ON THE NUMERICAL APPROXIMATION OF OPTIMAL DESIGN PROBLEMS

(preliminary version)

by

Dan TIBA *)

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*) Department of Mathematics, The National Institute for Scientific and Technical Creation, Bd.Păcii 220, 79622, Bucharest, Romania.

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1. Introduction

In this paper we discuss the approximation of the optimal packaging problem, which was first studied in [2].

We consider $\Omega =]0, b[\times]0,1[$, a domain in R^2, and $c>0$, $0<a<b$ are real numbers. We define:

$$U_{ad}=\{\alpha \in W^{1,\infty}(0,1);\ a\leq\alpha\leq b;\ |\alpha'| \leq c\},$$

$$\Omega(\alpha)=\{(x_1,x_2)\in\Omega;\ x_2\in]0,1[,\ 0<x_1<\alpha(x_2)\},$$

$$\Gamma(\alpha)=\{(x_1,x_2);\ x_1=\alpha(x_2),\ x_2\in[0,1]\},$$

$$K(\alpha)=\{v\in H_0^1(\Omega(\alpha));\ v\geq\varphi \text{ a.e. in } \Omega(\alpha)\},$$

where $\varphi\in H^2(\Omega)$, $\varphi\leq 0$ on $\Gamma=\partial\Omega$ and in $[a,b]\times[0,1]$.

Let $u(\alpha)\in K(\alpha)$ be the solution of the variational inequality

$$(1.1)\quad \int_{\Omega(\alpha)} \text{grad}\, u(\alpha)\,\text{grad}(v-u(\alpha))\geq 0,\quad \forall v\in K(\alpha)$$

which describes the equilibrium position of the membrane $\Omega(\alpha)$ in contact with the obstacle G given by $y=\varphi(x_1,x_2)$ and clamped on $\partial\Omega(\alpha)$.

Formally, $u(\alpha)$ satisfies:

$$(1.1)' \quad \begin{aligned} -\Delta u(\alpha) &\geq 0 && \text{in } \Omega(\alpha), \\ u(\alpha) &\geq \varphi && \text{in } \Omega(\alpha), \\ \Delta u(\alpha)\,(u-\varphi) &= 0 && \text{in } \Omega(\alpha), \\ u(\alpha) &= 0 && \text{in } \partial\Omega(\alpha). \end{aligned}$$

We denote by $z(\alpha)=\{x\in\Omega(\alpha);\ u(\alpha)(x)=\varphi(x)\}$ the coincidence set of (1.1).

The optimal packaging problem from [2] consists in minimizing the area of $\Omega(\alpha)$ such that the contact region $z(\alpha)$ contains a fixed subset D . In [7] we used the variational inequality technique for the numerical solution of this problem. Here we propose a different and simpler method, which is to be hoped to be applicable in other optimal design problems too.

We fix our attention on the simpler, but important, problem of the search of an admissible pair. That is, we want to solve

$$(1.2) \quad \text{Minimize} \int_D |u(\alpha)-\varphi|^2$$

subject to (1.1) and for $\alpha \in U_{ad}$.

If the optimal packaging problem has admissible pairs $\alpha\in U_{ad}$, $u(\alpha)$ given by (1.11) and satisfying the state constraint $z(u(\alpha))\supset D$, then these are optimal for the problem (1.2) with optimal value zero and conversely.

The idea of our approach has a geometric nature. If $\bar{\alpha}$, $\bar{u}$ is an optimal pair for (1.2), then we consider the surface $y=\bar{u}(x_1,x_2)$ and we extend it to $\tilde{u}(x_1,x_2)$ defined on the whole Ω.

If $\bar{u}$, $\tilde{u}$ are sufficiently regular then we may suppose $\tilde{u}$ as the solution of a similar variational inequality in the fixed domain Ω

$$(1.3)\quad -\Delta\tilde{u} + \beta(\tilde{u}-\varphi) \ni 0 \quad \text{in } \Omega$$
$$\tilde{u}/_{\Gamma} = w$$

where $w=0$ on $\Gamma \cap \partial\Omega(\bar{\alpha})$ and $0 \geq w \geq \varphi$ on $\Gamma \setminus \partial\Omega(\bar{\alpha})$. Here β is a maximal monotone graph in $R\times R$ of the form

$$(1.4)\quad \beta(r) = \begin{cases} 0 & r>0, \\ ]-\infty,\ 0] & r=0, \\ \emptyset & \text{otherwise.} \end{cases}$$

We also remark that the solution of (1.1) satisfies $u(\alpha) \geq 0$ a.e. in $\Omega(\alpha)$ (this may be obtained by taking $v=u^{+}(\alpha)$ in (1.1)). Therefore, for any $\alpha \in U_{ad}$, we have $u(\alpha) \geq 0$ a.e. in $E=[0,a]\times[0,1]$. This property is not necessarily satisfied by the solution of (1.3) and we have to impose it separately.

