FRIEDRICHS EXTENSION FOR NONCONVEX VARIATIONAL PROBLEM

by

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1. INTRODUCTION

It is well-known that many differential equations satisfy so-called minimum principles, namely they lead to the problem of finding a minimum for a certain functional or a family of functionals.

Let us consider X to be a real topological linear space, X^* its dual and a family of functionals $(F_f)_{f\in X^*}$, $F_f:X\to(-\infty, +\infty]$ which can be written as

$$F_f(v)=g(v)-\langle f,v\rangle \tag{1.1}$$

where $g:X\to(-\infty, +\infty]$ is a proper function. For a given $f\in X^*$ the minimum problem for F_f

(1.2) find $u \in X$ such that $F_f(u) \leq F_f(v)$ for all $v \in X$ is equivalent with the nonlinear equation

(1.3) find $u \in X$ such that $f \in \partial g(u)$

where $\partial g(u) = \{h \in X^* / g(v) - g(u) \geq \langle h, v-u \rangle \quad (\forall)\ v \in X\}$ is the subdifferential of g at u.

If X is a reflexive real Banach space and g is a convex, coercive, lower semicontinuous function then F_f is bounded from below and it exists a minimum point for F_f hence equation (1.3) has at least a solution for all $f \in X^*$. If X is a Banach space and g is a convex, coercive functional which is not lower semicontinuous then problems (1.2) or (1.3) may not have a solution. But if g is an uniform convex functional then all minimizing sequences for F_f have the same limit which is called the Sobolev solution of equation (1.3). In the papers of Ionescu [4], Ionescu, Rosca, Sofonea [6], Dinca, Mateescu [2] a variational characterization for the Sobolev solution is given, namely there exists $\tilde{g}: X \to (-\infty, +\infty]$ such that $\partial \tilde{g}$ extends ∂g (i.e. $D(\partial g) \subset D(\partial \tilde{g})$ $\partial \tilde{g}(x) = \partial g(x)$ for all $x \in D(\partial g)$) and for all $f \in X^*$ u is the Sobolev solution of equation (1.3) iff $f \in \partial \tilde{g}(u)$. Let $A: D(A) \subset X \to X^*$ be a dense defined, symmetric and positive definite linear operator. If we put $D(g) = D(A)$, $g(v) = \frac{1}{2}\langle Av, v \rangle$ then $A = \partial g$. In the paper of Ionescu, Rosca [5] it is proved that the Friedrichs extension $\tilde{A}$ of A is exactly $\partial \tilde{g}$. By the analogy with the linear case we can consider $\partial \tilde{g}$ as the Friedrichs extension of ∂g in the general nonlinear case.

The object of this paper is to study in a similar mode equation (1.3) when X is a topological linear space and g may not be a convex functional. For this kind of framework it is necessary to extend the concept of Sobolev solution; more precisely we say that u is a generalized solution for equation (1.3) if there exists a minimizing sequence for F_f which is converging to u. To be more specific we consider a very simple example with a non convex function g.

<u>EXAMPLE 1.1</u>. Let $g:R\to R$ be given by

$$(1.4)\qquad g(x)=\begin{cases} x^2 & \text{for } x\in(-\infty,0) \\ x^2+1 & \text{for } x\in[0,+\infty) \end{cases}$$

It is easy to see that g is not lower semicontinuous in x=0, $D(\partial g)=(-\infty,0)\cup[1,+\infty)$, $\partial g(x)=\{2x\}$ and $R(\partial g)=(-\infty,0)\cup[2,+\infty)$. If $f\in R(\partial g)$ then equation (1.3) has an unique classical solution. If $f\in(0,2)$ then all minimizing sequences for F_f have x=0 as a limit, hence x=0 is a Sobolev solution for (1.3) though equation (1.3) has no classical solution. If f=2 then $f\in\partial g(1)$, that is x=1 is a classical solution but x=1 is not a Sobolev solution because $x_n=-1/n$ is a minimizing sequence for F_f which has not x=1 as a limit. Moreover x=0 is a generalized solution for equation (3) but neither a classical nor a Sobolev solution. Let us consider $\overline{g}:R\to R$ to be the lower semicontinuous envelope of g given by

$$(1.5)\qquad \overline{g}(x)=\begin{cases} x^2 & \text{for } x\in(-\infty,0] \\ x^2+1 & \text{for } x\in(0,+\infty) \end{cases}$$

We remark that $D(\partial\overline{g})=(-\infty,0]\cup[1,+\infty)$, $\partial\overline{g}(x)=2x$ for $x\neq 0$ and $\partial\overline{g}(0)=[0,2]$. Moreover $R(\partial\overline{g})=R$ and for all $f\in R$ we have that $f\in\partial\overline{g}(u)$ iff u is the generalized solution of equation (1.3).

Hence we have obtained a variational characterization of the generalized solutions. Let us remark that $\partial\overline{g}$ is not a maximal monotone operator, hence we cannot construct a convex function $\overline{\overline{g}}$ sucht that $\partial\overline{g}\equiv\partial\overline{\overline{g}}$ i.e. we cannot have a variational characterization of the generalized solutions with a convex function when g is not convex.

In section 2 the definition of the generalized τ-solution and secventialy generalized τ-solution for equation (1.3) in a linear topological space X with the topology denoted by τ, are introduced. Two functions $\overline{g}$ and $\tilde{g}$ are constructed in order to give the variational characterization of the generalized τ-solutions and secventialy generalized τ-solutions respectively (Theorems 2.1 and 2.2). The link between this two extensions is given (Theorem 2.3). A particular but useful case in which this two extentions coincide is given (Theorem 2.4). The influence of the topology τ in this construction is studied (Theorem 2.5). In the last part of this section the convex case in a Banach space for the strong and the weak topology is considered (Theorem 2.6).

