FEEDBACK, ITERATION AND REPETITION

by

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In order to get an algebraic theory of computation one needs an axiomatic looping operation. This may be Kleene's repetition (cf. [6], for example), Elgot's iteration [7] or feedback [11,12,3]. The proper acyclic context for repetition seems to be a matrix theory (such a theory is equivalent with the theory of matrices over a semiring [8]), for iteration an algebraic theory in the sense of Lawvere and for feedback a (symmetric) strict monoidal category in the sense af MacLane [10].

The equational axioms for the looping operation are not easily codified. A regular algebra cf. Conway [6] is a structure which satisfies all the identities (written in terms of union, composition, repetition and constants 0, 1) which are valid in the algebra of regular events. The theory of matrices over a regular algebra is a matrix theory, but the axioms for repetition are yet unknown (by authors' knowledge). This algebra is intended as a model for the input-output behaviour of nondeterministic computation.

An iteration theory cf. Bloom, Elgot and Wright [1] is a structure which satisfies all the identities (written in terms of tupling, composition, iteration and constants I_a, 0_a, x_i^a) which are valid in the theory of regular trees. The axiomatization for iteration theories was found by Esik (see [9]). An iteration theory is an algebraic theory in which an iteration operation is given fulfiling some axioms. This algebra is intended as a model for the input behaviour of deterministic computation (we use the name "input behaviour" instead of the name "strong behaviour" used by Elgot).

A biflow is a structure which satisfies all the identities (written in terms of separated sum, composition, feedback and constants I_a, $\gamma_{a,b}$) which are valid in the algebra of flowchart schemes. An axiomatization for biflows is given in [12,3]. A biflow is a symmetric strict monoidal category in which a feedback operation is given fulfiling some axioms. This model is more related with the algorithms themselves than with their behaviours.

It is well known that we have some natural inclusions

matrix theories $\subseteq$ algebraic theories $\subseteq$ (symmetric) strict monoidal categories

and the inclusions are strict. It is also known that

matrix theories of regular algebras	$\subseteq$	iteration theories over matrix theories	$\subseteq$	biflows over matrix theories

and

iteration theories $\subseteq$ biflows over algebraic theories.

(It seems likely that one can prove that the above inclusions are strict - this was proved by Esik for the latter one.)

The aim of this paper is to give another passing between iterations and feedbacks than that previously given in [5]. Via this passing the axioms of iteration in an axiomatic system for algebraic theories with iterate (= biflows over algebraic theories) are translated in terms of feedback one-by-one.

When we combine the present passing with the known passing iterations -repetitions [14] we get an easy and natural passing between feedbacks, iterations and repetitions. This is used to give certain axiomatic systems for biflows over algebraic or matrix theories. More importantly, this passing is used in the concluded remarks to emphesize some new advantages of the use of feedback over the use of iteration or repetition than those initially given in [12].

BIFLOWS AND BIFLOWS OVER ALGEBRAIC AND MATRIX THEORIES

We assume the reader is familiar with the calculus of symmetric strict monoidal categories (cf. [10,4], for example), algebraic theories (cf. [7,4], for example) and matrix theories (cf. [8,4], for example).

Let us consider a category $(T,\cdot,I_a)$ having as objects the elements of a monoid $(M,+,\lambda)$. That is the composition satisfies

B1 $(fg)h = f(gh)$ B2 $I_a f = f = f I_b$.

The application of a function f in a point x is written xf, while the composite of $f:A \to B$ and $g:B \to C$ is written in the diagramatic order $f \cdot g$ (or fg).

A category as above is a <u>strict monoidal category</u> (<u>smc</u>, for short) if a sum $+ : T(a,b) \times T(c,d) \to T(a+c,b+d)$ is given fulfiling the axioms

B3 $(f+g)+h = f+(g+h)$ B5 $I_a + I_b = I_{a+b}$

B4 $I_\lambda + f = f = f + I_\lambda$ B6 $(f+g)(u+v) = fu+gv$

for $a \xrightarrow{f} b \xrightarrow{u} c$, $a' \xrightarrow{g} b' \xrightarrow{v} c'$.

An smc T is a <u>symmetric strict monoidal category</u> (<u>ssmc</u>, for short) if some constants $\gamma_{a,b} \in T(a+b,b+a)$ are given fulfiling the axioms

B7 $\gamma_{a,b}\, \gamma_{b,a} = I_{a+b}$ B9 $\gamma_{a,b+c} = (\gamma_{a,b} + I_c)(I_b + \gamma_{a,c})$

B8 $\gamma_{a,\lambda} = I_a$ B10 $(f+g)\gamma_{b,d} = \gamma_{a,c}(g+f)$

for $f:a \to b$, $g:c \to d$.

