. Similarly we get $g^1(x,y) = 0$ for any $x \in \partial D_1$, completing the proof.

We will denote by U the potential kernel of Brownian motion in D_1. It can be expressed as

(5.2) $Uf(x) = \int_{D_1} g^1(x,y)f(y)dy, \quad f \in \mathcal{B}_+(D_1), \quad x \in D_1.$

Lemma 5.2.

a) There exists a constant $C > 0$, such that

$\| Uf \| \leq C \| f \|, f \in \mathcal{C}^0(D_1)$.

b) If $f \in \mathcal{C}^1(D_1)$, then $D_i Uf = UD_i f$ for any $i \leq d - 1$.

Proof

Let $y \in D_1$ be fixed. The function $g(\cdot,(y',-y^d-2))$ is harmonic in D_1 and, on the boundary, we have

$$g(x,(y',-y^d-2)) = g(x,y), \quad \text{if } x^d = -1$$

$$g(x,(y',-y^d-2)) \leq g(x,y), \quad \text{if } x^d = 1.$$

From the maximum principle we get

$$g(x,(y',-y^d-2)) \leq g(x,y) - g^1(x,y), \quad x \in D_1.$$

Now we introduce the functions

$$l(u,s,t) = k(u^2 + (s - t)^2)^{(2-d)/2},$$

$$m(u,s,t) = \sum_{i \in \mathbf{Z}} (-1)^i l(u,s,2i + (-1)^i t),$$

so that $g(x,y) = l(|x' - y'|, x^d, y^d)$ and $g^1(x,y) = m(|x' - y'|, x^d, y^d)$.

The preceding inequality can be written as

$$m(u,s,t) \leq l(u,s,t) - l(u,s,-t-2),$$

for $s,t \;\; (-1,1)$, $u \in R_+$. Then, from the identity

$$(5.3)\quad a^{-p/2} - b^{-p/2} = (b - a)(b^{\frac{1}{2}} + a^{\frac{1}{2}})^{-1}\left(\sum_{i=0}^{p-1} a^{i/2} b^{(p-i-1)/2}\right) a^{-p/2} b^{-p/2},$$

we deduce

$$(5.4)\quad m(u,s,t) \leq 16k(d - 2)u^{2-d}(u^2 + 16)^{-1}.$$

The expression of Uf can be transformed by a change of variable in the integral, obtaining

$$(5.5)\quad Uf(x) = \int_{-1}^{1}\int_{R^{d-1}} m(|y'|, x^d, y^d) f(x' + y', y^d)\, dy' dy^d.$$

Then using the estimate (5.4) we get

$$\int_{-1}^{1}\int_{R^{d-1}} m(|y'|, x^d, y^d)\, dy' dy^d \leq 32k(d-2)\int_{R^{d-1}} |y'|^{2-d}(|y'|^2 + 16)^{-1} dy' < \infty.$$

This estimate and relation (5.5) imply both a) and b) in the statement, completing the proof.

For the next lemmas we need some more notation. Let $h : R \to R$ be so that

$$(5.6)\quad h \in \mathcal{C}^{\infty}(R),\ \operatorname{supp} h \subset (-1,1),\ h \geq 0,\ \int h = 1.$$

For a function $f : R^{d-1} \to R$ and $n \in N$ we define another function $J^n f : R^d \to R$ by $J^n f(x) = n f(x') h(n x^d)$.

<u>Lemma 5.3.</u>

There exists a constant $C > 0$ such that

$$\| U J^n f \| \leq C \| f \|,\ f \in \mathcal{C}^0(R^{d-1}),\quad n \in N.$$

<u>proof</u>

Using relation (5.5) and estimate (5.4) we get

$$| U J^n f(x) | \leq C \| f \| \int_{-1}^{1} n h(n y^d)\, dy^d = C \| f \|,$$

which proves the lemma.

The potential kernel of Brownian motion on the whole space R^d will be denoted by N. It can be expressed with the Green function given by (5.1) as

$$Nf(x) = \int_{R^d} g(x,y) f(y)\, dy,\ x \in R^d,$$

with $f \in \mathcal{B}_b(R^d)$ such that supp f is compact. It is known that Nf is bounded and continuous.

<u>Lemma 5.4.</u>

There exists a constant $C > 0$ such that

$$\| N J^n f \| \leq C r \| f \|,\ n \in N,$$

for any $f \in \mathcal{C}_c(R^{d-1})$ with supp $f \subset B'(r) = \{x \in R^{d-1} : |x| < r\}$.

Proof

Estimating $g(x,y) \leq k|x' - y'|^{2-d}$ we get

$$|NJ^n f(x)| \leq C\|f\| \int_{B(r)} |y'|^{2-d} dy' \int nh(ny^d) dy^d,$$

which leads to the inequality in the statement.

Lemma 5.5.

There is a constant $C > 0$ such that

$$\|D_d NJ^n f\| \leq C\|f\|, \quad f \in \mathcal{C}_c(R^{d-1}), \quad n \in N.$$

Proof

The function $J^n f$ has compact support and posseses a bounded derivative with respect to x^d. Therefore $D_d NJ^n f$ exists. In order to estimate it we first write

$$NJ^n f(x',s) = k\iint f(x' + y')nh(ny^d)(|y'|^2 + |s - y^d|^2)^{(2-d)/2} dy'dy^d.$$

We have to evaluate $NJ^n f(x',s) - NJ^n f(x',t)$. Then from the identity (5.3) we deduce

$$(|y'|^2 + |s - y^d|^2)^{(2-d)/2} - (|y'|^2 + |t - y^d|^2)^{(2-d)/2} \leq$$

$$\leq (d-2)|s - t| \cdot |s + t - 2y^d| \cdot |y'|^{2-d}(|y'|^2 + u^2)^{-1},$$

with $u = |s - y^d| \vee |t - y^d|$. The integral

$$I = \int_{R^{d-1}} |y'|^{2-d}(|y'|^2 + 1)^{-1} dy'$$

is finite and we have

$$\int |y'|^{2-d}(|y'|^2 + u^2)^{-1} dy' = u^{-1} I.$$

Since $|s + t - 2y^d| \leq 2u$, from the above considerations we get

$$|NJ^n f(x',s) - NJ^n f(x',t)| \leq 2(d - 2)|s - t| \, I \, \|f\|.$$

This inequality leads to $|D_d NJ^n f| \leq 2(d - 2) I \|f\|$, completing the proof.

Lemma 5. 6.