Conversely, if $\tilde{u} \geq 0$ in E and $\tilde{u}/_{\Gamma} = w$, since $w \leq 0$ on $b \times [0,1]$, then $\tilde{u}(x_1,x_2)=0$ defines some "curve" Λ in $[a,b] \times$ $\times [0,1]$ and we obtain a domain $\Sigma \subset \Omega$ on which $\bar{u}=\tilde{u}/_{\Sigma}$ satisfies (1.1) and $\bar{u}/_{\partial\Sigma}=0$. However, it is possible that $\Lambda \notin U_{ad}$.

But, it is wellknown that this definition of U_{ad} is required in order to have enough compactness to get existence in the problem (1.2). So, it is not prompted by a physical argument involved in the problem and we renounce it.

The above discussion motivates the introduction of the following boundary control problem:

$$(P)\quad \text{Minimize} \int_D |u-\varphi|^2$$

subject to

$$(1.3)\quad -\Delta u + \beta(u-\varphi) \ni 0 \qquad \text{in } \Omega,$$

(1.5) $u/_{\Gamma} = w$,

(1.6) $u \geqslant 0$ a.e. in E,

(1.7) $w \in W = \{ v \in L^2(\Gamma);\ v=0 \text{ on } \Gamma \cap \partial E,$
$0 \geqslant v \geqslant \varphi \text{ on } \partial\Omega \setminus \partial E \}$.

This approach in optimal design problems is different from those considered, for instance, in [6] and it has the advantage of simplicity. Boundary control problems via Dirichlet conditions without state constraints are studied in [1] under additional compactness assumptions on W and in [4] for a different type of nonlinear term in the state equation.

In the sequel we investigate the problem (P) and its solution will be interpreted as a minimizer in a certain class for the problem (1.2).

2. Existence for (P)

Due to the low regularity properties of $w \in L^2(\Gamma)$, we define a solution of (1.3)-(1.5) by transposition, as in Lions-Magenes [5] , Ch.II. We say that $u \in H^{1/2}(\Omega)$ is a solution of the variational inequality (1.3)-(1.5) if there is $\gamma \in L^2(\Omega)$, $\gamma \in \beta(u-\varphi)$ a.e. Ω, such that

(2.1) $$-\int_\Omega u\Delta\psi + \int_\Omega \gamma\psi = -\int_\Gamma w\frac{\partial\psi}{\partial n},$$

for all $\psi \in H^2(\Omega)$, $\psi=0$ on Γ.

We have

<u>Theorem 2.1.</u> <u>If</u> $w \in L^2(\Gamma)$, <u>then there is a unique</u> $u \in H^{1/2}(\Omega)$ <u>solution of</u> (1.3)-(1.5) <u>in the sense of</u> (2.1).

Proof

Assume first existence and prove uniqueness. Let $u_1, u_2 \in H^{1/2}(\Omega)$ be two solutions of (1.3)-(1.5) and $\gamma_1 \in \beta(u_1 - \varphi)$, $\gamma_2 \in \beta(u_2 - \varphi)$, $\gamma_1, \gamma_2 \in L^2(\Omega)$. We have

$$(2.2) \qquad -\int_\Omega (u_1 - u_2)\Delta\Psi + \int_\Omega (\gamma_1 - \gamma_2)\Psi = 0.$$

Then $v = u_1 - u_2$ is a solution by transposition of

$$\begin{aligned} -\Delta v &= \gamma_2 - \gamma_1 && \text{in } \Omega, \\ v &= 0 && \text{in } \Gamma \end{aligned}$$

and since $\gamma_2 - \gamma_1 \in L^2(\Omega)$, we get $v \in H^2(\Omega) \cap H^1_o(\Omega)$. With $\Psi = v$ in (2.2), by the monotonicity of β, we infer $v = 0$.