An smc T is an <u>algebraic theory</u> if some constants $0_a \in T(\lambda,a)$ and $V_a \in T(a+a,a)$ are given fulfiling the axioms

B11 $0_\lambda = I_\lambda$ B13 $V_a f = (f+f)V_b$

B12 $0_\lambda f = 0_a$ B14 $I_{a+b} = (I_a + 0_{b+a} + I_b)V_{a+b}$

In an algebraic theory T, defined as above, a tupling operation $\langle\, ,\, \rangle : T(a,c) \times T(b,c) \to T(a+b,c)$ and some constants $\langle a,b,c \rangle \in T(b,a+b+c)$ may be introduced as follows

$$\langle f,g \rangle = (f+g)V_c \qquad\qquad \langle a,b,c \rangle = 0_a + I_b + 0_c.$$

An algebraic theory may equivalently be introduced as a category T as above in which a tupling $\langle\, ,\, \rangle$ and some constants $\langle a,b,c \rangle$ are given fulfiling the axioms

T1 $T(\lambda,a)$ contains a unique element, denoted 0_a;

T2 $\langle\lambda,a,\lambda\rangle = I_a$ T3 $\langle a,b,c\rangle\langle d,a+b+c,e\rangle = \langle d+a,b,c+e\rangle$;

T4 for every $f \in T(a,c)$ and $g \in T(b,c)$ the morphism $\langle f,g\rangle$ is the unique $h \in T(a+b,c)$ such that $\langle\lambda,a,b\rangle h = f$ and $\langle a,b,\lambda\rangle h = g$.

In a such defined algebraic theory the sum of $f:a\to b$ and $g:c\to d$ is $\langle f\langle\lambda,b,d\rangle, g\langle b,d,\lambda\rangle\rangle$ and $V_a = \langle I_a,I_a\rangle$. We mention that every algebraic theory is an ssmc, where $\gamma_{a,b} = \langle\langle b,a,\lambda\rangle, \langle\lambda,b,a\rangle\rangle$.

An algebraic theory T is a <u>matrix theory</u> if some constants $\perp_a \in T(a,\lambda)$ and $\wedge_a \in T(a,a+a)$ are given fulfiling the axioms

B15 $\perp_\lambda = I_\lambda$ B17 $f\wedge_b = \wedge_a(f+f)$

B16 $f\perp_b = \perp_a$ B18 $\wedge_{a+b}(I_a+0_{b+a}+I_b) = I_{a+b}$

In a matrix theory T, defined as above, a target-tupling $[\ ,\ ] : T(a,b)\times T(a,c)\to T(a,b+c)$ and some constants $[a,b,c]\in T(a+b+c,b)$ may be introduced as follows

$$[f,g] = \wedge_a(f+g) \qquad [a,b,c] = \perp_a+I_a+\perp_c.$$

In a matrix theory T we may also define a union operation $\cup : T(a,b)\times T(a,b)\to T(a,b)$ and some constants $0_{a,b}\in T(a,b)$ as follows

$$f\cup g = \wedge_a(f+g)V_b \qquad 0_{a,b} = \perp_a 0_b$$

and ~~a~~ matrix building operation which maps $f:a\to c$, $g:a\to d$, $h:b\to c$ and $i:b\to d$ in $\begin{bmatrix} f & g \\ h & i\end{bmatrix}\in T(a+b,c+d)$ defined as being

$$\text{either } \langle[f,g],[h,i]\rangle \qquad \text{or } [\langle f,h\rangle,\langle g,i\rangle].$$

For given a,b,c and d every $j\in T(a+b,c+d)$ may be written in a unique way as $j = \begin{bmatrix} f & g \\ h & i\end{bmatrix}$ with f,g,h and i as above.

Let us consider the following axiomatic systems F1-2, I1-4 and R1-3.

Suppose a feedback operation $\uparrow^a : T(a+b,a+c) \to T(b,c)$ is given.

$F1_1$ $\uparrow^a \gamma_{a,a} = I_a$

$F1_2$ $\uparrow^b \uparrow^a f = \uparrow^{a+b} f$

$F1_3$ $\uparrow^{a+b}((\gamma_{a,b}+I_c)\, f\, (\gamma_{b,a}+I_d)) = \uparrow^{b+a} f$

$F1_4$ $(\uparrow^a f)g = \uparrow^a(f(I_a + g))$

$F1_5$ $g(\uparrow^a f) = \uparrow^a((I_a + g)f)$

$F1_6$ $\uparrow^a f + g = \uparrow^a(f + g)$

$F1_7$ $\uparrow^a I_a = I_\lambda$

$F2_1$ $\uparrow^a(\gamma_{a,a}(I_a + f)) = f$

$F2_2$ $\uparrow^b \uparrow^a f = \uparrow^{a+b} f$

$F2_3$ $\uparrow^{a+b}((\gamma_{a,b}+I_c)\, f\, (\gamma_{b,a}+I_d)) = \uparrow^{b+a} f$