In Section 3 we introduce the V-cercivity condition (a possible extension in a linear topological space of the coercivity condition used in a normed space) in order to obtain the existence of the (secvential) generalized τ-solution i.e. the surjectivity of $\partial\overline{g}$ ($\partial\tilde{g}$ respectively) (Theorem 3.1).

In section 4 the Sobolev τ-solutions are studied in a locally convex space. In order to obtain the existence of the Sobolev τ-solution (Theorem 4.1 and 4.2) the τ-uniform convexity condition (which is similar to the usual one used in normed space) is supposed.

In section 5 the K-variational problems are recalled from Ionescu, Rosca, Sofonea [6]. Since we consider that it is not so evident how the main results from [6] can be obtained using the theorems of the present paper we briefly indicate some tricks.

2. THE VARIATIONAL CHARACTERIZATION OF THE GENERALIZED SOLUTIONS

Let X be a real linear topological space with the topology denoted by τ which satisfies Hausdorff's axiom of separation and let X* be its dual. We consider $g:X\to(-\infty,+\infty]$ a proper functional (i.e. $D(g)=\{x\in X|\ g(x)<+\infty\}\neq\emptyset$) which is bounded from bellow by an afine function (i.e. there exists $x^*\in X^*$, $a\in R$ such that $g(x)\geq a+<x^*,x>$ for all $x\in X$). Let us construct the family of functionals $(F_f)_{f\in X^*}$ given by (1.1) and let $d:X^*\to[-\infty,+\infty)$ be given by

$$d(f)=\inf_{v\in X} F_f(v) \quad \text{for all} \quad f\in X^*. \tag{2.1}$$

One can easely see that $-d$ is the polar of g usely denoted by g^*.

We can also remark that $f\in\partial g(u)$ iff $F_f(u)=d(f)$.

<u>DEFINITION 2.1</u>. i) We say that u is a generalized (g.) τ-solution of the equation $f\in\partial g(x)$ iff there exists a generalized sequence $(u_\alpha)_{\alpha\in A}$ such that $u_\alpha\to u$ and $F_f(u_\alpha)\to d(f)$.

ii) We say that u is a secvential generalized (s.g.) τ-solution of the equation $f\in\partial g(x)$ iff there exists a sequence $(u_n)_{n\in N}$ such that $u_n\to u$ and $F_f(u_n)\to d(f)$.

REMARK 2.1. Every s.g. τ-solution is a g. τ-solution and if τ is metrizable then the two above definitions are equivalent. However, in general, a g.τ-solution is not a s.g.τ-solution which can be see in the following example

EXAMPLE 2.1. Let $(X,\|\ \|)$ be an infinite dimensional Banach space and $\tau=\sigma(X,X^*)$. If we consider $g(x)=\exp(-\|x\|)$ one can see that the equation $\theta_{X^*}\in\partial g(x)$ has no s.g.τ-solutions but all $u\in X$ are g.τ-solutions.

The following theorem states that if $\overline{g}$ is the lower semi-continuous envelope of g, then $\partial\overline{g}$ extends ∂g such that all g.τ-solutions of equation $f\in\partial g(x)$ are classical solutions for $f\in\partial\overline{g}(x)$ and conversely.

THEOREM 2.1. If $\overline{g}:X\to(-\infty,+\infty]$ is given by

$$(2.2)\qquad \overline{g}(x)=\underline{\lim}_{y\to x}\, g(y)=\sup_{V\in\mathcal{V}_\tau(x)}\ \inf_{y\in V} g(y)$$

then we have:

i) $\overline{g}\leq g$, $D(g)\subset D(\overline{g})$, $D(\partial g)\subset D(\partial\overline{g})$ and for all $u\in D(\partial g)$ we have $g(u)=\overline{g}(u)$ and $\partial g(u)=\partial\overline{g}(u)$.

ii) for all $f\in X^*$, $f\in\partial\overline{g}(u)$ iff u is a g.τ-solution of the equation $f\in\partial g(x)$.

COROLLARY 2.1. Let $u\in D(\partial g)$. Then for all $f\in X^*$ u is a classical solution of the equation $f\in\partial g(x)$ (i.e. $f\in\partial g(u)$) iff u is a g.τ-solution of the equation $f\in\partial g(x)$.

For all $f\in X^*$ let us denote by

$$(2.3)\qquad \overline{F}_f(v)=\overline{g}(v)-\langle f,v\rangle \qquad \text{for all} \quad v\in X$$

and let $\overline{d}:X^*\to[-\infty,+\infty)$ be given by

(2.4) $$\overline{d}(f)=\inf_{v\in X}\overline{F}_f(v)=-\overline{g}^*(f) \quad \text{for all} \quad f\in X^*$$

The following lemma will be useful in the proof of Theorem 2.1.

