ON INVARIANT SUBSPACES OF SEVERAL VARIABLE BERGMAN SPACES

by

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September 1988

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By using a natural localization method, one describes the finite codimensional invariant subspaces of the Bergman n-tuple of operators associated to some bounded pseudoconvex domains in $\mathbb{C}^n$, with a sufficiently nice boundary.

0. INTRODUCTION. Some recent investigations have been concerned with the structure and classification of the invariant subspaces of the Bergman n-tuple of operators, cf. Axler-Bourdon [2], Bercovici [3], Douglas [5]. Due to the richness of this lattice of invariant subspaces, the additional assumption on finite codimension was naturally adopted by the above mentioned authors as a first step towards a better understanding of its properties.

The present note arose from the observation that, when the L^2-bounded evaluation points of a pseudoconvex domain lie in the Fredholm resolvent set of the associated Bergman n-tuple, then the description of finite codimensional invariant subspaces is, at least conceptually, a fairly simple algebraic matter. This simplification requires only the basic properties of the sheaf model for systems of commuting operators discussed in [7]. In fact the localization method of Douglas [5] is quite similar, but in what concerns the particular question treated in this note the category of Fréchet modules seems to be more appropriate. Of course, the unitary classification of finite codimensional invariant subspaces of the Bergman space requires more elaborated techniques of Hilbert modules.

1. PSEUDOCONVEX DOMAINS. Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, $n\geq 1$, and let $L^2_a(\Omega)$ denote the corresponding Bergman space, i.e. the Hilbert space of all holomorphic and square summable functions defined on Ω. The n-tuple $M_{\Omega}=(z_1,\ldots,z_n)$ of multiplication operators on $L^2_a(\Omega)$ by the coordinate functions is referred to as the Bergman n-tuple of the domain Ω.

In this section we isolate a class of pseudoconvex domains which will be convenient for our techniques. A similar and deeper analysis is carried out in [2], in the one complex variable case.

In the sequel $\mathcal{O}$ denotes the sheaf of complex analytic functions on $\mathbb{C}^n$.

LEMMA 1. Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ and let $\lambda\in\partial\Omega$.

Assume that $\overline{\Omega}$ is a Stein compact and that $\mathcal{O}(\overline{\Omega})$ is dense in $L^2_a(\Omega)$. Then the following assertions are equivalent:

(i) The subspace $\sum_{j=1}^{n}(z_j-\lambda_j)L^2_a(\Omega)$ is dense in $L^2_a(\Omega)$.

(ii) There is no positive constant c, such that

(1) $|f(\lambda)|\leq c||f||_{2,\Omega}$

for every function $f \in \mathcal{O}(\overline{\Omega})$.

Proof. (i)$\Rightarrow$(ii). Assume that there exists a constant $c>0$, so that the estimate (1) holds. Then $\lim_{n\to\infty} f_m(\lambda)=:f(\lambda)$ exists for every convergent sequence $f_m \to f$, $f_m \in \mathcal{O}(\overline{\Omega})$, $f \in L_a^2(\Omega)$.

If assertion (i) would be true, then $f(\lambda)=0$ for every element $f \in L_a^2(\Omega)$, which is evidently a contradiction.

(ii)$\Rightarrow$(i). Assume that there exists a function $g \in L_a^2(\Omega)\setminus\{0\}$, orthogonal to the subspace $\sum_{j=1}^{n}(z_j-\lambda_j)L_a^2(\Omega)$.

Let $g_m \in \mathcal{O}(\overline{\Omega})$ be a sequence which approximates g in the norm of $L_a^2(\Omega)$. In view of the hypothesis on $\overline{\Omega}$ to possess a fundamental system of open pseudoconvex neighbourhoods, every function g_n can be decomposed as follows:

$$g_m(z)=g_m(\lambda)+\sum_{j=1}^{n}(z_j-\lambda_j)g_m^j(z) \quad ,$$

where $g_n^j \in \mathcal{O}(\overline{\Omega})$, $1 \leq j \leq n$, $1 \leq m$. This is possible by a standard application of Cartan's Theorem B. Accordingly

$$\langle g_m, g\rangle = g_m(\lambda)\langle 1, g\rangle \quad ,$$

whence we infer by passing to the limit $m\to\infty$ that $\langle 1,g\rangle \neq 0$.