$F2_4$ $\uparrow^a(f + 0_d) = \uparrow^a f + 0_d$

$F2_5$ $\uparrow^a \langle f,g \rangle = g\langle \uparrow^a \langle f, I_a + 0_c\rangle, I_c\rangle$

for $f:a \to a+c$, $g:b \to a+c$

A morfism $y:a \to b$ is called $\underline{\uparrow\text{-functorial}}$ if for every $f:a+c \to a+d$ and $g:b+c \to b+d$ the equality $f(y + I_d) = (y + I_c)g$ implies $\uparrow^a f = \uparrow^b g$.

Suppose an iteration operation $^\dagger : T(a,a+b) \to T(a,b)$ is given.

$I1_1$ $(f(V_a + I_b))^\dagger = f^{\dagger\dagger}$

$I1_2$ $(f(g + I_c))^\dagger = f\langle (gf)^\dagger, I_c\rangle$

for $f:a \to b+c$, $g:b \to a$

$I1_3$ $(f(I_a + g))^\dagger = f^\dagger g$

$I2_1$ $f\langle f^\dagger, I_b\rangle = f^\dagger$

$I2_2$ $(f(V_a + I_b))^\dagger = f^{\dagger\dagger}$

$I2_3$ $g(f(g + I_c))^\dagger = (gf)^\dagger$

for $f:a \to b+c$, $g:b \to a$

$I2_4$ $(f(I_a + g))^\dagger = f^\dagger g$

$I3_1$ $(0_a + I_a)^\dagger = I_a$

$I3_2$ $\langle f,g\rangle^\dagger = \langle f^\dagger \langle h, I_c\rangle, h\rangle$

where $h = (g\langle f^\dagger, I_{b+c}\rangle)^\dagger$

$I3_3$ $(\gamma_{a,b} f(\gamma_{b,a}+I_c))^\dagger = \gamma_{a,b} f^\dagger$

$I3_4$ $(f(I_a + g))^\dagger = f^\dagger g$

$I4_1$ $(0_a + f)^\dagger = f$

$I4_2$ $\langle f,g\rangle^\dagger = \langle f^\dagger \langle h, I_c\rangle, h\rangle$

where $h = (g\langle f^\dagger, I_{b+c}\rangle)^\dagger$

$I4_3$ $(\gamma_{a,b} f(\gamma_{b,a}+I_c))^\dagger = \gamma_{a,b} f^\dagger$

$I4_4$ $(f+0_c)^\dagger = f^\dagger + 0_c$

A morphism $y:a \to b$ is called $\underline{\dagger\text{-functorial}}$ if for every $f:a \to a+c$ and $g:b \to b+c$ the equality $f(y + I_c) = yg$ implies $f^\dagger = yg^\dagger$.

Suppose a repetition operation $^* : T(a,a) \to T(a,a)$ is given.

$R1_1$ $(f \cup g)^* = (f^* g)^* f^*$

$R1_2$ $(fg)^* = I_a \cup f(gf)^* g$

$R2_1$ $f^* = I_a \cup ff^*$

$R2_2$ $(f \cup g)^* = (f^* g)^* f^*$

$R2_3$ $(fg)^* f = f(gf)^*$

$R3_1$ $0_{a,a}{}^* = I_a$

$$R3_2 \quad \begin{bmatrix} f & g \\ h & i \end{bmatrix}^* = \begin{bmatrix} f^*gwhf^* \cup f^* & f^*gw \\ whf^* & w \end{bmatrix}, \text{ where } w = (hf^*g \cup i)^*$$

$R3_3$ $(\gamma_{a,b} f \gamma_{b,a})^* = \gamma_{a,b} f^* \gamma_{b,a}$.

A morfism $y{:}a \to b$ is called <u>*-functorial</u> if for every $f{:}a \to a$ and $g{:}b \to b$ the equality $fy = yg$ implies $f^*y = yg^*$.

A <u>biflow</u> is by definition an ssmc in which a feedback is given fulfiling the axioms $F1_{1-7}$. A <u>biflow over an algebraic theory</u> (resp. <u>over a matrix theory</u>) is an algebraic theory (resp. a matrix theory) considered with the natural structure of ssmc in which a feedback is given fulfiling the axioms $F1_{1-7}$.

As a corolary of the theorems in this paper we note that in an algebraic theory (resp. in a matrix theory) the axiomatic systems F1, F2, I1, I2, I3 and I4 (resp. F1, F2, I1, I2, I3, I4, R1, R2 and R3) are equivalent.

<u>Proposition.</u> In an algebraic theory the axiomatic systems I1-4 are equivalent.