The existence may be obtained as follows: let $w_n \in H^{3/2}(\Gamma)$, $w_n \to w$ in $L^2(\Gamma)$. Then (1.3)-(1.5) has a unique solution $u_n \in H^2(\Omega)$, corresponding to w_n, by Th.2.5, Kinderlehrer and Stampacchia [3], p.113. As $w \geq \varphi/_\Gamma$ we may assume that $w_n \geq \varphi/_\Gamma$. Otherwise, let $\phi_n \in H^2(\Omega)$ be such that $\phi_n/_\Gamma = w_n$. Then $\theta_n = \sup(\phi_n, \varphi) \in H^1(\Omega)$, $\theta_n \geq \varphi$ and we may suppose $\theta_n \in H^2(\Omega)$, $\theta_n \geq \varphi$, by using a mollifier. By the properties of the sup(.,.), θ_n is bounded in $H^1(\Omega)$ and $\theta_n/_\Gamma \to w$ strongly in $L^2(\Gamma)$.

Now, assume that in (1.3) β is replaced by β_ε, the Yosida approximation of β. Obviously $\beta_\varepsilon(u_n - \varphi) \in H^1_o(\Omega)$ because $w_n \geq \varphi$. Multiplying (1.3) by $\beta_\varepsilon(u_n - \varphi)$, integrating by parts and using the monotonicity of β_ε, we see that $\{\beta_\varepsilon(u_n - \varphi)\}$ is bounded in $L^2(\Omega)$ by a constant independent of ε and n. Letting $\varepsilon \to 0$ we have $\{\beta(u_n - \varphi)\}$ bounded in $L^2(\Omega)$. The properties of the Dirichlet map give $\{u_n\}$ bounded in $H^{1/2}(\Omega)$. We may pass to the limit in (2.1) and finish the proof.

Theorem 2.2. There is at least one optimal pair $[u^*, w^*] \in H^{1/2}(\Omega) \times L^2(\Gamma)$ of (P).

Proof

Let $w_n \in W$ be a minimizing sequence. Clearly $\{w_n\}$ is bounded in $L^2(\Gamma)$ and $\{u_n\}$ is bounded in $H^{1/2}(\Omega)$. So $u_n \to u$ strongly in $L^2(\Omega)$, $w_n \to w$ weakly in $L^2(\Gamma)$, $\beta(u_n - \varphi) \to \beta(u-\varphi)$ weakly in $L^2(\Omega)$ and we may pass to the limit in (2.1).

The pair $[u, w]$ satisfies $u \geq 0$ a.e. in E, $w \in W$ and it is admissible for (P), so it is optimal. We denote it $[u^*, w^*]$.

3. Penalization and regularization of (P)

Let χ and ξ be the characteristic functions of D, E in Ω and χ_ε, ξ_ε be their $C^\infty(\bar{\Omega})$ approximation. We may require that $\{\chi_\varepsilon\}, \{\xi_\varepsilon\}$ are bounded in $L^\infty(\Omega)$, $\chi_\varepsilon \geq 0$, $\xi_\varepsilon \geq 0$, $\chi_\varepsilon \to \chi$, $\xi_\varepsilon \to \xi$ strongly in $L^p(\Omega)$, $1 \leq p \leq \infty$.

We denote $\delta_\varepsilon > 0$ such that

$$\delta_\varepsilon^3 = \int_\Omega |\xi_\varepsilon - \xi|^2 dx \to 0$$

for $\varepsilon \to 0$ and we define the penalized problem

$$(P_\varepsilon) \quad \text{Minimize } \left\{ \int_\Omega \chi_\varepsilon |u-\varphi|^2 dx + \frac{1}{\delta_\varepsilon} \int_\Omega \xi_\varepsilon (u^-)^2 dx \right. ,$$

subject to (1.3), (1.5), (1.7).

The existence of optimal pairs $[u_\varepsilon, w_\varepsilon]$ for (P_ε) may be established as in section 2.

<u>Theorem 3.1. We have</u>

(3.1) $w_\varepsilon \to \tilde{w}$ <u>weakly in</u> $L^2(\Gamma)$,

(3.2) $u_\varepsilon \to \tilde{u}$ <u>strongly in</u> $L^2(\Omega)$,

<u>where</u> $[\tilde{u},\tilde{w}]$ <u>is an optimal pair for</u> (P).

