EXTENSIONS OF GROUPS AND SIMPLE C^*-ALGEBRAS (preliminary version)

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Since Powers ([9]) proved that the reduced C*-algebra of the free group on two generators $C_r^*(F_2)$ is simple with unique trace, several other classes consisting of discrete groups G such that $C_r^*(G)$ is simple with unique trace have been produced (see e.g. [1], [4]).

There are two types of combinatorial conditions which imply the simplicity of $C_r^*(G)$ and the uniqueness of its trace. Examples of such groups were given by Akemann and Lee - the groups which contain a free normal subgroup, with trivial centralizer and by P. de la Harpe - the groups which satisfy a certain combinatorial property, called Powers property. Moreover, in [3] we have introduced a weaker combinatorial condition, called weak Powers property and we showed that for a given exact sequence of groups $1 \to G_1 \to G \to G_2 \to 1$ with G_1 and G_2 weak Powers groups, it follows that $C_r^*(G)$ is simple with unique trace, although the group G may be not weak Powers group.

In this paper, we prove that the above assertion is still true in the four cases when G_1 and G_2 satisfy either the

Akemann-Lee condition or the weak Powers property.

Let G be a discrete group with unit 1, let $l^2(G)$ be the Hilbert space of the square summable complex functions on G. Denote by $\mathbb{C}[G]$ the group algebra of G, viewed as an operator algebra on $l^2(G)$. The elements of $\mathbb{C}[G]$ are finite sums $\sum_{g\in G} \lambda_g u(g)$, with $\lambda_g \in \mathbb{C}$, $u(g) \in \mathcal{B}(l^2(G))$ unitaries

$$(u(g)\xi)(s) = \xi(g^{-1}s), \quad \text{for } \xi \in l^2(G), \quad s\in G.$$

The uniform closure $C^*_r(G)$ of $\mathbb{C}[G]$ in $\mathcal{B}(l^2(G))$ coincides with the C*-algebra generated by the unitaries $u(g)$, $g\in G$. The norm on $\mathcal{B}(l^2(G))$ is denoted by $\|\cdot\|$ and the canonical trace on $C^*_r(G)$ is the unique extension of the positive functional

$$\tau : \mathbb{C}[G] \to \mathbb{C}, \quad \tau\Big(\sum_{g\in G} \lambda_g u(g)\Big) = \lambda_1 .$$

When $Y = \sum_{g\in G} \lambda_g u(g) \in \mathbb{C}[G]$, we denote $\operatorname{supp} Y = \{g \mid \lambda_g \neq 0\}$.

For $g_1, \ldots, g_n \in G$ one defines the unital complete positive map

$$\Phi : C^*_r(G) \to C^*_r(G), \quad \Phi(a) = \frac{1}{n}\sum_{i=1}^{n} u(g_i) a u(g_i)^* .$$

Such maps are called ([1]) averaging processes. Clearly, if Φ and Ψ are averaging processes, then $\Phi\circ\Psi$ is still an averaging process. In particular, for $g\in G$, $n\in N^*$, consider the following averaging process:

$$\theta_{gn} : C^*_r(G) \to C^*_r(G), \quad \theta_{gn}(a) = \frac{1}{n}\sum_{i=1}^{n} u(g)^i a u(g)^{-i} .$$

A subset X of a group G is called free if X is a basis for a subgroup of G. Recall the following important result ([2])

(1) For any free subset $\{x_1,\ldots,x_n\}$ of G, $\left\|\sum_{i=1}^{n} u(x_i)\right\| = 2\sqrt{n-1}$ in $C_r^*(G)$.

Recall also the following basic lemma (see e.g. [5, Proposition 1.9]).

(2) Let X be a subset of a group G. Then X is free if and only if $X \cap X^{-1} \neq \emptyset$ and no product $w = x_1 \ldots x_n$ is trivial, where $n \geq 1$, $x_1,\ldots,x_n \in X^{\pm 1}$ and all $x_i x_{i+1} \neq 1$.

We shall use two simple consequences of (2).

(3) Let G be a group, H a normal subgroup of G, G/H the quotient map and $Y = \{y_i\}_{i \in I}$ a free subset of G/H. If $x_i \in G$ and all $\pi(x_i) = y_i$, then $X = \{x_i\}_{i \in I}$ is a free subset of G.

(4) If $\{a,b\}$ is a free subset in G, then $\{a^n b^n\}_{n \geq 1}$ is also free. Note that (4) is exercise 12 in [6, page 51].

Using (3) and ideas from [1], we get the following lemma (for $H = \{1\}$, this lemma is just theorem 3 in [1]).