<u>Proof.</u> It is known from Esik [9] that $I4_{1-4}$ is equivalent with $I2_1$, $I4_{2-3}$ and $I2_4$. As $I4_1$ follows from $I3_1$ and $I3_4$ we get that I3 <==> I4 holds.

Note that $I3_3$ is a particular case of $I2_3$. By the Proposition B.1 of Appendix B in Stefanescu [13] the axiom $I3_2$ is equivalent with $I2_{2-3}$ in the presence of $I2_1$ and $I3_{3-4}$. Hence I2 <==> I3.

It is easy to see that I1 <==> I2. Indeed, $I1_2$ for $g = I_a$ gives $I2_1$; moreover, $g(f(g + I_c))^\dagger =$ (by $I1_2$) $gf\langle (gf)^\dagger, I_c\rangle =$ (by $I2_1$) $(gf)^\dagger$, hence $I1_2 \Rightarrow I2_3$. Conversely, $I2_1 + I2_3 \Rightarrow I1_2$; indeed, $(f(g + I_c))^\dagger =$ (by $I2_1$) $f(g + I_c)\langle (f(g + I_c))^\dagger, I_c\rangle = f\langle g(f(g + I_c))^\dagger, I_c\rangle =$ (by $I2_3$) $f\langle (gf)^\dagger, I_c\rangle$. ∎

ITERATIONS AND FEEDBACKS IN ALGEBRAIC THEORIES

Let T be an algebraic theory and It(T) (resp. Fd(T)) the set of all iterations (resp. feedbacks) defined on T. We define two applications

$$\alpha : Fd(T) \to It(T) \text{ and } \qquad \beta : It(T) \to Fd(T)$$

as follows

- $\uparrow\alpha$ maps $f \in T(a,a+b)$ in $\uparrow^a\langle f, I_a + 0_b\rangle$;
- $(\dagger\beta)^a$ maps $f = \langle f_1,f_2\rangle \in T(a+b,a+c)$ (with $f_1:a \to a+c$ and $f_2:b \to a+c$) in $f_2\langle f_1{}^\dagger,I_c\rangle$.

Let $Fd_r(T)$ (resp. $Fd_i(T)$) be the subset of all the feedbacks in Fd(T) that obey the axioms $F1_{4-6}$ (resp. $F2_5$) and $It_r(T)$ the subset of all the iterations in It(T) that obey the axiom $I3_4$. Finally, let us consider the restrictions $\alpha_r : Fd_r(T) \to It_r(T)$, $\beta_r : It_r(T) \to Fd_r(T)$, $\alpha_i : Fd_i(T) \to It(T)$ and $\beta_i : It(T) \to Fd_i(T)$ induced by α and β.

<u>Theorem.</u> a) The restrictions α_i, β_i, α_r and β_r are (totally defined) bijective functions. Moreover α_i is the converse of β_i and α_r of β_r.

b) For $k \in [4]$, $\dagger$ satisfied $I4_k$ iff $\dagger\beta$ satisfies $F2_k$.

c) For $k \in [3]$, $\dagger$ satisfies $I3_k$ iff $\dagger\beta$ satisfies $F1_k$.

d) y is $\dagger$-functorial iff y is $\dagger\beta$-functorial.

<u>Proof.</u> a) Note that $\uparrow = \dagger\beta$ satisfies $F2_5$; indeed, $g\langle\uparrow^a\langle f,I_a+0_b\rangle, I_b\rangle = g\langle(I_a+0_b)\langle f^\dagger,I_b\rangle, I_b\rangle = g\langle f^\dagger,I_b\rangle = \uparrow^a\langle f,g\rangle$. Consequently β_i is totally defined. Obviously $\dagger = \dagger\beta\alpha$. For the converse, note that $(\uparrow\alpha\beta)^a$ maps $\langle f_1,f_2\rangle \in T(a+b,a+c)$ (with $f_1:a \to a+c$ and $f_2:b \to a+c$) in $f_2\langle\uparrow^a\langle f_1,I_a+0_c\rangle, I_c\rangle$. Hence $\uparrow = \uparrow\alpha\beta$ for $\uparrow \in Fd_i(T)$.

For the second restriction, note that $\dagger$ satisfies $I3_4$ iff $\dagger\beta$, denoted $\uparrow$, satisfies $F1_4$. Indeed, $\uparrow$ satisfies $F1_4$ iff for every $f = \langle f_1,f_2\rangle : a+b \to a+c$ (with $f_1:a \to a+c$ and $f_2:b \to a+c$) and $g:c \to d$ $(\uparrow^a f)g = f_2\langle f_1{}^\dagger,I_c\rangle g = f_2\langle f_1{}^\dagger g,g\rangle$ is equal to $\uparrow^a(f(I_a+g)) = \uparrow^a\langle f_1(I_a+g), f_2(I_a+g)\rangle = f_2(I_a+g)\langle(f_1(I_a+g))^\dagger, I_d\rangle = f_2\langle(f_1(I_a+g))^\dagger, g\rangle$.