<u>Proof</u>

Obviously $\{w_\varepsilon\}$ is bounded in $L^2(\Gamma)$ and, on a subsequence, we may assume that $w_\varepsilon \to \tilde{w}$ weakly in $L^2(\Gamma)$, $\tilde{w} \in W$.

Multiplying (1.3) by $\beta_\varepsilon(u_\varepsilon-\varphi)\in H_0^1(\Omega)$, integrating by parts and using the inequality

$$\int_\Omega \beta(u_\varepsilon-\varphi)\beta_\varepsilon(u_\varepsilon-\varphi) \geq \int_\Omega \beta_\varepsilon(u_\varepsilon-\varphi)^2$$

we get

$$\int_\Omega \operatorname{grad}(u_\varepsilon-\varphi)\operatorname{grad}\beta_\varepsilon(u_\varepsilon-\varphi)+\int_\Omega \beta_\varepsilon(u_\varepsilon-\varphi)^2 \leq -\int_\Omega \Delta\varphi\;\beta_\varepsilon(u_\varepsilon-\varphi).$$

Then $\{\beta_\varepsilon(u_\varepsilon-\varphi)\}$ is bounded in $L^2(\Omega)$ and arguing as in the proof of Th.2.1, we see that $\{\beta(u_\varepsilon-\varphi)\}$ is bounded in $L^2(\Omega)$ and $\{u_\varepsilon\}$ is bounded in $H^{1/2}(\Omega)$, $u_\varepsilon \to \tilde{u}$ strongly in $L^2(\Omega)$. Passing to the limit in (2.1), we get that $\tilde{u}$ is the solution of (1.3), (1.5) corresponding to $\tilde{w}$.

Let $[u,w]$ be any admissible pair for the problem (P). Then it is also admissible for (P_ε) and we have

$$(3.3)\quad \int_\Omega \chi_\varepsilon|u_\varepsilon-\varphi|^2+\frac{1}{\delta_\varepsilon}\int_\Omega \xi_\varepsilon(u_\varepsilon^-)^2 \leq \int_\Omega \chi_\varepsilon|u-\varphi|^2+$$
$$+\frac{1}{\delta_\varepsilon}\int_\Omega \xi_\varepsilon(u^-)^2=\int_\Omega \chi_\varepsilon|u-\varphi|^2+\frac{1}{\delta_\varepsilon}\int_\Omega(\xi_\varepsilon-\xi)(u^-)^2.$$

By the Sobolev imbedding theorem, as $\Omega \subset R^2$, we infer $u \in L^4(\Omega)$, $u^- \in L^4(\Omega)$ and $(u^-)^2 \in L^2(\Omega)$. Then

$$\frac{1}{\delta_\varepsilon}\int_\Omega (\xi_\varepsilon - \xi)(u^-)^2 \leq \frac{1}{\delta_\varepsilon}\left(\int_\Omega (\xi_\varepsilon - \xi)^2\right)^{\frac{1}{2}}\left(\int_\Omega (u^-)^4\right)^{\frac{1}{2}} \leq$$

$$\leq \delta_\varepsilon^{1/2}\left(\int_\Omega (u^-)^4\right)^{1/2} \xrightarrow{\varepsilon \to 0} 0$$

Then (3.3) implies that $\frac{1}{\delta_\varepsilon}\int_\Omega \xi_\varepsilon (u_\varepsilon^-)^2$ is bounded, so

$$\int_E (\tilde{u}^-)^2 = \lim_{\varepsilon \to 0} \int_\Omega \xi_\varepsilon (u_\varepsilon^-)^2 = 0.$$

The pair $[\tilde{u}, \tilde{w}]$ is therefore admissible for the problem (P) and passing to the limit in (3.3), we get

$$\int_D |\tilde{u} - \varphi|^2 \leq \int_D |u - \varphi|^2$$

for any other admissible pair $[u, w]$. This shows that $[\tilde{u}, \tilde{w}]$ is an optimal pair and finishes the proof.

In order to get better differentiability properties, we introduce the regularization of the problem (P_ε) with parameter $\lambda > 0$:

(P_ε^λ) Minimize $\left\{ \int_\Omega \chi_\varepsilon |u - \varphi|^2 + \frac{1}{\delta_\varepsilon}\int_\Omega \xi_\varepsilon (u^-)^2 \right.$

subject to

(3.4) $\quad -\Delta u + \beta_\lambda (u - \varphi) = 0 \quad$ in Ω ,

and (1.5), (1.7).