Let $E = \{x \in R^d : |x'| < 1, |x^d| < 1\}$. There exists a constant $C > 0$ such that

$$|D_d u(x)| \leq C\|u\|, \; x \in E, \; |x'| < \tfrac{1}{2}, \; x^d \in (-1,1),$$

for any function $u \in \mathcal{C}^2(\bar{E})$ which is harmonic in E and satisfies the conditions $u(x',1) = u(x',-1) = 0$ if $|x'| < 1$.

Proof.

We define a function v, which extends u to the set $E' = \{x \in R^d : |x'| < 1, x^d \in (-3,3)\}$, as follows

$v(x',x^d) = u(x',x^d)$, if $x^d \in [-1,1]$,

$v(x',x^d) = -u(x',2 - x^d)$, if $x^d \in (1,3)$

$v(x',x^d) = -u(x', -2 - x^d)$, if $x^d \in (-3,-1)$.

Obviously v is continuous and computing left and right derivatives at a point $(x',1)$ we see that $D_d v$ exists and is continuous in E'. Since $u = 0$ on the set $\{x \in R^d : |x'| < 1, x^d = 1\}$, we have $D_i u = 0$, $D_{ij} u = 0$ on this set, for $i,j \leq d - 1$. This shows that the derivatives $D_i u$, $D_{ij} u$ exist and are continuous in E', for $i,j \leq d - 1$. Then we see that $D_d D_i v$ is also continuos in E'. From the relation $\Delta u = 0$ we deduce $D_{dd} u(x',1) = 0$, which leads to the existence and continuity of $D_{dd} v$ in E'. We conclude that $v \in \mathcal{C}^2(E')$. Now we choose a domain D with boundary of class $\mathcal{C}^{2+\alpha}$ such that $\{x \in R^d : |x'| \leq \tfrac{1}{2}, |x^d| \leq 1\} \subset D \subset E'$.
The inequality from the statement of the lemma follows from the estimate (1.10) applied to D. The lemma is proved .

The last three lemmas are needed in proving the following result, which will be essential in the future.

Lemma 5.7.

There exists a constant $C > 0$ such that

$$\|DdUJ^n f\| \leq C\|f\|, \quad f \in \mathcal{C}^o(R^{d-1}), \quad n \in N.$$

Proof

We choose a function $\varphi \in \mathcal{C}_c(R^{d-1})$ such that $\varphi(x') = 1$ if $|x'| \leq 1$ and $\varphi(x') = 0$ if $|x'| \geq 2$. Then for a given point $x \in D_1$ we write $x = (x', x^d)$ and define $l(y') = \varphi(y' - x')$, $y' \in R^{d-1}$. For a given $f \in \mathcal{C}^o(R^{d-1})$ we have

$$UJ^n f = UJ^n(lf) + UJ^n((1 - l)f).$$

The function $J^n((1 - l)f)$ vanishes in the set $\{y \in D_1 : |x' - y'| < 1\}$, and hence $UJ^n((1 - l)f)$ is harmonic in this set. Applying Lemma 5.6 we get

$$|DdUJ^n((1 - l)f)(x)| \leq C \| UJ^n((1 - l)f)\| \leq C \|f\|,$$

with a constant C which do not depend on x.

In order to estimate the derivative of $UJ^n(lf)$ we extend this function to $D' = \{y \in R^d : |y^d| < 3\}$. Namely we put

$$u(y) = UJ^n(lf)(y), \text{ if } y^d \in [-1,1],$$

$$u(y) = -UJ^n(lf)(y', 2 - y^d), \text{ if } y^d \in (1,3),$$

$$u(y) = -UJ^n(lf)(y', -2 - y^d), \text{ if } y^d \in (-3,-1).$$

The function h in the definition of J^n is such that $\delta = d(\text{supp}\, h, R \setminus (-1,1)) > 0$. Therefore $UJ^n(lf)$ is harmonic near the boundary of D_1. As in the proof of Lemma 5.6 we deduce that u is harmonic in the set $\{y \in R^d : |y^d| \in (1 - \delta, 1 + \delta)\}$. On the other hand $NJ^n(lf) - UJ^n(lf)$ is harmonic in D_1, and hence $NJ^n(lf) - u$ is harmonic in the set $D_{1+\delta} = \{y \in R^d : |y^d| < 1 + \delta\}$. From the estimate (1.10), applied with respect to the set $D_{1+\delta}$, we get

$$|D_d(NJ^n(lf) - u)(x)| \leq C\|NJ^n(lf) - u\| \leq \|NJ^n(lf)\| + \|UJ^n(lf)\| \leq C\|f\|.$$

The last inequlity from above follows from Lemma 5.4 and Lemma 5.3. Moreover applying Lemma 5.5 we get $\|D_d NJ^n(lf)\| \leq C\|f\|$, which combined with the preceding estiamte gives

$$|UJ^n(lf)(x)| \leq C\|f\|.$$

The constant C obtained here do not depend on x. Thus the proof is complete.

Now let us put $D_\beta = \{x \in R^d : |x^d| < \beta\}$ and denote by U^β the potential kernel of Brownian motion in D_β. From the last lemma we deduce the following corollary.

Corollary 5.8

There exists a constant C such that

$\| U^\beta J^n f \| \leq C \| f \|,$

$\| D_d U^\beta J^n \| \leq C \| f \|,$

for each $f \in \mathscr{C}^o(R^{d-1})$, $\beta \in (0,1]$ and $n > 1/\beta$.

Proof

For $t > 0$ we denote by F_t the map $F_t(x) = tx$. So, for $\beta \in (0.1]$, the map F_β applies diffeomorphically D_1 on D_β. We are going to prove the following relation

$$(*) \quad U^\beta f = \beta^2 (U^1(f \circ F_\beta)) \circ F_{\beta^{-1}}, \; f \in \mathscr{C}^o(D_\beta).$$

Obviously it suffices to prove this relation with $f \in \mathscr{C}^\alpha(D_\beta)$. For such a function f we put $u = U^\beta f$. We know that $u \in \mathscr{C}^{2+\alpha}(D_\beta)$, u vanishes on the boundary and $\frac{1}{2}\Delta u = -f$. Then $u \circ F_\beta \in \mathscr{C}^{2+\alpha}(D_1)$, this function vanishes on the boundary of D_1 and $\frac{1}{2}\Delta(u \circ F_\beta) = -\beta^2 f \circ F_\beta$. The unicity of the Poisson problem implies $u \circ F_\beta = \beta^2 U^1(f \circ F_\beta)$, which proves relation (*).