LEMMA. Let G be a discrete group and let H be a normal subgroup of G such that G/H contains a free normal subgroup F, with trivial centralizer in G/H. Then there exists a sequence $\{\theta_n\}_{n \geq 1}$ of averaging processes on G such that

$$\lim_{n \to \infty} \|\theta_n(u(g))\| = 0 \quad , \quad \forall\, g \in G \setminus H .$$

<u>Proof</u>. Assume first that $\{y_1,\ldots,y_k\}$ is a basis for the free group F. Let $x_i \in \pi^{-1}(y_i)$, $i=1,\ldots,k$ and set

$$\theta_n = \theta_{x_1^n} \circ \theta_{x_k^n} \circ \theta_{x_{k-1}^n} \circ \ldots \circ \theta_{x_1^n} \quad . \tag{5}$$

Clearly, θ_n are averaging processes on G.

Let $g \in G \setminus H$ and set $w = \pi(g) \in G/H \setminus \{\pi(1)\}$. In view of the assumption that the centralizer of F in G/H is trivial, there exists $i \in \{1,\ldots,k\}$ such that $w^{-1}y_i w \neq y_i$. Choose p be the first such index and denote either $q=p+1$, for $p<k$ or $q=1$ for $p=k$.

When $\{w^{-1}y_p w, y_p\}$ is free in G/H, (3) and (4) tell us that $\{(g^{-1}x_p g)^i x_p^{-i}\}_{i\geq 1}$ is free in G, so (1) applies and

$$\|\theta_n(u(g))\| \leq \|\theta_{x_p^n}(u(g))\| = \|u(g)*\theta_{x_p^n}(u(g))\| =$$

$$= \frac{1}{n}\left\|\sum_{i=1}^{n} u((g^{-1}x_p g)^i x_p^{-i})\right\| = \frac{2\sqrt{n-1}}{n} \quad .$$

When $\{w^{-1}y_p w, y_p\}$ is not free in G/H, $w^{-1}y_p w$ is a generator of F which commutes with y_p, hence $w^{-1}y_p w = y_p^{\pm 1}$. It follows that $w^{-1}y_p w = y_p^{-1}$ and $w^{-1}y_p^i w y_p^i = \pi(1)$. This implies that $g^{-1}x_p^i g x_p^i = h_i \in H$, for any i, and so

$$\theta_{x_q^n} \circ \theta_{x_p^n}(u(g)) = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} u(x_q^j x_p^i g x_p^{-i} x_q^{-j}) =$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} u(x_q^j g h_i x_p^{-2i} x_q^{-j}) \quad . \tag{6}$$

For every $i \in \{1,\ldots,n\}$ one obtains

$$\left\|\sum_{j=1}^{n} u(x_q^j g h_i x_p^{-2i} x_q^{-j})\right\| = \left\|\sum_{j=1}^{n} u(x_p^{2i} h_i^{-1} x_q^j g h_i x_p^{-2i} x_q^{-j})\right\| =$$

$$= \left\|\sum_{j=1}^{n} u((x_p^{2i} h_i^{-1} g^{-1} x_q g h_i x_p^{-2i})^j x_q^j)\right\| . \tag{7}$$

The element $y_p^{2i}w^{-1}y_q w y_p^{-2i} \in F$ commutes with y_q exactly when $y_p^{2i}w^{-1}y_q w y_p^{-2i} = y_q^s$, for an integer s and this is the case for at most one $i \in \{1,\dots,n\}$. Let i_o be such an index. Using (3) and (4) it follows that $\{(x_p^{2i}h_i^{-1}g^{-1}x_q g h_i x_p^{-2i})^j x_q^{-j}\}_{j\geq 1}$ is free in G, for $i \neq i_o$. Now, (1) and (7) imply

$$\left\| \sum_{j=1}^{n} u(x_q^j g h_i x_p^{-2i} x_q^{-j}) \right\| = 2\sqrt{n-1} \quad ,$$

and so, by (5):

$$\|\theta_n(u(g))\| \leq \|\theta_{x_q^n} \circ \theta_{x_p^n}(u(g))\| \leq$$

$$\leq \frac{1}{n^2} \left\| \sum_{j=1}^{n} u(x_q^j g h_i x_p^{-2i_o} x_q^{-j}) \right\| + \frac{n-1}{n^2} \cdot 2\sqrt{n-1} \leq \frac{1}{n} + \frac{n-1}{n^2} \cdot 2\sqrt{n-1} \quad .$$

Finally, in the case when $\{y_i\}_{i\geq 1}$ is a basis for F, take $x_i \in \pi^{-1}(y_i)$, $\theta_n = \theta_{x_n^n} \circ \theta_{x_{n-1}^{n-1}} \dots \circ \theta_{x_1^1}$. The same proof is still valid for $n > q = p+1$. □

A weak Powers group is a group G having the following property: given any non-empty finite subset F of $G \setminus \{1\}$ which is included into a conjugacy class, and any integer $N \geq 1$, there exist a partition $G = A \amalg B$ and elements $g_1, \dots, g_N$ in G such that

$$fA \cap A = \emptyset, \quad \text{for all} \quad f \in F \quad \text{and}$$

$$g_j B \cap g_k B = \emptyset, \quad \text{for} \quad j,k = 1,\dots,N, \quad j \neq k \quad .$$

Using the previous lemma and techniques from [3], we prove the main result in this paper.