LEMMA 2.1. i) For all $v\in X$ there exists a generalized sequence $(v_\alpha)_{\alpha\in A}$ such that $v_\alpha\to v$ and $g(v_\alpha)\to\overline{g}(v)$.

ii) for all $f\in X^*$ we have

(2.5) $$d(f)=\overline{d}(f)$$

Proof. If $v\notin D(\overline{g})$ then $\overline{g}(v)=g(v)=+\infty$ and we can put $v_\alpha=v$. It is well known (see for instance Laurent [7] p. 332 or Ekland Temam [3 p. 10]) that $Ep(\overline{g})=\overline{Ep(g)}$ where $Ep(g)$ is the epigraph of g i.e. $Ep(g)=\{(x,a)/g(x)\leq a\}$. If $v\in D(\overline{g})$ then $(u,\overline{g}(u))\in Ep(\overline{g})=\overline{Ep(g)}$ hence there exists $(v_\alpha,a_\alpha)_{\alpha\in A}\subset Ep(g)$ such that $v_\alpha\to v$ in X and $a_\alpha\to\overline{g}(u)$ in R. From the inequalities $\overline{g}(v_\alpha)\leq g(v_\alpha)\leq a_\alpha$ we deduce $\overline{g}(v)\leq\underline{\lim}\,\overline{g}(v_\alpha)\leq\underline{\lim}\,g(v_\alpha)\leq\overline{\lim}\,g(v_\alpha)\leq\lim a_\alpha=\overline{g}(v)$ hence $g(v_\alpha)\to\overline{g}(v)$.

Proof of Theorem 2.1. i) Let $u\in D(\partial g)$ and $f\in\partial g(u)$. From Lemma 2.1 we get $\overline{F}_f(u)\leq F_f(u)$, $d(f)=\overline{d}(f)\leq\overline{F}_f(u)$ hence $F_f(u)=\overline{F}_f(u)=\overline{d}(f)$ i.e. $g(u)=\overline{g}(u)$ and $f\in\partial\overline{g}(u)$. We have just proved that $D(\partial g)\subset D(\partial\overline{g})$ and for all $u\in D(\partial g)$ we have $g(u)=\overline{g}(u)$, $\partial g(u)\subset\partial\overline{g}(u)$. In order to prove that $\partial\overline{g}(u)\subset\partial g(u)$ for $u\in D(\partial g)$ let us consider $h\in\partial\overline{g}(u)$. From $d(h)=\overline{d}(h)=\overline{g}(u)-\langle h,u\rangle=g(u)-\langle h,u\rangle$ we deduce $d(h)=F_h(u)$ hence $h\in\partial g(u)$.

ii) Let $u\in D(\partial\overline{g})$ and $f\in\partial\overline{g}(u)$. From Lemma 2.1 we deduce that there exists $(u_\alpha)_{\alpha\in A}$ a generalized sequence such that $g(u_\alpha)\to\overline{g}(u)$ and $u_\alpha\to u$. Hence we have $F_f(v_\alpha)\to\overline{F}_f(u)=\overline{d}(f)=d(f)$ i.e. u is a $g.\tau$-solution of the equation $f\in\partial g(x)$. Conversely let

$u_\alpha \to u$ and $F_f(u_\alpha) \to d(f)$ hence $g(u_\alpha) \to d(f)+\langle f,u\rangle = \bar{d}(f)+\langle f,u\rangle$. Using the lower semicontinuity of $\bar{g}$ we get $\bar{g}(u) \leq \underline{\lim}\ g(u_\alpha) = \bar{d}(f)+\langle f,u\rangle$ hence $\bar{F}_f(u)=\bar{d}(f)$ i.e. $f \in \partial\bar{g}(u)$.

The following theorem gives a variational characterization of s.g.τ-solutions similar to Theorem 2.1.

<u>THEOREM 2.2</u>. There exists $\tilde{g}: X \to (-\infty, +\infty]$ such that

i) $\tilde{g} \leq g$, $D(\partial g) \subset D(\partial\tilde{g})$ and for all $u \in D(\partial g)$ we have $\tilde{g}(u)=g(u)$ and $\partial\tilde{g}(u)=\partial g(u)$.

ii) For all $f \in X^*$ we have that $f \in \partial\tilde{g}(u)$ iff u is a s.g.τ-solution of the equation $f \in \partial g(x)$.

<u>COROLLARY 2.2</u>. Let u D(∂g). Then for all f X*, u is a classical solution of the equation f $\partial\tilde{g}(x)$ (i.e. f $\partial\tilde{g}(u)$) iff u is a s.g.τ-solution of the equation f $\partial g(x)$.

<u>Proof of Theorem 2.2.</u> Let S be the set of all s.g.τ-solutions

(2.6) $$S=\{u \in X;\ (\exists)\ f\ X^*,\ (\exists)\ (u_n)_{n\in N} \subset X \text{ such that } u_n \to u \text{ and } F_f(u_n) \to d(f)\}$$

and let $\tilde{g}: X \to (-\infty, +\infty]$ be given by

(2.7) $$\tilde{g}(v) = \begin{cases} \bar{g}(v) & \text{if } v \in S \\ g(v) & \text{if } v \notin S. \end{cases}$$

We denote by $(\tilde{F}_f)_{f \in X^*}$ the following family of functional

(2.8) $$\tilde{F}_f(v) = \tilde{g}(v) - \langle f,v\rangle \quad \text{for all } v \in X$$

and let $\tilde{d}: X^* \to [-\infty, +\infty)$ be given by

$$(2.9) \qquad \tilde{d}(f) = \inf_{v \in X} \tilde{F}_f(v) = -\tilde{g}^*(f) \quad \text{for all} \quad f \in X^*$$

The proof of Theorem 2.2 follows the same way as the proof of Theorem 2.1 using instead of Lemma 2.1 the following lemma.

LEMMA 2.2. i) For all $v \in X$ there exists a sequence $(v_n)_{n \in N}$ such that $v_n \to v$ and $g(v_n) \to \tilde{g}(v)$.

ii) For all $f \in X^*$ we have $d(f) = \tilde{d}(f)$.