If $f \in \mathscr{C}(R^{d-1})$ and $n > 1/\beta$, then $\operatorname{supp} J^n f \subset D_\beta$. Applying relation (*) we get

$$(**) \quad U^\beta J^n f = \beta^2 U^1((J^n f) \circ F_\beta) \circ F_{\beta^{-1}}.$$

We observe that

$$(J^n f) \circ F_\beta(x) = f(\beta x') n\, h(n \beta x^d) = \beta^{-1} J'^n (f \circ F_\beta)(x),$$

where J'^n is defined similar to J^n. Namely the function $h'(t) = \beta h(\beta t)$ has the properties listed at (5.6) and J'^n is defined with respect to this function. Now we should note that in the preceding lemmas the constants obtained in the estimates do not depend on the function h. Therefore we can apply Lemma 5.3 and obtain

$$\| U^1(J'^n(f \circ F_\beta)) \| \leq C \| f \circ F_\beta \| = C \| f \|,$$

which implies the first inequality asserted by the corollary, on account of (**). Then from Lemma 5.7 we have

$$\| D_d U^1 (J'^n (f \circ F_\beta))\| \leq C \| f \|.$$

Since in general we have $D_d(u \circ F_{\beta^{-1}}) = \beta^{-1}(D_d u) \circ F_\beta$, from this inequality and (**) we get the secopnd inequality of the corollary. The proof is complete.

*

* *

Now we are going to state the main result of this section. First we fix some notation. Assume that L is an operator of the form (1.2) such that $c \equiv 0$ and $a^{dd} \equiv 1/2$. If $\beta \in (0,1]$ we denote by g^β the Green function associated with L in $D_\beta = \{ x \in R^d : |x^d| < \beta \}$. Besides we denote $\tilde{L} = \frac{1}{2}\Delta$ and $\tilde{g}^\beta$ will be the Green function associated with $\tilde{L}$ in D_β. Green potentials associated with g^β (resp. $\tilde{g}^\beta$) will be noted G^β_ν (resp. $\tilde{G}^\beta_\nu$). Identifying the hyperplane $\{x \in R^d : x^d = 0\}$ with R^{d-1} we have a Lebesgue measure (d-1 dimensional) in this hyperplane, which will be denoted by μ. Any function $f \in \mathcal{B}_d(R^{d-1})$ may be viewed as a function in this hyperplane so that the measure $f \cdot \mu$ is well defined. The main result of this section is the following lemma.

Lemma 5.9.

There exists a constant $C > 0$ such that

$$|G^\beta_{f\cdot\mu} - \tilde{G}^\beta_{f\cdot\mu}| \leq C\beta^2(\|f\|_1 + \beta\|f\|_2),$$

for any $f \in \mathcal{C}^2(R^{d-1})$ and $\beta \in (0,1]$.

Proof

We denote by V^β the potential kernel of an L-diffusion in D_β. It is given by the following relation

$$V^\beta f(x) = \int_{D_\beta} g^\beta(x,y) f(y) dy, \quad x \in D_\beta ,$$

for $f \in \mathcal{B}_+(D_\beta)$. Let h be a function satisfying the conditions (5.6) and $f \in \mathcal{C}^2(R^{d-1})$. Then

$J^n f \in \mathscr{C}^2(R^d)$ and if $n > \beta^{-1}$ we have supp $J^n f \subset D_\beta$. By Lemma 5.10 from below we have

$$V^\beta J^n f - U^\beta J^n f = V^\beta (L - \tilde{L}) U^\beta J^n f.$$

Since $J^n f \in \mathscr{C}^2(D_\beta)$, the function $U^\beta J^n f$ belongs at least to the class $\mathscr{C}^2(D_\beta)$ and the following relations hold

$$D_i U^\beta J^n f = U^\beta J^n D_i f, \quad D_{ij} U^\beta J^n f = U^\beta J^n D_{ij} f,$$

for each $i,j \leq d - 1$. Therefore we can write

$$V^\beta J^n f - U^\beta J^n f = V^\beta (u_n + v_n),$$

where

$$u_n = \sum_{i,j \leq d-1} (a^{ij} - \delta^{ij}/2) U^\beta J^n D_{ij} f + \sum_{i \leq d-1} b^i U^\beta J^n D_i f$$

$$v_n = \sum_{i \leq d-1} (a^{id} + a^{di}) D_d U J^n D_i f + b^d D_d U^\beta J^n f.$$

We have used here the fact that $a^{dd} = 1/2$. Applying Corollary 5.8 we get

$$\| u_n \| \leq C \beta \| f \|_2,$$

$$\| v_n \| \leq C \| f \|_1.$$

Further, using Lemma 5.11 from below, we obtain

$$\| V^\beta J^n f - U^\beta J^n f \| \leq C \beta^2 (\| f \|_1 + \beta \| f \|_2).$$

Now, in order to prove the deired inequality, it suffices to check that

$$V^\beta J^n f(x) \to G^\beta_{f \cdot \mu}(x) \text{ and } U^\beta J^n f(x) \to \overline{G}^\beta_{f \cdot \mu}(x)$$

as $n \to \infty$, for each $x \in D_\beta$. This can be done applying Lemma 1.25, the hypotheses of which are verified by a straightforward computation based on the estimate (1.13). The proof is complete.

Lemma 5.10

If V^β and U^β are the kernels appearing the proof of the preceding lemma, then the following equality holds

$$V^{\beta} f - U^{\beta} f = V^{\beta}(L - \tilde{L})U^{\beta} f,$$

for any $f \in \mathscr{C}^{\alpha}(D_{\beta})$.

<u>Proof</u>

First we observe that $U^{\beta} f \in \mathscr{C}^{2+\alpha}$ and hence the operator $L - \tilde{L}$ can be applied to this function. The right hand side of the equality in the statement may be written as

$$V^{\beta} L U^{\beta} f - V^{\beta} \tilde{L} U^{\beta} f.$$

Since U^{β} is the solution of the Poisson probelm, we have $\tilde{L}U^{\beta} f = -f$ and so we can see that the last term in the above expression becomes $V^{\beta} f$. Now let us denote by u the first term, i.e. $u = V^{\beta} L U^{\beta} f$. This function also may be viewed as the solution of the Poisson problem, so that $\tilde{L}u = -LU^{\beta} f$. The function $u + U^{\beta} f$ vanishes on the boundary of D_{β} and is L-harmonic. The maximum principle implies $u + U^{\beta} f = 0$, which shows that $V^{\beta} L U^{\beta} f = -U^{\beta} f$. The proof is complete.