THEOREM. Let $1 \to G_1 \to G \to G_2 \to 1$ be an exact sequence of discrete groups such that one of the following conditions is fulfilled:

i) G_1 and G_2 are weak Powers groups;

ii) G_1 is weak Powers group and G_2 contains a free normal subgroup with trivial centralizer;

iii) G_1 contains a free normal subgroup with trivial centralizer and G_2 is weak Powers;

iv) Any G_i contains a free normal subgroup with trivial centralizer.

Then $C_r^*(G)$ is simple, with unique trace.

<u>Proof</u>. G_1 is identified with a normal subgroup of G, G_2 with the quotient group G/G_1 and $\pi : G_1 \to G/G_1$ is the quotient map.

From a standard trick (see e.g. [4, Proposition 3]), it is enough to show that for any $Y = Y^* \in \mathbb{C}[G]$, $\tau(Y) = 0$ and for any $\varepsilon > 0$, there exist $g_1, \dots, g_N \in G$ such that

$$\left\| \frac{1}{N} \sum_{k=1}^{N} u(g_k) Y u(g_k)^* \right\| \leq \varepsilon \quad . \tag{8}$$

It is clear that $Y = \sum_{j=1}^{r} \lambda_j u(g_j) + \bar{\lambda}_j u(g_j)^*$, where $g_j \neq 1$, $j=1,\dots,r$. We may assume without loss of generality that $g_1, \dots, g_p \in G_1$ and $g_{p+1}, \dots, g_r \in G \setminus G_1$. Set $\tilde{Y} = Y_1 + \dots + Y_p$.

When G_1 is a weak Powers group, lemma 2.2 in [3] applies in a particular case ($A = \mathbb{C}$, the 2-cocycle c and the action α are trivial, hence $A \rtimes_{\alpha,c} G = C_r^*(G)$) and we find $h_1, \dots, h_n \in G_1$ such that

$$\left\| \frac{1}{n} \sum_{k=1}^{n} u(h_k) Y u(h_k)^* \right\| \leq \frac{p\varepsilon}{r} \tag{9}$$

If G_1 contains a free normal subgroup with trivial centralizer, the previous lemma in this paper applied for $H=\{1\}$ (theorem 3 in [1]) implies the same fact.

For G/G_1 weak Powers group, the statement follows as in [3, Proposition 2.10]. Take $\tilde{Y}_{p+1}=\frac{1}{n}\sum_{k=1}^{n} u(h_k) Y_{p+1} u(h_k)^*$. supp $\tilde{Y}_{p+1}$ is clearly included in $G \setminus G_1$. Since G/G_1 is a weak Powers group, it is not hard to observe (see the second case in the proof of proposition 1.5 in [3]) that for any finite set $M \subset G$ and for any integer $n_o \geq 1$, there exist $G=A \amalg B$ and $\gamma_1, \ldots, \gamma_{n_o} \in G$ such that

$$\gamma A \cap A = \emptyset, \quad \text{for any} \quad \gamma \in \{g g_{p+1} g^{-1} \mid g \in M\} \quad \text{and}$$

$$\gamma_j B \cap \gamma_k B = \emptyset, \quad \text{for} \quad j,k=1,\ldots,n_o,\ j \neq k .$$

As in the first part of the proof of lemma 2.2 in [3] we find $g_{11}, \ldots, g_{1n_1} \in G$ with

$$\left\| \frac{1}{n_1} \sum_{k_1=1}^{n_1} u(g_{1k_1}) \tilde{Y}_{p+1} u(g_{1k_1})^* \right\| \leq \frac{\varepsilon}{r} . \tag{10}$$

Putting this together with (9) we see that

$$\left\| \frac{1}{nn_1} \sum_{i=1}^{n} \sum_{k_1=1}^{n_1} u(g_{1k_1}) u(h_i) (\tilde{Y}+Y_{p+1}) u(h_i)^* u(g_{1k_1})^* \right\| \leq$$

$$\leq \left\| \frac{1}{n} \sum_{i=1}^{n} u(h_i) \tilde{Y} u(h_i)^* \right\| + \left\| \frac{1}{n_1} \sum_{k_1=1}^{n_1} u(g_{1k_1}) \tilde{Y}_{p+1} u(g_{1k_1})^* \right\| \leq \frac{(p+1)\varepsilon}{r} .$$

By an easy induction argument, we find $g_1,\dots,g_N \in G$ such that (8) is fulfilled.