Proof. If $v \notin S$ then we put $v_n \equiv v$ and if $v \in S$ then there exists $f \in X^*$ and $(v_n)_{n \in N}$ such $v_n \to v$ and $F_f(v_n) \to d(f)$ hence $g(v_n) \to d(f) + \langle f, u \rangle = \tilde{d}(f) + \langle f, u \rangle$. From the inequalities $\tilde{g}(u) = \overline{g}(u) \leqq \underline{\lim}\ \overline{g}(v_n) \leqq \lim g(v_n) = \tilde{d}(f) + \langle f, u \rangle \leqq \overline{g}(u) = \tilde{g}(u)$ we deduce $g(v_n) \to \tilde{g}(u)$.

The link between $\partial\tilde{g}$ and $\partial\overline{g}$ (i.e. between the s.g.τ-solutions and g.τ-solution) is given in the following theorem

THEOREM 2.3. $\overline{g} \leqq g \leqq g$, $D(\partial\tilde{g}) \subset D(\partial\overline{g})$ and for all $u \in D(\partial\tilde{g})$ we have $\tilde{g}(u) = \overline{g}(u)$ and $\partial\tilde{g}(u) = \partial\overline{g}(u)$.

COROLLARY 2.3. If u is a g.τ-solution of the equation $f \in \partial g(x)$ and it is also a s.g.τ-solution of the equation $h \in \partial g(x)$ then u is a s.g.τ-solution of the equation $f \in \partial g(x)$.

Proof of Theorem 2.3. From Theorem 2.2 one can easily deduce that $S = D(\partial\tilde{g}) \subset D(\partial\overline{g})$ and for all $u \in D(\partial\tilde{g})$ we have $\overline{g}(u) = \tilde{g}(u)$, $\partial\tilde{g}(u) \subset \partial\overline{g}(u)$. Let now $f \in \partial\overline{g}(u)$ for $u \in D(\partial\tilde{g})$. From lemma 2.1 and 2.2. we deduce that $\tilde{d}(f) = \overline{d}(f) = \overline{F}_f(u) = \tilde{F}_f(u)$ i.e. $f \in \partial\tilde{g}(u)$.

<u>REMARK 2.2</u>. If τ is metrizable one can deduce from Remark 2.1 and Theorems 2.1, 2.2, 2.3 that $\partial\overline{g}=\partial\tilde{g}$, but in general the equality does not hold (see for instance Example 2.1 where $\overline{g}\equiv 0$ and $\tilde{g}\equiv g$). However the following theorem gives sufficient conditions (in a particular choise of τ) in order to have the equality.

<u>THEOREM 2.4</u>. Let $(X,||\ ||)$ be a reflexive Banach space and $\tau=\sigma(X;X^*)$ be the weak topology of X. If $\lim g(x)=+\infty$ for $||x||\to+\infty$ then $\partial\overline{g}\equiv\partial\tilde{g}$, $\theta_{X^*}\in R(\partial\overline{g})$ (i.e. for all $f\in X^*$ we have that u is a g.$\sigma(X;X^*)$-solution of the equation $f\in\partial g(x)$ iff u is a s.g.$\sigma(X,X^*)$-solution of the equation $f\in\partial g(x)$).

<u>Proof</u>. Let us prove, for the beginning, that for all $v\in X$ there exists $(v_n)_{n\in N}$ such that $v_n\to v$ and $g(v_n)\to\overline{g}(v)$. If $v\notin D(\overline{g})$ then $g(u)=\overline{g}(u)=+\infty$ and we choose $v_n=v$. Let $v\in D(\overline{g})$, $r>0$ and $I=(\overline{g}(v)-r,\ \overline{g}(v)+r)\subset R$, $B=X\times I\subset X\times R$. Having in mind that $(v,\overline{g}(v))\in Ep(\overline{g})=\overline{Ep(g)}$ we deduce that $(v,\overline{g}(v))\in\overline{Ep(g)\cap B}$. Let us prove now that $Ep(g)\cap B$ is bounded in $X\times R$. Indeed if there exists $(w_n,b_n)\in Ep(g)\cap B$ such that $||w_n||\to+\infty$ we deduce that $g(w_n)\to+\infty$ but $g(w_n)\leq b_n\leq\overline{g}(v)+r<+\infty$ hence $Ep(g)\cap B$ is bounded in $X\times R$ which is reflexive. We can use a result of Browder [1, p. 81] to deduce that there exists $(v_n,a_n)\in Ep(g)\cap B$ such that $v_n\to v$ and $a_n\to\overline{g}(v)$. From the inequalities $\overline{g}(v)\leq\underline{\lim}\ \overline{g}(v_n)\leq\underline{\lim}\ g(v_n)\leq\overline{\lim}\ g(v_n)\leq\lim a_n=$ $=\overline{g}(v)$ we obtain $g(v_n)\to\overline{g}(v)$.

Let consider now $u\in D(\partial\overline{g})$ and $f\in\partial\overline{g}(u)$. From the first part of the proof we can construct $(u_n)_{n\in N}\subset X$ such that $u_n\to u$ and $g(u_n)\to\overline{g}(u)$. Since $F_f(u_n)\to\overline{F}_f(u)=\overline{d}(f)=d(f)$ we deduce that u is a s.g.$\sigma(X,X^*)$-solution of the equation $f\in\partial g(x)$. We can use Theorem 2.2 to obtain that $u\in D(\partial\tilde{g})$. Hence we get $D(\partial\overline{g})\subset D(\partial\tilde{g})$ and from Theorem 2.3 we deduce $\partial\bar{g}\equiv\partial\tilde{g}$.