<u>Lemma 5.11.</u>

There exists a constant $C > 0$ such that

$$\| U^{\beta} \| \leq C\beta^2, \quad \| V^{\beta} \| \leq C\beta^2, \quad \beta \in (0,1].$$

<u>Proof</u>

Let $u(x) = \exp ax^d + \exp(-ax^d)$. We have

$$Lu(x) = a(a\,a^{dd}(x) + b^d(x)) \exp ax^d + a(a a^{dd}(x) - b^d(x)) \exp(-ax^d).$$

We choose the constant a so that $a > \| b^d \| / \inf \{ a^{dd}(x) : x \in R^d \}$. Then there is a constant $k > 0$ such that $Lu(x) \geq k$, for any $x \in D_1$. Further, for a given $\beta \in (0,1]$, we put $v(x) = e^{a\beta} + e^{-a\beta} - u(x)$. Obviously $v(x) = 0$, if $x \in \partial D_{\beta}$ and the function $f(x) = -Lv(x)$ satisfies $f \geq k$. From relation (1.9) we get $v = V^{\beta} f$. Then we can write

$$V^{\beta} 1 \leq k^{-1} V^{\beta} f = k^{-1} v.$$

An elementary estimate shows that $v(x) \leq e^{a\beta} + e^{-a\beta} - 2 \leq C\beta^2$, with a constant independent of β. Therefore we have $\| V^\beta \| = \| V^\beta 1 \| \leq Ck^{-1} \beta^2$. The inequality with U^β is similar. The proof is complete.

Lemma 5.12

Let $p : D_1 \to R_+$ be an excessive function with respect to an L-diffusion in D_1. Suppose that p is bounded and $P_{D_\varepsilon} p = p$, for some $\varepsilon \in (0,1)$. Then there exists $f \in \mathcal{C}^q(D_1)$, $f \geq 0$ such that $V^1 f$ is bounded and $p \leq V^1 f$.

Proof

The function v defined in the proof of the preceding lemma, in the case $\beta = 1$, can be written as $v = V^1 f$. On the other hand it is easy to see that $v(x) \geq e^a + e^{-a} - e^{a\varepsilon} - e^{-a\varepsilon} > 0$, fro any $x \in D_\varepsilon$. If we choose a constant b such that $bv \geq p$ on the set D_ε we will have $bv \geq p$ in D_1. Thus the function bf will possess the required properties.

6. Potentials Supported by Small Sets

Lemma 6.1.

Let X be a standard process with state space E. Let $A=(A_t)$ be an additive functional such that its potential $u(x)=E^x(A_\infty)$ is finite for each $x\in E$. Suppose that K is a closed set which supports A in the sense that $A_T=0$, a.s. with $T=T_K$. Assume that D is an open set such that $K\subset D$ and p is an excessive function such that

$$u(x)-P_{E\setminus D}u(x)+P_{E\setminus D}p(x)\leq p(x),$$

for each $x\in K$. Then the following inequality holds

$$u(x)\leq p(x),\ x\in E.$$

Proof.

We introduce the stopping times $T=T_K$, $S=T_{E\setminus D}$, $R_1=T+S\circ\theta_T$, $R_{n+1}=R_1+R_n\circ\theta_{R_1}$, $n=1,2,\ldots$ It is easy to see that $R_{n+1}=R_n+R_1\circ\theta_{R_n}$, and hence

$$R_n\leq R_{n+1}+T\circ\theta_{R_n}\leq R_{n+1}\ .$$

From the equality $X(R_n+T\circ\theta_{R_n})=X_T\circ\theta_{R_n}$, it follows that $X(R_n+T\circ\theta_{R_n})\in K$ if $R_n+T\circ\theta_{R_n}<\infty$. Similarly $X(R_n)\in E\setminus D$ if $R_n<\infty$. These show that $\lim_n R_n\geq\zeta$. Therefore $u(x)=\lim_n E^x(A(R_n))$, because $A=A_{\zeta-}$ by definition of an additive functional of X. Thus the lemma follows once we have proved that

$$(6.1)\qquad E^x(A(R_n)) \leq p(x),\ x \in E.$$

Now we are going to prove this relation by induction. Let us suppose it is true with n. In order to check it with n+1, we first write

$$A(R_{n+1}) = A(R_n)\circ\theta_{R_1} + A(R_1) = (A(R_n)\circ\theta_S + A_S)\circ\theta_T.$$

Then, using (6.1) with n, we obtain

$$E^x(A(R_n)\circ\theta_S) = E^x(E^{X_S}(A(R_n)) \leq E^x(p(X(S))) = P_{E\setminus D}p(x).$$

On the other hand we have $E^x(A_S) = u(x) - P_{CD}u(x)$, and hence we can deduce

$$E^x(A(R_{n+1})) = E^x(E^{X(T)}(A(R_n)\circ\theta_S + A_S)) \leq$$

$$\leq E^x(P_{E\setminus D}p(X_T) + (u - P_{E\setminus D}u)(X_T)).$$

Since $X_T \in K$ if $T < \infty$, we can use the inequality in the hypothesis of the lemma and deduce that the last term is dominated by $E^x(p(X_T)) \leq p(x)$, because p is excessive. So we have proved relation (6.1) with n+1, completing the proof.

Lemma 6.2.

Let L be an operator of the form (1.2) in R^d, $d \geq 3$, such that $c \equiv 0$. Then there exist $C > 1$ and $r > 0$ such that

$$u \leq (1 - C^2\delta^{d-2})^{-1} \| u - P_{R^d\setminus B(a,t)} u \| ,$$

for each $a \in R^d$, $0<t<r$, $0<\delta< C^{-\frac{2}{d-2}}$ and u of the form $u(x)= =E^x(A_\infty)$, with A an additive functional supported by $\overline{B(a,\delta t)}$ in the sense that $A_T=0$, a.s. for $T=T_{\overline{B(a,\delta t)}}$.

(We remark that $\delta< 1$, because $C^{-\frac{2}{d-2}}<1$, and hence $\overline{B(a,\delta t)}\subset B(a,t)$).

