A STUDY OF INTRINSIC PROPERTIES OF RATIONAL HOMOTOPY TYPES VIA DEFORMATION THEORY

by

Martin MARKL*), Fel CVUT**) and Stefan PAPADIMA***)

October 1988

*) Matematický ustav ČSAV , Žitna 25 , 11567 Praha 1, Czechoslovakia.

**) Katedra matematiky, Scuhbatarova 2, 16627 Praha 6, Czechoslovakia.

***) Department of Mathematics, The National Institute for Scientific and Technical Creation , Bd. Păcii 220, 79622, Bucharest, Romania.

A STUDY OF INTRINSIC PROPERTIES OF RATIONAL HOMOTOPY TYPES VIA DEFORMATION THEORY

Martin MARKL

Matematický ústav ČSAV, Žitná 25, 115 67 Praha 1, Czechoslovakia

and

FEL ČVUT, Katedra matematiky, Suchbátarova 2, 166 27 Praha 6, Czechoslovakia

Ştefan PAPADIMA

INCREST, Math. Dept., Bd. Păcii 220, 79 622 Bucharest, Romania

INTRODUCTION AND STATEMENT OF MAIN RESULTS

Rational homotopy theory associates to 1-connected locally finite complexes X two (Eckmann-Hilton dual) types of graded algebras as invariants which will be simultaneously treated in this paper, namely $H^*(X;\mathbb{Q})$ and $\pi_*(\Omega X)\otimes\mathbb{Q}$. In what follows we shall denote by B^* a graded algebra over an arbitrary characteristic zero field k, which is supposed to be of finite type ($\dim_k B^n < \infty, \forall n$); B^* will be either graded commutative and 1-connected ($B^0 = k$ and $B^1 = 0$) or a graded Lie algebra, which is also supposed to be 1-connected ($B^0 = 0$). $\mathcal{B}^*X$ will stand either for $H^*(X;k)$ or for $\pi_*(\Omega X)\otimes k$. Among other things, rational homotopy theory ([16], [19]) provides, for $k = \mathbb{Q}$, the existence of a canonical X which realizes any given A^* (i.e. $\mathcal{B}^*X = A^*$), namely the so-called <u>formal</u> ($\mathcal{B}^* = H^*$) or <u>coformal</u> ($\mathcal{B}^* = \pi_*\Omega$) space associated to A^* (see e.g. [12]), and also provides an algebraic parametrization of spaces within A^* (i.e. spaces Y with $\mathcal{B}^*Y = A^*$). Much attention has been paid to the study of the (co)formality of spaces and of the corresponding <u>intrinsic</u> property for algebras: B^* is said to be <u>intrinsically (co)formal</u> if it is realized by exactly one $\mathbb{Q}$-homotopy type (namely by the (co)formal one) - see for example [19], [6],

[1], [17], [20], [12], [9], [15], [2], [18], [10],... Various other approximations to (co)formality have also been considered. We shall focus here on the topologically meaningful and algebraically tractable property of (co)spherical generation. Given X, consider the rational Hurewicz homomorphism $h: \pi_*(\Omega X)\otimes Q \longrightarrow H_{*+1}(X;Q)$. As it is well known, $\mathrm{Im}(h) \subset PH_*$ (PH_* = primitives of the homology coalgebra) and $[\pi_*, \pi_*] \subset \ker(h)$ ($[\pi_*, \pi_*]$ = derived loop homotopy Lie algebra), the inclusions being strict in general. We shall say that X is spherically (cospherically) generated if the first (second) inclusion is an equality. It is also well known that the (co)formality of X implies the (co)spherical generation property, without being in general equivalent to it. One may thus consider the corresponding intrinsic property for algebras: B^* is said to be intrinsically (co)spherically generated if any X within B^* is (co)spherically generated (co in the case B^* = graded Lie algebra L^*).