Consequently if $\dagger$ satisfies $I3_4$, then $\uparrow$ satisfies $F1_4$ and if $\uparrow$ satisfies $F1_4$, then by using I_a+0_c for f_2 above we conclude that $\dagger$ satisfies $I3_4$. Hence we have a bijective correspondence between $It_r(T)$ and the subset of all the feedbacks in $Fd(T)$ that satisfy $F2_5 + F1_4$. The conclusion follows if we show that $F2_5 + F1_4 \Longleftrightarrow F1_{4-6}$. Note that:

$F2_5 \Longrightarrow F1_5$; indeed, if $f = \langle f_1,f_2\rangle : a+b \to a+c$ (with $f_1 : a \to a+c$ and $f_2 : b \to a+c$) and $g : d \to b$, then $\uparrow^a((I_a+g)f) = \uparrow^a\langle f_1, gf_2\rangle =$ (by $F2_5$) $gf_2\langle \uparrow^a\langle f_1, I_a+0_c\rangle, I_c\rangle$ = (by $F2_5$) $g \uparrow^a\langle f_1,f_2\rangle = g\uparrow^a f$;

$F2_5 + F1_4 \Longrightarrow F1_6$; indeed, if $f = \langle f_1,f_2\rangle : a+b \to a+c$ (with $f_1 : a \to a+c$ and $f_2 : b \to a+c$) and $g : d \to e$, then $\uparrow^a(f+g) = \uparrow^a\langle f_1+0_e, f_2+g\rangle =$ (by $F2_5$)$(f_2+g)\langle \uparrow^a\langle f_1+0_e, I_a+0_{c+e}\rangle, I_{c+e}\rangle$ = (by $F1_4$)$(f_2+g)\langle \uparrow^a\langle f_1, I_a+0_c\rangle+0_e, I_{c+e}\rangle$ $= (f_2+g)(\langle \uparrow^a\langle f_1,I_a+0_c\rangle, I_c\rangle + I_e) = f_2\langle \uparrow^a\langle f_1, I_a+0_c\rangle, I_c\rangle + g =$ (by $F2_5$) $\uparrow^a f + g$.

$F1_{4-6} \Longrightarrow F2_5$; indeed, if $f : a \to a+c$ and $g : b \to a+c$, then $\uparrow^a\langle f,g\rangle$ $= \uparrow^a[(I_a+g)(\langle f, I_a+0_c\rangle + I_c)(I_a+V_c)] =$ (by $F1_{4-6}$) $g(\uparrow^a\langle f, I_a+0_c\rangle + I_c)V_c = g\langle \uparrow^a\langle f, I_a+0_c\rangle, I_c\rangle$.

b) Let $\uparrow$ and $\dagger$ be such that $\uparrow = \dagger\beta$. The equivalence in the case $k = 1$ holds since for $f : b \to c$ $(0_a + f)^\dagger = \uparrow^a\langle 0_a+f, I_a+0_c\rangle = \uparrow^a(\gamma_{a,a}(I_a+f))$.

For $k = 2$, note that if $f : a \to a+b+c$, $g : b \to a+b+c$ and $i : d \to a+b+c$, then $\uparrow^{a+b}\langle f,g,i\rangle$ $= i\langle\langle f,g\rangle^\dagger, I_c\rangle$ and $\uparrow^b\uparrow^a\langle f,g,i\rangle = \uparrow^b(\langle g,i\rangle\langle f^\dagger, I_{b+c}\rangle)$ $= \uparrow^b\langle g\langle f^\dagger, I_{b+c}\rangle, i\langle f^+, I_{b+c}\rangle\rangle$ $= i\langle f^+, I_{b+c}\rangle\langle h, I_c\rangle = i\langle\langle f^\dagger\langle h, I_c\rangle, h\rangle, I_c\rangle$, where $h = (g\langle f^\dagger, I_{b+c}\rangle)^\dagger$. Consequently $\uparrow$ satisfies $F2_2$ iff $\dagger$ satisfies $I4_2$.