Proof

We take $C>1$ and $h>0$ so that the inequalities (1.13) and (1.14) are satisfied. Now let $a\in R^d$ be fixed and put $U(t)= =\{x\in R^d : g(x,a)>t\}$. From (1.13) we deduce that $U(Ct^{2-d})\subset B(a,t)$ for any $t>0$ and from (1.14) we obtain $B(a,t)\subset U(C^{-1}t^{2-d})$ if $t<r$. Therefore we have

$$B(a,\delta t)\subset U(C^{-1}(\delta t)^{2-d})\subset U(Ct^{2-d})\subset B(a,t),$$

provided $\delta^{2-d}>C^2$ and $t<r$. Then we put

$$q(x)=g(x,a)\wedge C^{-1}(\delta t)^{2-d}$$

so that we have

(6.2) $$C^{-1}(\delta t)^{2-d}\leq q(x),\ x\in \overline{B(a,\delta t)},$$

(6.3) $$q(x)\leq Ct^{2-d},\ x\in R^d\setminus B(a,t).$$

We inted to apply the preceding lemma and therefore denote $D=B(a,t)$ and $K=\overline{B(a,\delta t)}$. From relation (6.3) we have

$$P_{R^d\setminus D}\, q \leq C t^{2-d} ,$$

which leads to

$$q(x) - P_{R^d\setminus D}\, q(x) \geq (C^{-1}\delta^{2-d} - C)\, t^{2-d} ,$$

for each $x \in K$. The number $b = C^{-1}\delta^{2-d} - C$ is strictly positive. Putting $p = t^{d-2} b^{-1} q$, we have

$$p(x) - P_{R^d\setminus D}\, p(x) \geq 1, \ x \in K.$$

Thus, if u is as in the statement, we can write

$$u(x) - P_{R^d\setminus D}\, u(x) \leq \| u - P_{R^d\setminus D}\, u \| \, (p(x) - P_{R^d\setminus D}\, p(x))$$

for each $x \in K$. Now we may apply Lemma 6.1 and get

$$u \leq \| u - P_{R^d\setminus D}\, u \| \, p.$$

Since $q = C^{-1}\delta^{2-d} t^{2-d}$, we get $\| p \| = C^{-1}\delta^{2-d} b^{-1} = (1 - C^2\delta^{d-2})^{-1}$ and the preceding inequality leads to the inequality asserted by the lemma.

7. The Case of a Hypersurface

To treat the case of a hypersurface we shall transform a neighbourhood of a piece of surface by a diffeomorphism into a neighbourhood of a piece of a hyperplane. So locally we reduce the problem to that studied in Section 3. Therefore if one wants only local results the surface may be arbitrary. However we intend to prove global results, particularlly an L^2-estimate analogous to (3.2) and (3.5). For this reason we restrict our attention to the case of a compact hypersurface. First we recall some elementary facts from differential geometry.

Let $f\in\mathcal{C}^{3+\alpha}(R^{d-1})$ and set $K=\{x\in R^d : x^d=f(x')\}$. We define $H:R^{d-1}xR\rightarrow R^d$ by

$$(7.1) \qquad H(y,\ t)=(y,f(y))+tn(y),$$

where $n(y)$ is the upper normal vector to the hipersurface K at the point $(y,f(y))$. This normal vector is expressed by

$$n^i(y)=-D_i f(y)\Big(1+\sum_{j=1}^{d-1}(D_j f(y))^2\Big)^{-1/2}\ ,\ i\leqslant d-1\ ,$$
$$n^d(y)=\Big(1+\sum_{j=1}^{d-1}(D_j f(y))^2\Big)^{-1/2}\ .$$

The map H is of class $\mathcal{C}^{2+\alpha}$. The following lemma is known by geometers. Though we are unable to give a precise reference, we omit the proof.

Lemma 7.1

Let γ be a constant such that

$$\sum_{i,j=1}^{d-1} D_{ij}f(y)\xi^i\xi^j \leq \gamma|\xi|^2 , \quad y,\xi\in R^{d-1}.$$

Then the following relations hold

(7.2) $\quad |H(y,t)-H(y,0)| = |t| \ , \ y\in R^{d-1}, \ t\in R$.

(7.3) $\quad |H(y,t)-H(z,0)| > t, \ y,z\in R^{d-1}, y\neq z, \ |t|<\gamma^{-1}$.

Now let us compute the differential of H:

$$dH=\begin{pmatrix} & I+tN & & n^1 \\ & & & \vdots \\ D_1 f+tD_1n^d, & \dots, & D_{d-1}f+tD_{d-1}n^d , & n^d \end{pmatrix}$$

where the matrix N has the components $D_i n^j$, $i,j\leq d-1$. It follows that we can choose $\beta > 0$ such that $dH(y,t)$ is non-singular for each $y\in R^{d-1}$, provided that $|t|<\beta$.

From the above lemma we deduce that H is injective on the strip $R^{d-1}\times(-\gamma^{-1},\gamma^{-1})$. Therefore if we suppose that the number β is choosen such that $\beta\leq\gamma^{-1}$, then H is a diffeomorphism from $D_\beta = R^{d-1}\times(-\beta,\beta)$ onto the open set $E=H(D_\beta)$. Moreover we have

(7.4) $\quad \{x\in E : d(x,K)=\varepsilon\} = H(\{x\in R^d : |x^d| = \varepsilon\}), \ 0<\varepsilon<\beta$.

Let us denote by $F=H^{-1}:E\to D$. We may consider that F is of class $\mathcal{C}^{2+d}$, eventually changeing β to be smaller. Since $F^d\circ H(y,t)=t$, the partial derivatives of the left side term at the point $(y,0)$ will be

$$(D_i F^d)\circ H(y,0)+(D_d F^d)\circ H(y,0)D_i f(y)=0, \quad i\leq d-1,$$

$$\sum_{i=1}^{d}(D_i F^d)\circ H(y,0)n^i(y)=1.$$

From these relations it follows

(7.5) $(D_i F^d)\circ H(y,0)=n^i(y),\ i=1,\dots,d.$

Now let us recall the formula of the "area" on the hypersurface K. It is a measure supported by K wich we will denote by μ. If $u\in\mathcal{B}_+(K)$, then the integral with respect to μ is known to be

$$\int_K u(x)\mu(dx)=\int_{R^{d-1}} u(y,f(y))\,(1+\sum_{i=1}^{d-1}(D_i f(y))^2)^{1/2}dy.$$

Denoting by JH the determinant of the matrix dH we may write

(7.6) $\displaystyle\int u(x)\mu(dx)=\int_{R^{d-1}} u\circ H(y,0)|JH(y,0)|\,dy.$

Further we are going to describe the transport of an L-diffusion by a diffeomorphism. Let us suppose that L is an operator of the form (1.2) with $c\equiv 0$ in $R^d (d\geq 3)$. Suppose that E and D are open sets in R^d and $F:E\to D$ is a diffeomorphism onto D of class $\mathcal{C}^{2+\alpha}$. Let $u:D\to R$ possess second order derivatives. Then one has

(7.7) $\qquad L(u\circ F)=(\hat{L}u)\circ F,$

where

$$\hat{L}=\sum_{i,j=1}^{d}\hat{a}^{ij}D_{ij}+\sum_{i=1}^{d}\hat{b}^{i}D_{i}\ ,$$

$$\hat{a}^{ij}=\sum_{p,q=1}^{d}(a^{pq}D_pF^iD_qF^j)\circ F^{-1},$$

$$\hat{b}^{i}=(LF^{i})\circ F^{-1}\ .$$

The coefficients $\hat{a}^{ij}$, $\hat{b}^{i}$ are of class $\mathcal{C}^{\alpha}(D)$. The fact that L is defined only in the set D should not produce any trouble. For example, if D has boundary of class $\mathcal{C}^{2+\alpha}$ we can extend L in a neighbourhood of $\bar{D}$ by symmetry with respect to the boundary. Then an extension to R^d may obviously be done such that L is of the form (1.2). If D has not a smooth boundary we may restrict our attention to a smaller set with boundary of desired type.