Assume finally that G/G_1 contains a free normal subgroup with trivial centralizer. Note that supp $\sum_{k=1}^{n} u(h_k)(Y-\tilde{Y})u(h_k) \subset G \setminus G_1$. The lemma applies and there exist $(\Theta_n)_{n\geq 1}$ averaging processes on G such that

$$\lim_{k\to\infty} \|\Theta_k(\frac{1}{n}\sum_{k=1}^{n} u(h_k)(Y-\tilde{Y})u(h_k)^*)\| = 0 ,$$

from which we find $\gamma_1,\dots,\gamma_m \in G$ with

$$\|\frac{1}{m}\sum_{i=1}^{m} u(\gamma_i)(\frac{1}{n}\sum_{k=1}^{n} u(h_k)(Y-\tilde{Y})u(h_k)^*)u(\gamma_i)^*\| \leq \frac{(r-p)\varepsilon}{r} . \tag{11}$$

From (9) and (11) we see that

$$\|\frac{1}{nm}\sum_{i=1}^{m}\sum_{k=1}^{n} u(\gamma_i h_k)Yu(\gamma_i h_k)^*\| \leq$$

$$\leq \|\frac{1}{n}\sum_{k=1}^{n} u(h_k)\tilde{Y}u(h_k)^*\| +$$

$$+ \|\frac{1}{m}\sum_{i=1}^{m} u(\gamma_i)(\frac{1}{n}\sum_{k=1}^{n} u(h_k)(Y-\tilde{Y})u(h_k)^*)u(\gamma_i)^*\| \leq$$

$$\leq \frac{p\varepsilon}{r} + \frac{(r-p)\varepsilon}{r} = \varepsilon .$$

Letting N=nm, one obtains (8). □

REMARK. The previous theorem is still true if in ii, iii or iv we assume only that G_i contains a set of non-abelian, free, normal subgroups $\{F_j\}_{j\in J}$ (J countable), with $\bigcap_{j\in J} H_j=\{1\}$, where H_j is the centralizer of F_j in G_i (using same arguments and corolary 5 from [1]).

The two types of combinatorial properties of groups considered above are very useful in studying groups G with $C_r^*(G)$ simple with unique trace. In several concrete cases it is easy to show that the group satisfy one of this conditions, but it is more difficult to decide the validity of the other.

EXAMPLES. 1) Consider the non-trivial automorphism θ of Z_3 of order 2, the action $\alpha: F_2 \to \mathrm{Aut}\,(Z_3)$, $\alpha_s=\mathrm{id}$, $\alpha_t=\theta$ (s,t are the generators of F_2) and let G be the semi-direct product of Z_3 by F_2 relative to α.

It is obvious that F_2 is a free subgroup of G with trivial centralizer, of finite index ($[G:F_2]=3$), but $C_r^*(G)$ is not simple and has at least two traces because Z_3 is normal in G and amenable (see [8, Proposition 1.6]). This shows us that in theorem 3 from [1], the normality condition is essential. Also, questions (1) and (3) in [4, page 234, respectively page 239] remains open only for <u>normal</u> subgroups.

2) The direct product $F_2 \times F_2$ is clearly a weak Powers group ([3, Proposition 1.4]) which is not free ([5, Observation at page 177]), but it is not clear if $F_2 \times F_2$ contains or not a free normal subgroup with trivial centralizer.

3) The infinite product $G=G_1 \times G_2 \times \ldots$, where G_i are weak Powers groups is a weak Powers group (see e.g. [3, Proposition 1.4 and Lemma 3.1]) which is weak commutative (for every finite

set $F \subset G \setminus \{1\}$, there exists an $g \in G \setminus \{1\}$ such that $gf=fg$, $\forall f \in F$), hence it is inner amenable ([7, Lemma 6.1.1]), but it is not obvious whether G satisfies the Akemann-Lee conditions.

4) Consider the automorphism θ of F_2 defined by $\theta(s)=t^{-1}s^{-1}t^2$, $\theta(t)=t^{-1}s^{-1}t$. Then $\theta^3=id_{F_2}$ and every other element of order 3 in Aut (F_2) is conjugate to θ (compare with [5, Proposition 4.6]). If G is the semi-direct product of F_2 by Z_3 relative to θ, then it is clear that F_2 is a normal subgroup of G with trivial centralizer, but it seems to be difficult to test whether G has the weak Powers property.

5) Let F_Z be the free group on countable many generators indexed by Z, θ be the automorphism of F_Z that shifts the generators by one and G be the semi-direct product of F_Z by Z relative to θ . Then F_Z is a free normal subgroup of G with trivial centralizer, hence $C^*_r(G)$ is simple with unique trace, but it is not clear whether G has the weak Powers property.

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