Various classes of examples of intrinsically (co)formal algebras have been exhibited. The most suitable approach to this problem turned out to be the use of deformation theory (in the spirit of [3], [13]) - see for example [6], [1], [17], [20], [7], [9],... The most general characterizations of the intrinsic (co)formality property for an algebra B^* (B^* = graded commutative algebra H^* or B^* = graded Lie algebra L^*) have been formulated as rigidity theorems, namely as an equivalence (under certain additional assumptions) between the intrinsic property and the vanishing of a deformation-theoretic H^1 group associated to B^*; two points are to be stressed here: first that the linear condition $H^1 = 0$ always implies the intrinsic property, without being in general equivalent to it, and second that the relevant H^1 is part of a classical cohomology theory associated to the graded algebra B^* (the same as in the classical examples of deformation theory [3], [13]).

As we are going to formulate our results in classical cohomology theory terms, we pause a little for recalling the definitions and making some notational conventions. Let then B^* be a graded algebra as before and let M^* be a (left) graded B^*-module. If $B^* = H^*$ (commutative algebra case) we shall consider the (bigraded) Harrison cohomology $\mathrm{Harr}^{*,*}(H,M)$ as defined in [20] and regrade it by defining

$$H^i_j(H,M) = \mathrm{Harr}^{i+j+1,i}(H,M), \text{ for } i,j \geqslant 0,\ i+j > 0.$$

If $B^* = L^*$ (graded Lie algebra case) we shall consider the bigraded classical Lie cohomology $H^{n,p}(L,M) = \mathrm{Ext}^{n,p}_{UL}(k,M)$ (where n = resolution degree and p = total degree, see e.g. [21]) and regrade it by defining

$$H^i_j(L,M) = H^{i+j+1,-i-1}(L,M), \text{ for } i,j \geqslant 0,\ i+j > 0.$$

The examples of natural B-modules which arise in connection with our deformation theory are: B^* as a left B^*-module, the B^*-submodule of decomposables of B^*, to be denoted by D^*B ($DB = B^+.B^+$, if $B^* = H^*$, $DB = [B,B]$ if $B^* = L^*$), the quotient B^*-module of indecomposables of B^*, to be denoted by $Q^*B = B^*/D^*B$, and finally B^*-modules of the form $M^* = N^{>n}$ (for some n and some B^*-module N^*).

Our point of view in this paper, which has been begun in [14], will be the <u>skeletal</u> one, namely to consider both the intrinsic properties and the deformation theory not only for B^* but for all <u>skeleta</u> of B^*, in a systematic way. We thus define, for any n, the <u>n-skeleton</u> of B^*, by just truncating above degree n, and denote it by $B^*(n) = B/B^{>n}$; it will be a quotient algebra of B^*. If $B^* = \mathcal{B}^*(X)$ then $B^*(n) = \mathcal{B}^*(X(n))$, where $X(n)$ denotes the n-skeleton of a minimal CW-decomposition of X (in the case $\mathcal{B}^* = H^*$), respectively the n-th Postnikov stage of X (in the case $\mathcal{B}^* = \pi_*\Omega$). Finally let us say that B^* is <u>skeletally intrinsically (co)formal</u> (respectively <u>(co)spherically generated</u>) if all skeleta $B^*(n)$ have the corresponding intrinsic property. We may now state our skeletal results. The main common feature is that certain linear conditions, expressible in terms of classical graded algebra cohomology, which in general are only sufficient for the existence of the intrinsic properties, become also necessary in the skeletal framework.

__Theorem A__. The following conditions are equivalent

(i) B^* is skeletally intrinsically (co)formal

(ii) $H^1_{\geq 1}(B(n),B(n)) = 0$, for any n

(iii) The natural map $H^1_{\geq 1}(B(n),B^n(n)) \longrightarrow H^1_{\geq 1}(B(n),B(n))$ is zero, for any n.

The proof will be given in Proposition 2.2. We ought to point out that if $H^1_{\geq 1}(B,B) = 0$ then B must be intrinsically (co)formal and that the converse does not hold in general (see the remarks of 1.5). In Section 1 we also offer two (nonskeletal) characterizations of the intrinsic (co)formality of B: a general (but nonlinear) one in Proposition 1.1 and a linear one in Proposition 1.3 (under the additional assumption that $\dim_k B < \infty$). With the same hypothesis, we show that the gap between the intrinsic formality of B and of $B \otimes_k \bar{k}$, where $\bar{k}$ is the algebraic closure of k, is measured by an interesting rationality property of the variety of structure constants for the deformation theory of B, in Proposition 1.4 (see also 1.5).