Above β_λ is the Yosida approximation of β. In applications, we shall also take a smoothing of β_λ.

<u>Proposition 3.2.</u> <u>Let</u> $[u_\varepsilon^\lambda, w_\varepsilon^\lambda]$ <u>be an optimal pair for</u> (P_ε^λ). <u>Then</u>, <u>as</u> $\lambda \to 0$, <u>we have</u>

(3.5) $w_\varepsilon^\lambda \to w_\varepsilon$ <u>weakly in</u> $L^2(\Gamma)$,

(3.6) $u_\varepsilon^\lambda \to u_\varepsilon$ <u>strongly in</u> $L^2(\Omega)$

<u>where</u> $[u_\varepsilon, w_\varepsilon]$ <u>is an optimal pair for</u> (P_ε).

<u>Proof</u>

As $\{w_\varepsilon^\lambda\}$ is bounded in $L^2(\Gamma)$ we may assume (3.5) on a subsequence again denoted w_ε^λ (ε is fixed here). We may prove, as before, that $\{u_\varepsilon^\lambda\}$ is bounded in $H^{1/2}(\Omega)$ and $\{\beta_\lambda(u_\varepsilon^\lambda - \varphi)\}$ is bounded in $L^2(\Omega)$ with respect to $\lambda > 0$ and we get (3.6).

For the moment, the pair $[w_\varepsilon, u_\varepsilon]$ which appears in (3.5), (3.6) is only admissible for (P_ε). To see that it is optimal, we remark that

$$(3.7)\quad \int_\Omega \chi_\varepsilon |u_\varepsilon^\lambda - \varphi|^2 + \frac{1}{\delta_\varepsilon}\int_\Omega \xi_\varepsilon (u_\varepsilon^{\lambda\,-})^2 \leq \int_\Omega \chi_\varepsilon |u^\lambda - \varphi|^2 + \frac{1}{\delta_\varepsilon}\int_\Omega \xi_\varepsilon (u^{\lambda\,-})^2,$$

where u^λ is the solution of (3.4), (1.5), corresponding to some fixed $w \in W$.

By the same argument as usual, we infer that $u^\lambda \to u$ strongly in $L^2(\Omega)$, with u the solution of (1.3), (1.5) corresponding to $w \in W$.

Letting $\lambda \to 0$ in (3.7) we end the proof.

Remark. The same statement is valid when β_λ is replaced by β^λ, its regularization, defined as follows

$$\beta^\lambda(r) = \int_{-\infty}^{\infty} \beta_\lambda(r-\lambda\tau)\rho(\tau)\,d\tau,$$

where ρ is a Friedrichs mollifier.

4. An algorithm

We use a gradient algorithm to solve the problem (P_ε^λ).

Let $\theta: L^2(\Gamma) \to L^2(\Omega)$ denote the correspondence $w \to u$ defined by (3.4), (1.5) and $u=\theta(w)$, $u_\mu = \theta(w+\mu v)$, $v \in L^2(\Gamma)$.

Proposition 4.1. The mapping θ is Gateaux differentiable and $r = \nabla\theta(w)v$ satisfies

$$(4.1) \qquad -\Delta r + \nabla\beta^\lambda(u-\varphi)\, r = 0 \qquad \text{in } \Omega,$$

$$(4.2) \qquad r/_\Gamma = v.$$

Proof

We have

$$-\Delta u + \beta^\lambda(u-\varphi) = 0,$$

$$-\Delta u_\mu + \beta^\lambda(u_\mu - \varphi) = 0,$$

$$u/_\Gamma = w, \quad u_\mu/_\Gamma = w + \mu v,$$

which should be understood in the sense of (2.1).