For $k = 3$, note that if $f = \langle f_1,f_2\rangle : b+a+c \to b+a+d$ (with $f_1 : b+a \to b+a+d$ and $f_2 : c \to b+a+d$), then $\uparrow^{a+b}((\gamma_{a,b}+I_c)f(\gamma_{b,a}+I_d))$ $= \uparrow^{a+b}\langle \gamma_{a,b}f_1(\gamma_{b,a}+I_d), f_2(\gamma_{b,a}+I_d)\rangle$ $= f_2(\gamma_{b,a}+I_d)\langle(\gamma_{a,b}f_1(\gamma_{b,a}+I_d))^\dagger, I_d\rangle$ $= f_2\langle \gamma_{b,a}(\gamma_{a,b}f_1(\gamma_{b,a}+I_d))^\dagger, I_d\rangle$ and $\uparrow^{b+a}f = f_2\langle f_1{}^\dagger, I_d\rangle$. Since $\gamma_{a,b}\gamma_{b,a} = I_{a+b}$ it follows that $I4_3 \Longleftrightarrow F2_3$.

For $k = 4$, note that the axioms $F2_4$ and $I4_4$ may be written as $\uparrow^a(f(I_{a+c}+0_d))$ $= (\uparrow^a f)(I_c+0_d)$ and $(f(I_{a+b}+0_c))^\dagger = f^\dagger(I_b+0_c)$, respectively. Now the equivalence $F2_4 \Longleftrightarrow I4_4$ directly follows from the above proof of the equivalence $F1_4 \Longleftrightarrow I3_4$.

The proof of c) is covered by the above proof of b).

d) Suppose that $y:a \to b$ is $\dagger$-functorial and $f = \langle f_1, f_2\rangle : a+c \to a+d$ (with $f_1 : a \to a+d$ and $f_2 : c \to a+d$) and $g = \langle g_1, g_2\rangle : b+c \to b+d$ (with $g_1 : b \to b+d$ and $g_2 : c \to b+d$) are such that $f(y + I_d) = (y + I_c)g$. Then $f_1(y+I_d) = yg_1$ and $f_2(y+I_d) = g_2$. By the $\dagger$-functoriality of y $f_1{}^{\dagger} = yg_1{}^{\dagger}$. Hence $\uparrow^a f = f_2\langle f_1{}^{\dagger}, I_d\rangle = f_2\langle yg_1{}^{\dagger}, I_d\rangle = f_2(y+I_d)\langle g_1{}^{\dagger}, I_d\rangle = g_2\langle g_1{}^{\dagger}, I_d\rangle = \uparrow^b g$.

Conversely, suppose that $y:a \to b$ is $\uparrow$-functorial and $f:a \to a+c$ and $g:b \to b+c$ are such that $f(y+I_c) = yg$. Then $\langle f, I_a+0_c\rangle(y+I_c) = \langle f(y+I_c), y+0_c\rangle = \langle yg, y+0_c\rangle = (y+I_a)\langle g, y+0_c\rangle$. By the $\uparrow$-functoriality of y $\uparrow^a\langle f, I_a+0_c\rangle = \uparrow^b\langle g, y+0_c\rangle$. As $\uparrow^a\langle f, I_a+0_c\rangle = (I_a+0_c)\langle f^{\dagger}, I_c\rangle = f^{\dagger}$ and $\uparrow^b\langle g, y+0_c\rangle = (y+0_c)\langle g^{\dagger}, I_c\rangle = yg^{\dagger}$ the result follows. ∎

<u>Corollary.</u> In an algebraic theory the axiomatic systems F1, F2, I1, I2, I3 and I4 are equivalent. ∎

REPETITIONS, ITERATIONS AND FEEDBACKS IN MATRIX THEORIES

Let T be a matrix theory and Rp(T) the set of all repetitions defined on T. We use the applications in [13]

$$\sigma : It(T) \to Rp(T) \quad \text{and} \quad \tau : Rp(T) \to It(T)$$

defined as follows

- $\dagger\sigma$ maps $f \in T(a,a)$ in $[f, I_a]^{\dagger}$;
- $*\tau$ maps $f = [f_1, f_2] \in T(a, a+b)$ (with $f_1 : a \to a$ and $f_2 : a \to b$) in $f_1^* f_2$.

Finally, let us consider the restrictions $\sigma_r : It_r(T) \to Rp(T)$ and $\tau_r : Rp(T) \to It_r(T)$ induced by σ and τ.

<u>Theorem.</u> a) The restrictions σ_r and τ_r are (totally defined) bijective functions. Moreover, σ_r is the converse of τ_r.

b) For $k \in [3]$, $*$ satisfies $R3_k$ iff $*\tau$ satisfies $I3_k$.

c) For $k \in [3]$, $*$ satisfies $R2_k$ iff $*\tau$ satisfies $I2_k$.

d) For $k\in[2]$, $*$ satisfies $R1_k$ iff $*\tau$ satisfies $I1_k$.

e) y is $*$-functorial iff y is $*\tau$-functorial.