Let $f \in \mathcal{O}(\overline{\Omega})$. By arguing as above we obtain

$$|f(\lambda)\langle 1,g\rangle| = |\langle f,g\rangle| \leq ||f||\cdot||g|| \quad ,$$

which proves that assertion (ii) is not true.

The above lemma was intended to bring forward the following two classes of examples.

EXAMPLE 1. A boundary point of a strictly pseudoconvex domain with smooth boundary satisfies condition (ii) above.

Let Ω be a strictly pseudoconvex domain with smooth boundary and let $\lambda\in\partial\Omega$ be fixed. Assume that there exists a constant $c>0$, such that the estimate (1) holds for λ. Then repeating an argument given in the previous proof,

$$f(\lambda)=\lim_{m\to\infty} f_m(\lambda)$$

exists whenever $f=\lim_m f_m$ in $L_a^2(\Omega)$ and $f_n\in\mathcal{O}(\bar{\Omega})$. Moreover, in this case relation (1) holds for f.

By Theorem 3.4.9 of [6], the algebra $\mathcal{O}(\bar{\Omega})$ is dense in $L_a^2(\Omega)$. In particular any element of $A^\infty(\Omega)=\mathcal{O}(\Omega)\cap\mathcal{C}^\infty(\bar{\Omega})$ is approximable in the $L^2(\Omega)$-norm by functions belonging to $\mathcal{O}(\bar{\Omega})$.

Since λ is a peak point for the algebra $A^\infty(\Omega)$, see for instance [4], there exists $h\in A^\infty(\Omega)$ with the properties $h(\lambda)=1$ and $|h(z)|<1$ for $z\in\bar{\Omega}\setminus\{\lambda\}$. Then

$$1\leq c||h^m||_{2,\Omega}$$

for every natural m, and $\lim_{m\to\infty} ||h^m||_{2,\Omega}=0$. This contradicts our assumption and thus condition (ii) is verified.

EXAMPLE 2. Let $\Omega=\Omega_1\times\ldots\times\Omega_n$ be a polydomain whose factors Ω_j satisfy the following requirement: no connected component of $\partial\Omega_j$ is reduced to a point, $1\leq j\leq n$. Then every point $\lambda\in\partial\Omega$ verifies condition (ii).

In order to prove this fact we need the identification $L_a^2(\Omega)\cong L_a^2(\Omega_1)\tilde{\otimes}\ldots\tilde{\otimes} L_a^2(\Omega_n)$; where "$\tilde{\otimes}$" denotes the hilbertian tensor product.

In virtue of Theorem 5 of [2], the subspace $(z_j-\lambda_j)L_a^2(\Omega_j)$ is dense in $L_a^2(\Omega_j)$ for every λ_j in $\partial\Omega_j$.

Fix a point $\lambda\in\partial\Omega$. Since one of its entries λ_j belongs to $\partial\Omega_j$, we get

$$\Big[\sum_{j=1}^{n}(z_j-\lambda_j)L_a^2(\Omega)\Big]^- \supset \Big[L_a^2(\Omega_1)\,\tilde{\otimes}\,\dots\,\tilde{\otimes}\,(z_j-\lambda_j)L_a^2(\Omega_j)\,\tilde{\otimes}\,\dots\,\tilde{\otimes}\,L_a^2(\Omega_n)\Big]^- = L_a^2(\Omega).$$

By an inspection of the proof of the previous lemma it follows that the implication (i) $\Rightarrow$ (ii) remains true without any additional assumption on Ω. Therefore condition (ii) is verified for the point $\lambda\in\partial\Omega$.

DEFINITION 1. Let $\mathcal{C}$ denote the class of those bounded pseudoconvex domains Ω in $\mathbb{C}^n$, which fulfil the following condition: for every point λ in $\partial\Omega$ there is no constant $c>0$ with the property:

$$|f(\lambda)|\le c\,||f||_{2,\Omega} \quad , \quad f\in\mathcal{O}(\bar{\Omega}) .$$

The preceding examples provide elements of $\mathcal{C}$. Also it is worth to mention that the class $\mathcal{C}$ is closed under cartesian products and analytic isomorphisms which extend to the boundary.