Let $v_n \in H^{3/2}(\Gamma)$, $v_n \to v$ strongly in $L^2(\Gamma)$ and $u_\mu^n = \theta(w+\mu v^n)$, $z_\mu^n = \dfrac{u_\mu^n - u}{\mu}$. Then

$$(4.3) \qquad -\int_\Omega z_\mu^n \Delta\psi + \int_\Omega \frac{\beta^\lambda(u_\mu^n-\varphi) - \beta^\lambda(u-\varphi)}{\mu}\psi = -\int_\Gamma v_n \frac{\partial\psi}{\partial n}$$

for any $\Psi \in H^2(\Omega) \cap H_o^1(\Omega)$. By Th.2.5 of Stampacchia and Kinderlehrer [3], p.113, $z_\mu^n \in H^2(\Omega)$ and satisfies

$$- \Delta z_\mu^n + \frac{\beta^\lambda(u_\mu^n - \varphi) - \beta^\lambda(u-\varphi)}{\mu} = 0 \text{ in } \Omega,$$

$$- z_\mu^n / \Gamma = v_n .$$

We denote by $S_n \in H^2(\Omega)$ the solution of $-\Delta S_n = 0$, $S_n/\Gamma = v_n$ and $t_\mu^n = z_\mu^n - S_n$. Then $t_\mu^n \in H^2(\Omega) \cap H_o^1(\Omega)$ and we have

$$(4.4) \qquad -\Delta t_\mu^n + \frac{\beta^\lambda(u_\mu^n - \varphi) - \beta^\lambda(u-\varphi)}{\mu} = 0$$

We multiply (4.4) by t_μ^n, integrate by parts and use the fact that β^λ is Lipschitzian of constant $\frac{1}{\lambda}$ and monotone:

$$\int_\Omega (\text{grad } t_\mu^n)^2 + \int_\Omega \frac{\beta^\lambda(u_\mu^n - \varphi) - \beta^\lambda(u-\varphi)}{\mu} z_\mu^n = $$
$$= \int_\Omega \frac{\beta^\lambda(u_\mu^n - \varphi) - \beta^\lambda(u-\varphi)}{\mu} S_n$$

Then

$$| t_\mu^n|^2_{H_o^1(\Omega)} \leq \frac{1}{\lambda} |z_\mu^n|_{L^2(\Omega)} |S_n|_{L^2(\Omega)} \leq$$

$$\frac{1}{\lambda} |t_\mu^n|_{L^2(\Omega)} |S_n|_{L^2(\Omega)} + \frac{1}{\lambda} |S_n|^2_{L^2(\Omega)} .$$

Since $\{S_n\}$ is bounded in $H^{1/2}(\Omega)$, we get that $\{t_\mu^n\}$ is bounded in $H_o^1(\Omega)$ and $\{z_\mu^n\}$ is bounded in $H^{1/2}(\Omega)$ with respect to n and μ. We can pass to the limit in (4.3) $n \to \infty$, $\mu \to 0$ and obtain (4.1), (4.2).

<u>Remark</u>. We have $r \in H^{1/2}(\Omega)$.

We define the adjoint state $p_\varepsilon^\lambda \in H^2(\Omega) \cap H_o^1(\Omega)$ by

$$(4.5)\quad -\Delta p_\varepsilon^\lambda + \nabla\beta(u_\varepsilon^\lambda-\varphi)\,p_\varepsilon^\lambda = 2\,\chi_\varepsilon(u_\varepsilon^\lambda-\varphi) - \frac{2}{\delta_\varepsilon}\,\xi_\varepsilon\, u_\varepsilon^{\lambda-},$$

$$(4.6)\quad p_\varepsilon^\lambda/_\Gamma = 0.$$

The existence, uniqueness and regularity of p_ε^λ is wellknown.

Let us obtain the maximum principle for the problem (P_ε^λ). We have:

$$(4.7)\quad \int_\Omega \chi_\varepsilon|u_\varepsilon^\lambda-\varphi|^2 + \frac{1}{\delta_\varepsilon}\int_\Omega \xi_\varepsilon(u_\varepsilon^{\lambda-})^2 \leq \int_\Omega \chi_\varepsilon|u_\mu-\varphi|^2 + \frac{1}{\delta_\varepsilon}\int_\Omega \xi_\varepsilon(u_\mu^-)^2$$

where u_μ is the solution of (3.4) with $u_\mu/_\Gamma = w_\varepsilon^\lambda - \mu(w_\varepsilon^\lambda - v)$ for any $v\in W$, $\mu>0$.