THEOREM 2.5. Let us consider two topologies on X denoted by τ_1 and τ_2 such that $\tau_1\leqq\tau_2$ and $X^*=X^*_{\tau_1}=X^*_{\tau_2}$. We denote by $\bar{g}^i$, $\tilde{g}^i$, i=1,2 the functions constructed in theorems 2.1 and 2.2 for $\tau\equiv\tau_i$. Then we have

i) $\bar{g}^1\leqq\bar{g}^2$, $D(\partial\bar{g}^2)\subset D(\partial\bar{g}^1)$, $\partial\bar{g}^1(u)=\partial\bar{g}^2(u)$ for all $u\in D(\partial\bar{g}^2)$.

ii) $\tilde{g}^1\leqq\tilde{g}^2$, $D(\partial\tilde{g}^2)\subset D(\partial\tilde{g}^2)$, $\partial\tilde{g}^1(u)=\partial\tilde{g}^2(u)$ for all $u\in D(\partial\tilde{g}^2)$.

COROLLARY 2.4. i) If u is a g.τ_2-solution (s.g.τ_2-solution) then u is a g.τ_1-solution (s.g.τ_1-solution) of the equation $f\in\partial g(x)$.

ii) If u is a g.τ_1-solution (s.g.τ_1-solution) of the equation $f\in\partial g(x)$ and it is also a g.τ_2-solution (s.g.τ_2-solution) of the equation $h\in\partial g(x)$ then u is a g.τ_2-solution (s.g.τ_2-solution) of the equation $f\in\partial g(x)$.

REMARK 2.3. In general $\bar{g}^1<\bar{g}^2$ and $D(\partial\bar{g}^2)$ $D(\partial\bar{g}^1)$. This can be seen in example 2.1 for $\tau_1=\sigma(X,X^*)$ and τ_2=norm topology in which we have $\bar{g}^1\equiv 0$, $\bar{g}^2\equiv g$, $D(\partial\bar{g}^1)=\{\theta_X\}$ and $D(\partial\bar{g}^2)=\emptyset$.

Proof of Theorem 2.5. $\bar{g}^1\leqq\bar{g}^2$ by construction and since $S_2\subset S_1$ (S_i is given by (2.6) by replacing τ with τ_i, i=1,2) we get $\tilde{g}^1\leqq\tilde{g}^2$. From Corollary 2.4 i) which is obvious and theorem 2.1 we deduce that $D(\partial\bar{g}^2)\subset D(\partial\bar{g}^1)$ and for all $u\in D(\partial\bar{g}^2)$ we have $\partial\bar{g}^2(u)\subset\partial\bar{g}^1(u)$. Let now $u\in D(\partial\bar{g}^2)$, $h\in\partial\bar{g}^1(u)\cap\partial\bar{g}^2(u)$ and $f\in\partial\bar{g}^1(u)$. Using lemma 2.1 ii) we get $\bar{g}^1(u)=d(h)+\langle h,u\rangle=\bar{g}^2(u)$ and since $\bar{g}^1(u)=d(f)+\langle f,u\rangle$ we obtain $\bar{g}^2(u)=d(f)+\langle f,u\rangle$ i.e. $f\in\partial\bar{g}^2(u)$. In order to prouve (ii) the same technique can be used.

Let $(X,|| \ ||)$ be a Banach space and we denote by s the norm topology and by w the weak, $\sigma(X,X^*)$, topology of X. Let $\bar{g}^s$, $\tilde{g}^s$ and $\bar{g}^w$, $\tilde{g}^w$ be the function constructed above for $\tau=s$ and $\tau=w$ respectively. Then we have the following theorem

THEOREM 2.6. If g is convex then

i) $\bar{g}^w \equiv \bar{g}^s$, $\tilde{g}^w \equiv \tilde{g}^s$ and $\bar{g}^s$ is convex

ii) $\partial\bar{g}^w \equiv \partial\bar{g}^s \equiv \partial\tilde{g}^s \equiv \partial\tilde{g}^w$ is a maximal monotone operator.

Proof. i) Since g is convex we deduce that Ep(g) is a convex set hence $Ep(\bar{g}^s)=\overline{Ep(g)}^s=\overline{Ep(g)}^w=Ep(\bar{g}^w)$ i.e. $\bar{g}^s \equiv \bar{g}^w$ is convex. Let S_s and S_w be given by (2.6) for $\tau=s$ and $\tau=w$ respectively. It is obvious that $S_s \subset S_w$ and in order to prove that $S_w \subset S_s$ one can use the Mazur lemma and the convexity of F_f. Since $S_w=S_s$ from (2.7) we deduce that $\tilde{g}^w \equiv \tilde{g}^s$.

ii) Using remark 2.2 we get $\partial\tilde{g}^s \equiv \partial\bar{g}^s$ and from theorems 2.3 and 2.5 we have $\partial\bar{g}^s=\partial\tilde{g}^s$ $\partial\tilde{g}^w$ $\partial\bar{g}^w=\partial\bar{g}^s$.

REMARK 2.3. The fact that $\partial g^w \equiv \partial\bar{g}^s \equiv \partial\tilde{g}^s \equiv \partial\tilde{g}^w$ (i.e. the (s.)g. strong and weak solutions coincides) is important in applications. Indeed the V-coercivity condition (see the next section) with V a precompact set which assures the existence of the generalized solution is working in the weak topology (since $B(\theta,1)$ is compact for X a reflexive space). From the above equalities we have that if a g. weak solution exists then it is a strong one, hence we have strongly converging minimizing sequences. In the next we give an example in which g is not convex but however the above equalities hold .