2. THE MAIN RESULT. In complete analogy with the first part of [2] we can state the next.

THEOREM 1. *Let Ω be a domain belonging to the class $\mathcal{C}$. Any M_Ω-invariant subspace S of finite codimension in $L_a^2(\Omega)$ has the form*

$$S = \sum_{j=1}^{k} P_j L_a^2(\Omega) \quad ,$$

where P_j are polynomials having a finite number of common zeroes, all contained in Ω.

Proof. Consider an invariant subspace S as in the statement and denote $Q=L_a^2(\Omega)/S$. Instead of working with n-tuples of commuting operators, we adopt the equivalent point of view of $\underline{\mathcal{O}}(\mathbb{C}^n)$-modules.

It was proved in [7], as a byproduct of Hörmander's L^2-estimates for the $\bar\partial$-operator, that $L_a^2(\Omega)$ is a module with property (β). Roughly speaking, that means that $L_a^2(\Omega)$ is suitable for localization in the category of Fréchet $\underline{\mathcal{O}}(\mathbb{C}^n)$-modules. Since $\dim Q<\infty$, the module Q has also property (β). From the exact sequence

$$0 \to S \to L_a^2(\Omega) \to Q \to 0$$

one deduces that S has the same property, too. Let denote by $\underline{\mathcal{S}}$, $\underline{\mathcal{F}}$ and $\underline{\mathcal{Q}}$ the corresponding Fréchet quasi-coherent $\underline{\mathcal{O}}$-modules. They are related by a similar exact sequence:

$$0 \to \underline{\mathcal{S}} \to \underline{\mathcal{F}} \to \underline{\mathcal{Q}} \to 0$$

Recall that $\underline{\mathcal{F}}$ was called in [7] the sheaf model of the system of operators M_Ω.

Among other things, it was proved in Section IV of [7] that there exists an exact sequence of Fréchet $\underline{\mathcal{O}}$-modules, derived from the Dolbeault complex:

$$0 \to \underline{\mathcal{F}} \to \underline{\mathcal{M}}^0 \to \underline{\mathcal{M}}^1 \to \ldots \to \underline{\mathcal{M}}^n \to 0 \, ,$$

with $\underline{\mathcal{M}}^j|_\Omega \cong \underline{\mathcal{E}}^{(0,j)}|_\Omega$, $0 \leq j \leq n$. Consequently $\underline{\mathcal{F}}|_\Omega \cong \underline{\mathcal{O}}|_\Omega$.

The last isomorphism can equally be obtained by a direct computation on the Koszul complex of M_Ω.

Let $\lambda \in \text{Supp}\,\underline{Q}$ and denote by $\underline{m}_\lambda$ the ideal of germs of analytic functions vanishing at λ. Because $\underline{\mathcal{O}}/\underline{m}_\lambda$ is a quotient module of $\underline{Q}$, at the level of global sections there exists a continuous $\underline{\mathcal{O}}(\mathbb{C}^n)$-linear map:

$$\underline{\varepsilon}_\lambda : L^2_a(\underline{\Omega}) \to \mathbb{C} \quad ,$$

with the property $\varepsilon_\lambda(f) = f(\lambda)$ for every function $f \in \underline{\mathcal{O}}(\bar{\Omega})$. By taking into account the hypothesis $\Omega \in \underline{\mathcal{C}}$ one gets $\lambda \in \text{Supp}\,\underline{Q}$. In conclusion we have proved the inclusion $\text{Supp}\,(\underline{Q}) \subset \Omega$.

As a finite length module, Q is even a module over the algebra of polynomials. Hence by Hilbert's Syzygies Theorem it admits a finite resolution of the form:

$$(2) \qquad 0 \to \underline{\mathcal{O}}^r \to \ldots \to \underline{\mathcal{O}}^k \xrightarrow{(P_1,\ldots,P_k)} \underline{\mathcal{O}} \to \underline{Q} \to 0 ,$$

where P_j are polynomials. Moreover, $P_1(\lambda) = \ldots = P_k(\lambda) = 0$ if and only if $\lambda \in \text{Supp}\,\underline{Q}$.