We divide by μ in (4.7) and let $\mu\to 0$; then, we infer

$$(4.8)\quad \int_\Omega \Big(2\,\chi_\varepsilon(u_\varepsilon^\lambda-\varphi) - \frac{2}{\delta_\varepsilon}\xi_\varepsilon u_\varepsilon^{\lambda-}\Big)\nabla\theta(w_\varepsilon^\lambda)(v-w_\varepsilon^\lambda) \geq 0$$

for all $v\in W$.

We notice that the adjoint operator $\nabla\theta(w_\varepsilon^\lambda)^*$: $L^2(\Omega)\to L^2(\Gamma)$ is given by

$$(4.9)\quad \nabla\theta(w_\varepsilon^\lambda)^*(f) = -\frac{\partial p}{\partial n},$$

where p is the solution of (4.5), (4.6) with the right-hand side equal to $f\in L^2(\Omega)$. This may be seen by multiplying in (4.5) with $r=\nabla\theta(w_\varepsilon^\lambda)\ell$, $\ell\in L^2(\Gamma)$. By (4.1) and (2.1), we have

$$(\nabla\theta(w_\varepsilon^\lambda)^* f,\ell)_{L^2(\Gamma)} = (f,\nabla\theta(w_\varepsilon^\lambda)\ell)_{L^2(\Omega)} = -\int_\Omega r\Delta p + \int_\Omega \nabla\beta(u_\varepsilon^\lambda-\varphi)\,rp = -\int_\Gamma r\frac{\partial p}{\partial n},$$

since $p\in H^2(\Omega)\cap H_0^1(\Omega)$. This yields (4.9).

We have proved:

Proposition 4.2. The optimality conditions for the problem (P_ε^λ) are given by (3.4), (1.5), (4.5), (4.6) and

$$(4.10) \int_\Gamma \frac{\partial p_\varepsilon^\lambda}{\partial n}(w_\varepsilon^\lambda - v) \geq 0 \qquad \forall v \in W.$$

Obviously, (4.10) is a direct consequence of (4.8) and (4.9). It may be equivalently rewritten as

$$(4.11) \qquad \frac{\partial p_\varepsilon^\lambda}{\partial n} \in \partial I(w_\varepsilon^\lambda),$$

where $I : L^2(\Gamma) \to]-\infty, +\infty]$ is the indicator function of W in $L^2(\Gamma)$.

In order to apply a usual gradient algorithm, we penalize the control constraints $w \in W$ by adding to the cost functional of (P_ε^λ) a term of the form $I_\varepsilon(w)$, a smooth regularization of $I(w)$ (this may be easily done due to the pointwise definition of W).

We denote this new problem again by (P_ε^λ) and we preserve all the other notations. All the above results may be reobtained in the same way with the only modification that (4.11) becomes

$$(4.12) \quad \frac{\partial p_\varepsilon^\lambda}{\partial n} = \nabla I_\varepsilon(w_\varepsilon^\lambda).$$

The algorithm is as follows:

Step 1. Let w_o be given and set $n := 0$.

Step 2. Compute u_n from (3.4), (1.5).

Step 3. Test if the pair $[u_n, w_n]$ is satisfactory.

If YES, then STOP, otherwise GO TO step 4.

Step 4. Compute p_n from (4.5), (4.6).

Step 5. Compute w_{n+1} by

$$(4.13) \quad w_{n+1}=w_n-\sigma_n(\nabla l_\varepsilon(w_n)-\frac{\partial p_n}{\partial n}).$$

Step 6. n:=n+1 and GO TO step 2.

The test involved in step 3 may compare two consecutive iterations and σ_n from step 5 is a real parameter obtained via a line search.

Remark. If $w_n \in H^{3/2}(\Gamma)$, then we have already mentioned that $u_n \in H^2(\Omega)$. So the right-hand side in (4.5) is in $H^1(\Omega)$ and the regularity results for (4.5), (4.6) give $p_n \in H^3(\Omega)$. Then $\frac{\partial p_n}{\partial n} \in H^{3/2}(\Gamma)$ and, by (4.13), $w_{n+1} \in H^{3/2}(\Gamma)$.

Therefore, if we start the algorithm with a regular iteration w_o, this regularity is preserved during all the iterations. So u_n is a strong solution of (3.4) and we may consider its restriction to a subdomain to obtain the solution of the optimal design probem as explained in section 1.

Remark. In our simple approach we replace a problem over a variable domain (which is its greatest difficulty) by a problem over a fixed domain in a direct manner, without any supplementary complications.

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