Proof. a) Note that $*\tau$, denoted $\dagger$, satisfies $I3_4$; indeed, if $f=[f_1,f_2]:a\to a+b$ (with $f_1:a\to a$ and $f_2:a\to b$) and $g:b\to c$, then $(f(I_a+g))^\dagger=[f_1,f_2g]^\dagger=f_1^*f_2g=f^\dagger g$. Consequently τ_r is totally defined. Obviously $*=*\tau\sigma$. For the converse note that $\dagger\sigma\tau$ maps $f=[f_1,f_2]=[f_1,I_a](I_a+f_2)\in T(a,a+b)$ (with $f_1:a\to a$ and $f_2:a\to b$) in $[f_1,I_a]^\dagger f_2$. Hence $\dagger=\dagger\sigma\tau$ for $\dagger\in It_r(T)$.

b) Let $\dagger$ and $*$ be such that $\dagger=*\tau$. The equivalence in the case $k=1$ holds since $(0_a+I_a)^\dagger=[0_{a,a},I_a]^\dagger=0_{a,a}^*I_a=0_{a,a}^*$.

Let $k=2$, note that if $f=[f_1,f_2,f_3]:a\to a+b+c$ (with $f_1:a\to a$, $f_2:a\to b$ and $f_3:a\to c$) and $g=[g_1,g_2,g_3]:b\to a+b+c$ (with $g_1:b\to a$, $g_2:b\to b$ and $g_3:b\to c$), then

$$\langle f,g\rangle^\dagger=\begin{bmatrix}f_1 & f_2 & f_3\\ g_1 & g_2 & g_3\end{bmatrix}^\dagger=\begin{bmatrix}f_1 & f_2\\ g_1 & g_2\end{bmatrix}^*\begin{bmatrix}f_3\\ g_3\end{bmatrix}\text{ and } h:=(g\langle f^\dagger,I_{b+c}\rangle)^\dagger=$$

$$=\left([g_1\ g_2\ g_3]\begin{bmatrix}f_1^*f_2 & f_1^*f_3\\ I_b & 0\\ 0 & I_c\end{bmatrix}\right)^\dagger=[g_1f_1^*f_2\cup g_2\quad g_1f_1^*f_3\cup g_3]^\dagger=w(g_1f_1^*f_3\cup g_3)\text{ , where}$$

$w=(g_1f_1^*f_2\cup g_2)^*$, hence

$$\langle f^\dagger\langle h,I_c\rangle,h\rangle=\begin{bmatrix}f_1^*f_2h\cup f_1^*f_3\\ h\end{bmatrix}=\begin{bmatrix}f_1^*f_2wg_1f_1^*\cup f_1^* & f_1^*f_2w\\ wg_1f_1^* & w\end{bmatrix}\begin{bmatrix}f_3\\ g_3\end{bmatrix}.$$

Consequently, if $*$ satisfies $R3_2$ then $\dagger$ satisfies $I3_2$. If $\dagger$ satisfies $I3_2$, then applying

$$\begin{bmatrix}f_1 & f_2\\ g_1 & g_2\end{bmatrix}^*\begin{bmatrix}f_3\\ g_3\end{bmatrix}=\begin{bmatrix}f_1^*f_2wg_1f_1^*\cup f_1^* & f_1^*f_2w\\ wg_1f_1^* & w\end{bmatrix}\begin{bmatrix}f_3\\ g_3\end{bmatrix}$$

for $f_3=I_a$, $g_3=0_{b,a}$ and $f_3=0_{b,a}$, $g_3=I_b$ we get $R3_2$.

For $k=3$, note that if $f=[f_1,f_2]:b+a\to b+a+c$ (with $f_1:b+a\to b+a$ and $f_2:b+c\to c$), then $(\gamma_{a,b}f(\gamma_{b,a}+I_c))^\dagger=[\gamma_{a,b}f_1\gamma_{b,a},\gamma_{a,b}f_2]^\dagger=(\gamma_{a,b}f_1\gamma_{b,a})^*\gamma_{a,b}f_2$ and $\gamma_{a,b}f^\dagger=\gamma_{a,b}f_1^*f_2$. Since $\gamma_{b,a}\gamma_{a,b}=I_{b+a}$ it follows that $R3_3\Longleftrightarrow I3_3$.

c) Let $\dagger$ and $*$ be such that $\dagger=*\tau$. For $k=1$, note that if $f=[f_1,f_2]:a\to a+b$ (with $f_1:a\to a$ and $f_2:a\to b$), then $f^\dagger=f_1^*f_2$ and $f\langle f^\dagger,I_b\rangle=f_1f_1{}^*f_2\cup f_2=(f_1f_1{}^*\cup I_a)f_2$. Hence $R2_1\Longleftrightarrow I2_1$.