Since the sheaf $\underline{\mathcal{F}}$ is isomorphic to $\underline{\mathcal{O}}$ in a neighbourhood of $\text{Supp}\,\underline{Q}$, one finds $\underline{Q} \otimes_{\underline{\mathcal{O}}} \underline{\mathcal{F}} \cong \underline{Q}$ and $\text{Tor}_p(\underline{Q}, \underline{\mathcal{F}}) = 0$ for $p > 0$. Accordingly, the sequence (2) remains exact after tensor multiplication with $\underline{\mathcal{F}}$:

$$0 \to \underline{\mathcal{F}}^r \to \ldots \to \underline{\mathcal{F}}^k \xrightarrow{(P_1,\ldots,P_k)} \underline{\mathcal{F}} \to \underline{Q} \to 0 .$$

But the sheaf $\underline{\mathcal{F}}$ is acyclic on $\mathbb{C}^n$, so that by passing to global sections one finally obtains the exact sequence:

$$\left[L_a^2(\Omega)\right]^k \xrightarrow{(P_1,\dots,P_k)} L_a^2(\Omega) \to Q \to 0 .$$

This completes the proof of Theorem 1.

The same lines of the preceding proof can be used in order to obtain the next result.

PROPOSITION 1. Let T be a commutative n-tuple of operators with property (β) acting on the Banach space X. Assume that for every $\lambda\in\sigma(T)$, either $\sum_{j=1}^{n}(T_j-\lambda_j)X$ is dense in X or

$$\dim(X/\sum_{j=1}^{n}(T_j-\lambda_j)X)=1 .$$

Then every T-invariant subspace S of finite codimension in X has the form

$$S=\sum_{i=1}^{k}P_i(T)X ,$$

where P_i are polynomials having only a finite number of zeroes, all lying in $\sigma(T)\setminus\sigma_{ress}(T)$.

3. REMARKS. Theorem 1 establishes a one to one correspondence between the M-invariant subspaces S of finite codimension in $L_a^2(\Omega)$ and the ideals $I\subset\mathbb{C}[z_1,\dots,z_n]$, with the property that the natural restriction map

$$\mathbb{C}[z_1,\dots,z_n]/I \to L_a^2(\Omega)/S$$

is an isomorphism.

Thus the integers $a_k(\lambda)=\text{length}\ \mathrm{Tor}_k^{\mathbb{C}[z_1,\dots,z_n]}(I,\mathbb{C}_\lambda)$, $k\in\mathbb{N}$, $\lambda\in\mathbb{C}$, $\mathbb{C}_\lambda=\mathbb{C}[z_1,\dots,z_n]/\underline{m}_\lambda$, are invariants for the class of isomorphism of the subspace S, up to similarity. For n=1 these integers classify completely S. In the multidimensional case, however, it is likely that they not suffice for determining S up to similarity.

It was pointed out in [2], and the same conclusion remains valid for any n>1, that there exist domains $\Omega\subset\mathbb{C}^n$ not belonging to the class $\mathcal{C}$, such that Theorem 1 fails to be true.

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REFERENCES

1. O.P. Agrawal, D.N. Clark, R.G. Douglas, Invariant subspaces in the polydisk, Pacific J. Math. 121(1986), 1-11.
2. S. Axler, P. Bourdon, Finite codimensional invariant subspaces of Bergman spaces, Trans. Amer. Math. Soc. 306:2(1988), 805-817.
3. H. Bercovici, A question on invariant subspaces of Bergman spaces, Proc. Amer. Math. Soc. (to appear).
4. J. Chaumat, A.M. Chollet, Characterisation et propriétes des ensembles localement pic de $A^{\infty}(D)$, Duke Math. J. 47(1980), 763-787.
5. R.G. Douglas, On Silov resolution of Hilbert modules, in the volume "Special Classes of Linear Operators and Other Topics", pp. 51-60, Birkhäuser, Basel-Boston--Stuttgart, 1988.
6. L. Hörmander, L^2-estimates and existence theorems for the $\bar{\partial}$-operator, Acta Math. 113(1-2)(1965), 89-152.
7. M. Putinar, Spectral theory and sheaf theory. II, Math. Z. 192(1986), 473-490.