Let us consider a process $X=(\Omega,\mathcal{M},\mathcal{M}_t,X_t,\theta_t,P^x)$ which is an L-diffusion in E. Then we can construct another process $\hat{X}=(\Omega,\mathcal{M},\mathcal{M}_t,\hat{X}_t,\theta_t,\hat{P}^x)$ as follows

$$(7.8)\qquad \hat{X}_t=F\circ X_t,\quad \hat{P}^x=P^{F^{-1}(x)},\ x\in D$$

It is easy to see that $\hat{X}$ is an $\hat{L}$-diffusion in D. If V^E is the potential kernel of X and $\hat{V}^D$ is the potential kernel of $\hat{X}$, then for each $u\in\mathcal{B}_+(D)$ we have

$$(7.9)\qquad V^E(u\circ F)=(\hat{V}^Du)\circ F.$$

The corresponding Green functions g^E and $\hat{g}^D$ are related by the following formula, which follows from the preceding one

$$(7.10)\quad \hat{g}^D(x,y)=g^E(F^{-1}(x),\ F^{-1}(y))\left|\mathcal{J}F^{-1}(y)\right|,\ x,y\in D.$$

Now we state the main result of this paper.

Theorem 7.2

Let L be an operator of the form (1.2) in $R^d (d \geq 3)$ such that $a^{ij} \in \mathcal{C}^{2+\alpha}(R^d)$, $b^i \in \mathcal{C}^{1+\alpha}(R^d)$, $i,j=1,\dots,d$, and $c \equiv 0$. Let K be a compact hypersurface of class $\mathcal{C}^{3+\alpha}$ in R^d and set

$$a(x) = 2\sum_{i,j=1}^{d} a^{ij}(x) n^i(x) n^j(x), \quad x \in K,$$

where $n^i(x)$, $i=1,\dots,d$ are the components of a unit vector normal to the surface K at the point x. Let μ be the measure "surface area" on K and set $\lambda = a.\mu$. Assume that X is an L-diffusion in R^d and A^ε is the functional defined by (2.1) for each $\varepsilon > 0$. Then there exists a CAF, A, such that

$$(7.11) \qquad \lim_{\varepsilon \to 0} \sup_t |A_t^\varepsilon - A_t| = 0, \text{ a.s.},$$

$$(7.12) \qquad E^x(\sup_t |A_t^\varepsilon - A_t|^2)^{1/2} \leq C\varepsilon^{1/4}(\ln 1/\varepsilon)^{1/2}, \ x \in R^d, \varepsilon \in (0, \varepsilon_o),$$

where ε_o is a constant independent of x. The functional A is determined by the following relation

$$(7.13) \qquad E^x(A_\infty) = \int_K g(x,y)\lambda(dy), \ x \in R^d.$$

The proof of this theorem will be similar to that of Theorem 3.1 once we have established an estimate for $\| u - G_\lambda \|$, where $u(x) = E^x(A_\infty^\varepsilon)$. As in the case of Theorem 3.3, the difficult point is just this estimate. The estimate will be obtained by several lemmas and, at a certain point, we use the analogous estimate obtained in the proof of Theorem 3.3.

Now let x_o be an arbitrary point of K. The discussion in the first part of this section shows that we may choose an open neighbourhood B of x_o and a diffeomorphism $F:B \to \hat{B}$ onto a ball $\hat{B}=\{x\in R^d : |x|<\beta\}$ such that $F(x_o)=0$, $F(K\cap B)=\{x\in\hat{B}: x^d=0\}$ and

$$F(\{x\in B: d(x,K)=\varepsilon\})=\{x\in B: |x^d|=\varepsilon\},\ \varepsilon<\beta .$$

The last relation follows from (7.4). We denote by $\hat{L}$ the operator defined by (7.7) in $\hat{B}$. We fix an L-diffusion in R^d and denote by $\bar{X}$ the restriciton of X to B. Then we transport $\bar{X}$ by F according to formula (7.8), obtaining in $\hat{B}$ a process which we denote by $\hat{X}$. If $t<T_{R^d\setminus B}(\omega)$, the functional $A_t^{\varepsilon}(\omega)$ coincide with the corresponding functional $\hat{A}_t^{\varepsilon}$ considered with respect to $\hat{X}$ relative to $\hat{K}=\{x\in\hat{B}: x^d=0\}$. In the proof of Theorem 3.3 we observed that the potential u is bounded (in a strip). Thus we conclude that the function

$$(7.14)\qquad E^x(A^{\varepsilon}(T_{R^d\setminus B})),\ x\in B$$

is bounded. This allows us to a deduce the following lemma.

<u>Lemma 7.3</u>

If ε is small, the function u is bounded.

<u>Proof.</u>

For a point $x_o\in K$ we choose B,F and $\hat{B}$ as above. Then we take $r>0$ such that $\overline{B(x_o,r)}\subset B$. In the proof of Lemma 6.2 we saw that there exist $\delta\in(0,1)$ and a bounded excessive function $\bar{p}$ such that

$$p(x)-P_{R^d\setminus B(x_o,r)}p(x)\geq 1,\ x\in B(x_o,\delta r).$$

We set $K'=K\cap\overline{B(x_o,\delta r)}$ and $A'_t=\int_{[0,t]}1_{K'}(X_t)\,dA^{\varepsilon}_t$. Since x_o is arbitrary a compacity argument reduces the boundedness of u to the boundedness of the function u' defined by $u'(x)=E^x(A'_\infty)$, $x\in R^d$. To prove that u' is bounded we first remark that the function v defined by