For $k = 2$, note that if $f = [f_1,f_2,f_3] : a \to a+a+b$ (with $f_1:a\to a$, $f_2:a\to a$ and $f_3:a\to b$), then $f^{\dagger\dagger} = [f_1^* f_2, f_1^* f_3]^\dagger = (f_1^* f_2)^* f_1{}^* f_3$ and $(f(V_a+I_b))^\dagger = [f_1 \cup f_2, f_3]^\dagger = (f_1 \cup f_2)^* f_3$. Hence $R2_2 \Longleftrightarrow I2_2$.

For $k = 3$, note that if $f = [f_1,f_2] : a \to b+c$ (with $f_1:a\to b$ and $f_2:a\to c$) and $g:b\to a$, then $g(f(g+I_c))^\dagger = g[f_1 g,f_2]^\dagger = g(f_1 g)^* f_2$ and $(gf)^\dagger = [gf_1,gf_2]^\dagger = (gf_1)^* gf_2$. Hence $R2_3 \Longleftrightarrow I2_3$.

d) The case $k = 1$ is covered by c). For $k = 2$, note that if $f = [f_1,f_2] : a \to b+c$ (with $f_1:a\to b$ and $f_2:a\to c$) and $g:b\to a$, then $(f(g+I_c))^\dagger = [f_1 g,f_2]^\dagger = (f_1 g)^* f_2$ and $f\langle (gf)^\dagger,I_c\rangle = [f_1,f_2]\langle (gf_1)^* gf_2,I_c\rangle = f_1(gf_1)^* gf_2 \cup f_2 = (I_a \cup f_1(gf_1)^* g)f_2$. Hence $R1_2 \Longleftrightarrow I1_2$.

e) Suppose that $y:a\to b$ is $*$-functorial and $f = [f_1,f_2] : a \to a+b$ (with $f_1:a\to a$ and $f_2:a\to c$) and $g = [g_1,g_2] : b\to b+c$ (with $g_1:b\to b$ and $g_2:b\to c$) are such that $(f(y+I_c) = yg$. Then $f_1 y = yg_1$ and $f_2 = yg_2$. By the $*$-functoriality of y $f_1^* y = yg_1^*$. Consequently, $yg^\dagger = yg_1^* g_2 = f_1^* yg_2 = f_1^* f_2 = f^\dagger$. Conversely, suppose that $y:a\to b$ is $\dagger$-functorial and $f:a\to a$ and $g:b\to b$ are such that $fy = yg$. Then $[f,y](y+I_b) = [fy,y] = [yg,y] = y[g,I_b]$, hence $[f,y]^\dagger = y[g,I_b]^\dagger$. Therefore $f^*y = yg^*$. ∎

Note that the composites $\alpha\sigma$ and $\tau\beta$ work as follows:

- $\uparrow\alpha\sigma$ maps $f\in T(a,a)$ in $\uparrow^a \begin{bmatrix} f & I_a \\ I_a & 0_{a,a} \end{bmatrix}$;
- $(*\tau\beta)^a$ maps $f = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \in T(a+b,a+c)$ (with $f_{11}:a\to a$, $f_{12}:a\to c$, $f_{21}:b\to a$ and $f_{22}:b\to c$) in $f_{21}f_{11}{}^* f_{12} \cup f_{22}$.

<u>Corollary.</u> a) The restrictions $\alpha_r\sigma_r$ and $\tau_r\beta_r$ are (totally defined) bijective functions. Moreover $\alpha_r\sigma_r$ is the converse of $\tau_r\beta_r$.

b) For $k\in[3]$, $*$ satisfies $R3_k$ iff $*\tau\beta$ satisfies $F1_k$.

c) y is $*$-functorial iff y is $*\tau\beta$-functorial. $\blacksquare$

<u>Corollary.</u> In a matrix theory the axiomatic systems F1, F2, I1, I2, I3, I4, R1, R2 and R3 are equivalent. $\blacksquare$

CONCLUDED REMARKS

Here we give some advantages of the use of feedback over the use of iteration or repetition.

First, the proper acyclic context for the use of feedback is a symmetric strict monoidal category, for iteration an algebraic theory and for repetition a matrix theory. Hence feedback may be used in a more general context than iteration or repetition.

Second, in the context of matrix theories there is a bijection between iterations that obey the axiom $I3_4$ and repetitions. Hence iteration is better than repetition since it displays some properties of the looping operation which are hiddened by repetition. Analogously, in the context of algebraic theories there is a bijection between feedbacks that obey the axiom $F2_5$ and iterations. Hence feedback is better than iteration (resp. repetition) since it displays some properties of the looping operation which are hiddened by iteration (resp. repetition). Naturally, the proofs in terms of feedbacks are longer.

Finally, ler us note that some properties are easier to express in terms of feedback, e.g. the property expressed by the "matrix formula" $R3_2$ or by the "pairing axiom" $I3_2$ is expressed in terms of feedback as $F1_2$.

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