EXAMPLE 2.2. Let $\Omega=(0,\pi)$, $X=H_0^1(\Omega)$ and $g:X\to R$ be given by $g(v)=\int_\Omega|\nabla v_x|^2dx+\int_\Omega\phi(v(x))dx$ where ϕ is given by (1.4). It is not so difficult to prove that $\overline{g}^w(v)=\overline{g}^s(v)=\int_\Omega|\nabla v|^2(x)dx+\int_\Omega\overline{\phi}(v(x))dx$ where $\overline{\phi}$ is given by (1.4). Using theorem 2.4 we get that $\partial\overline{g}^w\equiv\partial\tilde{g}^w\equiv\partial\overline{g}^s\equiv\partial\tilde{g}^s$.

3. THE EXISTENCE OF THE GENERALIZED SOLUTIONS

In reflexive Banach spaces the usual coercivity condition $\lim g(x)/||x||=+\infty$ for $||x||\to\infty$ assures the existence of the s.g.$\sigma(X,X^*)$-solutions. In this section we give a possible generalization of this condition in a linear topological space in order to obtain the existence of the g.τ-solutions (s.g.τ-solutions) (i.e. the surjectivity of the extension $\partial\overline{g}(\partial\tilde{g})$).

Let X be a real linear topological space with the topology denoted by τ which satisfies Hausdorff's axiom of separation, X^* its dual and let V be a subset of X.

DEFINITION 3.1. We say that g is V-coercive if $D(g)\subset\bigcup_{n\geq 0} nV$ and $\liminf_{v\in W_n} g(v)/n=+\infty$ where $W_n=(n+1)V\setminus nV$ and $W_0=V$.

REMARK 3.1. It is easy to see that if X is a normed space and $g(x)/||x||\to+\infty$ for $||x||\to+\infty$ then g is B(0,1)-coercive. ($B(0,1)=\{x/||x||\leq 1\}$).

The main result of this section is the following theorem

THEOREM 3.1. Let g be a V-coercive function.

i) If V is a precompact set then for all $f\in X^*$ there exists at least one g.τ-solution of the equation $f\in\partial g(x)$ i.e. $R(\partial\bar{g})=X^*$.

ii) If V is a secvential precompact set then for all $f\in X^*$ there exists at least one s.g.τ-solution of the equation $f\in\partial g(x)$ i.e. $R(\partial\tilde{g})=R(\partial\bar{g})=X^*$.

<u>COROLLARY 3.1</u>. Let $(X,\|\ \|)$ be a Banach space and $\lim g(x)/\|x\|=+\infty$ when $\|x\|\to\infty$.

i) If X is a reflexive space then for all $f\in X^*$ there exists at least one s.g.$\sigma(X,X^*)$-solution (i.e. $R(\partial\tilde{g})=R(\partial\bar{g})=X^*$).

ii) If X is the dual of a normed space Z and $\tau=\sigma(X,Z)$ is the topology on X then for all $f\in X^*\equiv Z$ there exists at least one g.τ-solution of the equation $f\in\partial g(x)$ (i.e. $R(\partial\bar{g})=X^*$). Moreover if Z is separable then for all $f\in X^*=Z$ there exists at least one s.g.τ-solution of the equation $f\in\partial g(x)$ (i.e. $R(\partial\tilde{g})=X^*$).

In order to prove theorem 3.1 the following lemma will be an useful tool.

<u>LEMMA 3.1</u>. Let g be a V-coercive function and $f\in X^*$. If $\sup_{v\in V}|\langle f,v\rangle|<+\infty$ then for all $a\in R$ there exists $k\in N$ such that $F_f^{\leq}(a)=\{x/F_f(x)\leq a\}\subset\bigcup_{i=0}^{k}W_i$.

<u>Proof</u>. Let us suppose that for all $k\in N$ there exists $v_k\in F_f^{\leq}(a)$ such that $v_k\notin\bigcup_{i=0}^{k}W_i$. Since $F_f^{\leq}(a)\subset D(g)\subset\bigcup_{n\geq 0}W_n$ there exists $n_k\geq k$ such that $v_k\in W_{n_k}$ hence $v_k=(n_k+1)w_k$ with $w_k\in V$. If we denote by $A_f=\sup_{v\in V}|\langle f,v\rangle|$ then we have $a\geq F_f(v_k)\geq g(v_k)-(n_k+1)A_f\geq$ $\geq n_k\left[\inf_{v\in W_{n_k}} g(v)/n_k-(1+\frac{1}{n_k})A_f\right]$ and passing to the limit we obtain $a\geq+\infty$, contradiction.

Proof of Theorem 3.1. i) Let $f\in X^*$ and let $(v_n)_{n\geq 0}$ be a minimizing sequence for F_f. If we put $a>d(f)$ then there exists n_o such that $(u_n)_{n\geq n_o}\subset F_f^{\leq}(a)$. Using now lemma 3.1 we get that $(v_n)_{n\geq n_o}$ belongs to a precompact set hence there exists a generalized subsequence $(u_\alpha)_{\alpha\in A}$ of $(u_n)_{n\geq n_o}$ and $u\in X$ such that $u_\alpha\to u$ i.e. u is a g.τ-solution of the equation $f\in\partial g(x)$.

ii) can be proved in a similar manner.

4. SOBOLEV τ-SOLUTIONS

In this section X will be a locally convex, secventialy complete space with the topology τ generated by the family of seminorms $(p_\alpha)_{\alpha\in A}$.

DEFINITION 4.1. We say that u is a Sobolev (S.) τ-solution of the equation $f\in\partial g(x)$ if all minimizing sequences for F_f converge to u.