$$v(x)=E^x(A'_R),\ x\in B(x_o,r),$$

with $R=T_{R^d\setminus B(r_o,r)}$, is bounded, because it is dominated by the function of (7.14). Further we will obtain that u' is bounded by the same method as in the proof of Lemma 6.1. So we put

$$Q=T_{K'},\ R_1=Q+R\circ\theta_Q,\ R_{n+1}=R_1+R_n\circ\theta_{R_1}.$$

Then we have $R_n\nearrow\infty$ a.s. We choose a constant $C>0$ such that $v+2\varepsilon\leq C(p-P_Rp)$ on K' and assert that $u'\leq Cp$. This last estimate follows from

$$E^x(A'(R_n))\leq Cp(x),\ x\in R^d,\ n=1,2,\ldots \tag{7.15}$$

Now we are going to prove relation (7.15) by induction. From Lemma 2.3 we get

$$A'(R_{n+1})\leq A'(R_n)\circ\theta(R_1)+A'(R_1)+\varepsilon,\ \text{a.s.}$$

Since $\theta(R_1)=\theta(R)\circ\theta(Q)$ and $A'(R_1)\leq A'(R)\circ\theta(Q)+\varepsilon$, we deduce

$$A'(R_{n+1})\leq A'(R_n)\circ\theta_R\circ\theta_Q+A'(R)\circ\theta(Q)+2\varepsilon,\ \text{a.s.},$$

and hence

$$E^x(A'(R_{n+1}))\leq E^x(E^{X(Q)}(A'(R_n)\circ\Theta(R)+A'(R))+2\varepsilon).$$

If we suppose that relation (7.15) holds with n, we deduce

$$E^x(A'(R_n)\circ\Theta(R))\leq CP_Rp(x),\ x\in R^d.$$

Combining this with the preceding relation we obtain

$$E^x(A'(R_{n+1}))\leq E^x(CP_Rp(X_Q)+v(X_Q)+2\varepsilon)\leq$$

$$\leq E^x(Cp(X_Q))\leq Cp(x),$$

which is relation (7.15) for n+1. The lemma is proved.

Now we choose a number $\gamma\in(0,\beta)$ and set $\hat{O}=\{x\in R^d: |x|<\gamma\}$, $O=F^{-1}(\hat{O})$. We have $\hat{O}\subset\hat{B}$ and $O\subset B$. The next step in the proof of Theorem 7.2 is the following lemma.

<u>Lemma 7.4.</u>

There exist $C>0$ and $\varepsilon_o>0$ such that

$$\| u-P_{R^d\setminus O}u-G_\lambda^O\|\leq C\sqrt{\varepsilon\ln 1/\varepsilon}\ ,\quad \varepsilon\in(0,\varepsilon_o).$$

<u>Proof</u>

Let $R=T_{R^d\setminus O}$ and set $v(x)=E^x(A_R^\varepsilon)$. By Lemma 2.3 we have $u-\varepsilon\leq v+P_Ru\leq u$, which implies

$$|u-P_Ru-v|\leq\varepsilon$$

Now with respect to the process $\hat{X}$ in $\hat{B}$ we set $\hat{R}=\hat{T}_{\hat{B}\setminus\hat{O}}$, $\hat{v}(x)=\hat{E}^x(\hat{A}^\varepsilon(\hat{R}))$. Then obviously $\hat{v}(x)=v(F^{-1}(x))$ for each $x\in O$. Obviously we may assume $\beta\leq 1$ so that $\hat{B}\subset D_1$.

We also assume that $\hat{L}$ is extended to an operator of the form (1.2) in R^d with $c\equiv 0$. We denote by $\hat{\hat{X}}$ an $\hat{L}$-diffusion in D_1 and by $\hat{\hat{A}}^\varepsilon$ the functional given by (2.1) relative to $\hat{\hat{X}}$ with respect to $\hat{\hat{K}}=\{x\in R^d:x^d=0\}$. Further we set $\hat{\hat{u}}(x)=\hat{\hat{E}}^x(\hat{\hat{A}}^\varepsilon_\infty)$, $x\in D_1$ and $\hat{\hat{v}}(x)=\hat{\hat{E}}^x(\hat{\hat{A}}^\varepsilon(\hat{\hat{R}}))$, $x\in\hat{O}$, with $\hat{\hat{R}}=\hat{\hat{T}}_{D_1\setminus\hat{O}}$. Then we have

$$|\hat{\hat{u}}-\hat{\hat{P}}_{\hat{\hat{R}}}\hat{\hat{u}}-\hat{\hat{v}}|\leq\varepsilon$$

Since the restriction of $\hat{\hat{X}}$ to $\hat{B}$ has the same transition function as $\hat{X}$, they have identical hitting distributions. The functions $\hat{v}$ and $\hat{\hat{v}}$ both are represented in terms of hitting distributions by means of a formula analogous to (3.7′). Therefore we have $\hat{v}=\hat{\hat{v}}$, and hence

$$\text{(7.16)}\qquad |u-P_R u-(\hat{\hat{u}}-\hat{\hat{P}}_{\hat{\hat{R}}}\hat{\hat{u}})\circ F|\leq 2\varepsilon .$$

Further we apply estimation (3.13) and we get

$$|\hat{\hat{u}}-\hat{G}^1_{\hat{\lambda}}|\leq C\sqrt{\varepsilon\ln 1/\varepsilon},\qquad \text{for } \varepsilon\in(0,\varepsilon_o),$$

where $\hat{\lambda}(dx)=2\hat{a}^{dd}(x',0)dx'$, and $\hat{G}^1_{\hat{\lambda}}$ is the Green potential associated with the Green function $\hat{g}^1$ of $\hat{L}$ in D_1.

From this inequality we immediately get

$$\text{(7.17)}\qquad |\hat{\hat{u}}-\hat{\hat{P}}_{\hat{\hat{R}}}\hat{\hat{u}}-\hat{G}^O_{\hat{\lambda}}|\leq C\sqrt{\varepsilon\ln 1/\varepsilon}$$

Let us look now at the measure $\hat{\lambda}$ and its Green potential. By relation (7.5) we have

$$\hat{a}^{dd}(y,0)=\sum_{p,q=1}^{d} a^{pq}(F^{-1}(y,0))n^{p}(F^{-1}(y,0))n^{q}(F^{-1}(y,0))=1/2\,a(F^{-1}(y,0)),$$

where a is the function defined in the statement of the theorem. Using relation (7.10) and then (7.6) we get

$$\hat{G}^{\hat{O}}_{\hat{\lambda}}(F(x))=\int_{\hat{O}}\hat{g}^{\hat{O}}(F(x),z)\hat{\lambda}(dz)=$$
$$=\int_{\hat{O}} g^{O}(x,F^{-1}(y,0))\,|JF^{-1}(y,0)|\,a(F^{-1}(y,0))dy=G^{O}_{\lambda}(x)\ .$$

From this relation and the estimates (7.16) and (7.17) we get the inequality asserted by the lemma.

Now let us look at the function u. By Lemma 2.4 we know that $P_D u=u$ for each open set D such that $\overline{V^{\varepsilon}}\subset D$. Since u is bounded we can find a function $f\in\mathcal{B}_+(R^d)$ with compact support such that $Vf\geq u$. In particular u is a natural potential. By Theorem 1.23 we have a measure ν supported by $\overline{V^{\varepsilon}}$ such that $u=G_\nu$. We set $\nu(x,r)=1_{B(x,r)}\cdot\nu$

<u>Lemma 7.5</u>

There exist $C>0$ and $\delta\in(0,1)$ such that

$$G_{\nu(x,r)}\leq C(r+\sqrt{\varepsilon\ln 1/\varepsilon}),\quad \varepsilon\in(0,\varepsilon_0),$$

for each $x\in O$, and $r>0$ such that $B(x,\delta^{-1}r)\subset O$, with ε_0 given by the preceding lemma.

<u>Proof.</u>

Let δ be given by Lemma 6.2 and set $E=B(x,\delta^{-1}r)$. By the preceding lemma we get

$$|u-P_{R^d\setminus E}u-G_\lambda^E| \leq C\sqrt{\varepsilon \ln 1/\varepsilon}$$

Obviously we have $G_\lambda^E \leq G_{1_E\cdot\lambda}$. Then a strightforward computation based on estimation (1.13) shows that the last potential is bounded by Cr, where $C>0$ is a constant independent of x and r. Thus we get

$$|u-P_{R^d\setminus E}u| \leq C(r+\sqrt{\varepsilon \ln 1/\varepsilon}).$$

On the other hand we have $u-P_{R^d\setminus E}u=G_\nu^E \geq G_{\nu(x,r)}^E$. Applying Lemma 6.2 we obtain the desired estimate. The lemma is proved.

<u>Lemma 7.6</u>

There exist $C>0$ and $\varepsilon_o>0$ such that

$$|u-G_\lambda| \leq C\sqrt{\varepsilon}\,(\ln 1/\varepsilon)^{3/2}, \quad \varepsilon \in (0,\varepsilon_o).$$

<u>Proof</u>

We are going to show that for each function $f\in\mathcal{C}^\infty(R^d)$ with the properties supp $f\subset\mathcal{O}$ and $0\leq f\leq 1$, there exist C and ε_o such that

$$(7.18) \qquad |G_{f.\nu}-G_{f.\lambda}| \leq C\sqrt{\varepsilon}\,(\ln 1/\varepsilon)^{3/2}, \quad \varepsilon\in(0,\varepsilon_o).$$

Once we have proved this estimate, the lemma will follow using a finite family $\{f_i\}_{i\in I}$ of functions of this type such that $\nu=\sum_{i\in I} f_i\cdot\nu$ and $\lambda=\sum_{i\in I} f_i\cdot\lambda$. The existence of such a finite family is ensured by the fact that both ν and λ have compact supports and the point x_o and its neighbourhood $\mathcal{O}$ are arbitrary.

Now let us prove relation (7.18). For each $x \in R^d$ we have

$$G_{f.\nu}(x)-G_{f.\lambda}(x)=\int_O g(x,y)f(y)(\nu-\lambda)(dy).$$

It suffices to estimate the above expression on a compact set M such that supp $f \subset \overset{\circ}{M}$, (on account of Lemma 1.24(a)). We fix such a compact M, which also is included in O. Let $r_o=d(M,CO)$ and δ furnished by Lemma 7.5. If $r<\delta r_o$ we have

$$\int_{B(x,r)} g(x,y)\nu(dy) \leq C(r+\sqrt{\varepsilon \ln 1/\varepsilon}).$$

As we have already mentioned (in the proof of the preceding lemma) we have

$$\int_{B(x,r)} g(x,y)\lambda(dy) \leq Cr.$$

Now we choose a function $\ell \in \mathcal{C}^{\infty}(R)$ such that $0 \leq \ell \leq 1$, $\ell(t)=0$ if $t \leq 1/2$, and $\ell(t)=1$ if $t \geq 1$. We set $\ell_r(y)=\ell(r^{-1}|x-y|)$ and this function will satisfy

$$0 \leq 1-\ell_r \leq 1_{B(x,r)} .$$

By the preceeding estimates we get

$$\left|\int_O (1-\ell_r(y))g(x,y)f(y)(\nu-\lambda)(dy)\right| \leq C(r+\sqrt{\varepsilon \ln 1/\varepsilon}).$$

This implies

$$|G_{f\cdot\nu}(x)-G_{f\cdot\lambda}(x)|\leq C(r+\sqrt{\varepsilon\ln 1/\varepsilon})+\left|\int_{\mathcal{O}} h(y)(\nu-\lambda)(dy)\right|,$$

where $h(g)=\ell_r(y)g(x,y)f(y)$. The function h vanish in the ball $B(x,r/2)$ and outside M. By Lemma 1.27 we have $h\in\mathcal{E}^2(\mathcal{O})$. Then Lemma 1.28 implies

$$\int_{\mathcal{O}} h(y)(\nu-\lambda)(dy)=-\int_{\mathcal{O}}(G^{\mathcal{O}}_{\nu}(y)-G^{\mathcal{O}}_{\lambda}(y))L^{*}h(y)\,dy.$$

By Lemma 7.4 we have $\|G^{\mathcal{O}}_{\nu}-G^{\mathcal{O}}_{\lambda}\|\leq C\sqrt{\varepsilon\ln 1/\varepsilon}$. On the other hand, a strightforward computation together with the estimates for $g(x,y)$, $D^{y}_{i}g(x,y)$ and $D^{y}_{ij}g(x,y)$ lead to

$$|L^{*}h(y)|\leq C\,\|f\|_2\,|x-y|^{-d},$$

provided that $|x-y|\geq r/2$. It follows that

$$\int_{\mathcal{O}}|L^{*}h(y)|\,dy\leq C\ln 1/r,$$

and hence

$$\left|\int_{\mathcal{O}} h(y)(\nu-\lambda)(dy)\right|\leq C\sqrt{\varepsilon\ln 1/\varepsilon}\,(\ln 1/r).$$

Putting $r=\varepsilon$ we get estimate (7.18), which completes the proof of the lemma.

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