REMARK 4.1. It is easy to see that if u is a S.τ-solution then u is a s.g.τ-solution and it is unique (no other g.τ-solution exists). As we can see in Example 1.1 the converse is false.

The following example points out the influence of the choise of the topology τ in the definition of S.τ-solution

EXAMPLE 4.1. Let $X=\ell^2$, $g:X\to R$ given by $g(v)=\sum_{n=1}^{\infty}\frac{1}{n}v_n^2+h(v)$ where $h(v)=0$ if $||v||\leq 1$ and $h(v)=||v||^2-1$ if $||v||>1$. One can prove that $u=\theta_X$ is the S.$\sigma(X,X^*)$-solution of the equation $\theta_{X^*}\in\partial g(x)$ but it is not a Sobolev strong solution (the sequence $x_n=(\delta_i^n)_{i\geq 1}$ is a minimizing one but, since $||x_n||=1$, x_n converges not to θ_X).

If X is a normed space then it is known that the coercivity and the uniform convexity of g are sufficient conditions for the existence of the Sobolev solutions in the norm topology (see Vainberg [9], Ionescu, Rosca, Sofonea [6], Dinca, Mateescu [2]). In order to give sufficient conditions for the existence of Sobolev τ-solutions we give the following definition of the uniform convexity in a localy convex space, similar to the usual one in a normed space (see Zalinescu [10]).

<u>DEFINITION 4.2</u>. We say that g is a τ-uniformly convex function on the set $B \subset X$ if for all $\alpha \in A$, $\varepsilon>0$, there exists $\delta>0$ such that for all $u,v \in B$ and $\lambda \in (0,1)$ if $\lambda g(u)+(1-\lambda)g(v)-g(\lambda u+(1-\lambda)v)<$ $<\delta\lambda(1-\lambda)$ then $p_\alpha(u-v)<\varepsilon$.

<u>EXAMPLE 4.2</u>. Let us consider $X=\ell^2$, $g:X\to R$ given by $g(v)=\sum_{n=1}^{\infty}\frac{1}{n}v_n^2$. One can prove that g is $\sigma(X,X^*)$-uniformly convex on all bounded sets but as can be seen in Vainberg [9] g is not uniformly convex in the norm topology.

<u>REMARK 4.2</u>. The τ-uniform convexity of g on a set B given in Definition 4.2 is equivalent with the following one: for all $\alpha \in A$ there exists $m_\alpha \in \mathcal{M}=\{m:R_+\to R_+/\ m$ is increasing and lower semicontinuous with $m(0)=0\}$ such that

$$\lambda g(u)+(1-\lambda)g(v)-g(\lambda u+(1-\lambda)v)\geqq\lambda(1-\lambda)m_\alpha(p_\alpha(u-v)) \tag{4.1}$$

for all $\lambda \in (0,1)$ and $u,v \in B$. In order to prove this statement one can use the same techniques as in Rosca [8] or in Zalinescu [10].

<u>THEOREM 4.1</u>. If g is τ-uniformly convex on X then

i) $\overline{g}$ is τ-uniformly convex on X and $\partial\overline{g}$ is one to one (i.e. $\partial g(x_1) \cap \partial g(x_2) = \emptyset$ for $x_1 \neq x_2$).

ii) If $d(f) > -\infty$ then there exists an unique S.τ-solution of the equation $f \in \partial g(x)$.

<u>COROLLARY 4.1</u>. If g is τ-uniformly convex on X then for all $f \in X^*$ we have that u is a S.τ-solution of the equation $f \in \partial g(x)$ iff u is a s.g.τ-solution iff u is a g.τ-solution iff $f \in \partial \overline{g}(u)$.

<u>Proof of Theorem 4.1.</u> (i) Let $u,v \in X$ and $(v_\alpha)_{\alpha \in C}$, $(v_\beta)_{\beta \in D}$ such that $u_\alpha \to u$, $v_\beta \to v$ and $g(v_\alpha) \to \overline{g}(u)$, $g(v_\beta) \to \overline{g}(v)$ (see Lemma 2.1). We consider $F = C \times D$ with $\gamma_1 = (\alpha_1, \beta_1) \leq \gamma_2 = (\alpha_2, \beta_2)$ for $\alpha_1 \leq \alpha_2$ and $\beta_1 \leq \beta_2$. Let $(u'_\gamma)_{\gamma \in F}$ and $(v'_\gamma)_{\gamma \in F}$ be two subsequences of $(u_\alpha)_{\alpha \in A}$ and $(v_\beta)_{\beta \in D}$ respectively given by $u'_{(\alpha,\beta)} = u_\alpha$, $v'_{(\alpha,\beta)} = v_\beta$. Passing to the limit in the inequality

$$\lambda g(u'_\gamma) + (1-\lambda) g(v'_\gamma) \geq g(\lambda u'_\gamma + (1-\lambda) v'_\gamma) + \lambda(1-\lambda) m_\alpha (p_\alpha (u'_\gamma - v'_\gamma))$$

and having in mind that $\underline{\lim}\, g(\lambda v'_\gamma + (1-\lambda) v'_\gamma) \geq \overline{g}(\lambda u + (1-\lambda) v)$ we get 4.1. The injectivity of $\partial \overline{g}$ is a direct concequence of the strict convexity of $\overline{g}$.

ii) Let $(u_n)_{n \in N}$ be a minimizing sequence for F_f. From 4.1 we have

$$\frac{1}{4} m_\alpha (p_\alpha (u_n - u_m)) \leq \frac{1}{2} F_f(u_n) + \frac{1}{2} F_f(u_m) - F_f(\frac{1}{2}(u_n + v_m)) \leq$$

$$\leq \frac{1}{2}(F_f(u_n) - d(f)) + \frac{1}{2}(F_f(u_m) - d(f))$$

and since $F_f(u_n) \to d(f)$ we deduce that $(u_n)_{n \geq 1}$ is a Cauchy sequence hence there exists $u \in X$ such that $u_n \to u$. Let now $(v_n)_{n \geq 1}$ be another minimizing sequence for F_f. Using the same technique one can prove that there exists $v \in X$ such that $v_n \to v$. In this way we obtain that u and v are s.g.τ-solutions of the equation $f\in\partial g(x)$ hence $f\in\partial\overline{g}(u)$ and $f\in\partial\overline{g}(v)$. But $\partial\overline{g}$ is one to one hence u=v is the unique Sobolev τ-solution.

THEOREM 4.2. If g is τ-uniformly convex on all bounded sets and g is V-coercive with V a bounded set then for all $f\in X^*$ the equation $f\in\partial g(x)$ has a unique Sobolev τ-solution (i.e. $R(\partial\overline{g})=R(\partial\tilde{g})=X^*$).

Proof. Let $(u'_n)_{n\geq 1}$ be a minimizing sequence for F_f. Since g is V-coercive with V a bounded set we can use Lemma 3.1 in order to prove that $(v_n)_{n\geq 1}$ belongs to a bounded set. We can use now the same technique like in the proof of Theorem 4.1 ii) to deduce that $(u_n)_{n\geq 1}$ is a Cauchy sequence hence there exists $u\in X$ such that $u_n\to u$. Let now $(v_n)_{n\geq 1}$ be another minimizing sequence for F_f. From the first part of the proof we obtain that there exists $v\in X$ such that $v_n\to v$. We consider $(w_n)_{n\geq 1}$ given by $w_{2n}=u_n$, $w_{2n+1}=v_n$. Since $(w_n)_{n\geq 1}$ is still a minimizing sequence for F_f we deduce that $(w_n)_{n\geq 1}$ is Cauchy, hence u=v is the unique Sobolev τ-solution of the equation $f\in\partial g(x)$.

5. K-VARIATIONAL PROBLEMS

Let $(X,\|\ \|)$ be a real Banach space and $K:D(k)\subset X\to X$ be a linear closed operator. Let $g:X\to(-\infty,+\infty]$ be a proper function with $D(g)\subset D(K)$ and $P:D(P)\subset X\to 2^{X^*}$ be a multivalued nonlinear operator. We recall from Ionescu, Rosca, Sofonea [6] the following definition.

DEFINITION 5.1. The pair (P,g) is called a K-variational problem (K-v.p.) if for all $f\in X^*$ we have $u\in D(P)$ and $f\in Pu$ iff

$G_f(u) \leq G_f(v)$ for all $v \in X$, where

(5.1) $\qquad G_f(v)=g(v)-<f,Kv> \quad$ for all $\quad v \in X$.

Let $(Z,||| \ |||)$ be $D(K)$ endowed with the graph norm of K (i.e. $|||u|||^2=||u||^2+||Ku||^2$) which is a Banach space. Since $K:Z \to X$ is bounded we can consider $K^*:X^* \to Z^*$ the adjoint of K. One can easely remark that (P,g) is a K-v.p. iff $D(P)=\{u \in X/$ $(\exists)$ $f \in X^*$ such that $K^*f \in \partial g(u)\}$ and $P=K^{*-1}\partial g$ where the subgradient in considered in Z and K^{*-1} is a multivalued operator. Let now construct $\overline{g}$ and $\tilde{g}$ as in section 2 for τ=the norm topology in Z. Since $\partial\overline{g}=\partial\tilde{g}$ extends ∂g we have that $\overline{P}=K^{*-1}\partial\overline{g}$ extends P and $(\overline{P},\overline{g})$ is a K-v.p. If we examine now the K-uniform convexity imposed on g in [6] we see that it is exactly the uniform convexity of g with respect to Z. Hence from Theorem 4.1 i) we get that $\overline{g}$ is also K-uniformly convex. If we use the same argument as in Lemma 7 of [6] we deduce that if $\lim g(x)/||Kx||=+\infty$ when $||Kx|| \to +\infty$ then for all $f \in X^*$ we have $\inf_{v \in X} G_f(x)=\inf_{v \in Z} F_{K^*f}(v)=$ $=d(K^*f)>-\infty$. We can use now Theorem 4.1 ii) to obtain that all minimizing sequence for $G_f \equiv F_{K^*f}$ are convergent in Z at the same limit called in [6] the K-generalized solution of the equation $f \in Px$ which is the Sobolev strong (in Z) solution of the equation $K^*f \in \partial g(x)$.

We have just proved the following theorem which is the main result of [6].

<u>THEOREM 5.1</u>. Let (P,g) be a k-v.p. with g a K-uniformly convex function and $\lim g(x)/||Kx||=+\infty$ when $||Kx|| \to +\infty$. Then there exists $(\overline{P},\overline{g})$ another K-v.p. with $\overline{g}$ a K-uniformly convex function such that

i) $\overline{P}$ extends P

ii) for all $f\in X^*$ u is the K-generalized solution of the equation $f\in Px$ iff u $D(\overline{P})$ and $f\in\overline{P}u$

iii) for all $f\in X^*$ there exists an unique K-generalized solution (i.e. $\overline{P}$ is one to one and $R(\overline{P})=X^*$)

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