__Theorem B__. The assertions below are equivalent

(i) B^* is skeletally intrinsically (co)spherically generated

(ii) $H^1_{\geq 1}(B(n),QB(n)) = 0$, for any n

(iii) The natural map $H^1_{\geq 1}(B(n),Q^nB(n)) \longrightarrow H^1_{\geq 1}(B(n),QB(n))$ is zero, for any n

(iv) $H^1_{\geq 1}(B,QB) = 0$.

The proof is to be found in Proposition 2.3 (see also 2.4 for an example explaining the main difference between the two skeletal properties, which is due to condition (iv) above). Again it has to be noted that the vanishing of $H^1_{\geq 1}(B,QB)$ implies the intrinsic (co)-spherical generation of B, but is not in general equivalent to it (see 1.9 and 1.10). One may also find a general (nonlinear) characterization of nonskeletal intrinsic spherical generation in Proposition 1.8.

Our last two results are related to the gap between (co)formality and (co)spherical generation (see also [2]).

Theorem C. We have the equivalences

(i) B^* is skeletally intrinsically (co)formal

(ii) B^* is skeletally intrinsically (co)spherically generated and the natural map $H^1_{\geq 1}(B(n),DB(n)) \longrightarrow H^1_{\geq 1}(B(n),B(n))$ is zero, for any n

(iii) B^* is skeletally intrinsically (co)spherically generated and the natural map $H^1_{\geq 1}(B(n),D^nB(n)) \longrightarrow H^1_{\geq 1}(B(n),B(n))$ is zero, for any n.

Proof: see Proposition 2.5. Supposing that B^* is intrinsically (co)spherically generated, a classical (nonskeletal) rigidity theorem corresponding to the equivalence of (i) and (ii) above reads: if in addition $H^2_{\geq 2}(B,DB) = 0$ then the intrinsic (co)formality of B^* is equivalent to the vanishing of $H^1_{\geq 1}(B,DB) \longrightarrow H^1_{\geq 1}(B,B)$ - see 1.12, 1.13. Another variant is contained in the statement below (whose proof is given in 1.11 and 1.12).

Proposition D. Suppose that $H^1_{\geq 1}(B,DB) \longrightarrow H^1_{\geq 1}(B,B)$ is zero. Given X with $\mathcal{B}^*(X) = B^*$, X is (co)formal if and only if X is (co)spherically generated.

The paper is divided into two sections, the second one being devoted to the skeletal properties. Both parts are written in the language of deformation theory, and consequently all topological statements are proven in greater generality. The deformation theoretic framework is set up in the first section.

This paper may be regarded as the (weighted) sum of the unpublished [14] and [8]: the skeletal point of view of [14] plus the suitable deformation theoretic framework of [8] = the present major double revision. Both authors are grateful to the natural circumstances (i.e. the relatively short distance between Bucharest and Prague) which made mail cooperation work satisfactorily.

1. GENERAL RIGIDITY THEOREMS

Rational homotopy theory offers the possibility of faithfully translating questions from the homotopy category to a differential graded algebra setting ([16],[19]). In particular, given a graded Q-algebra B as in the introduction, it can be realized as the cohomology (loop space homotopy) algebra of a space; moreover, the central problem of classifying homotopy types with prescribed cohomology (homotopy) algebra has been recognized to admit a succesful reformulation in deformation theoretic terms ([6],[1],[17],[20],[7],...). This later setting has, among other things, the following useful features: it can be described for an arbitrary characteristic zero coefficient field k and, under suitable finiteness restrictions on B, the methods of algebraic geometry are available over the algebraic closure $\overline{k}$ ([3],[13]); rationality properties may be deduced from the unipotence of the linear groups which are in general involved here ([19],[11],[12],[6]).

We are thus going to describe first a convenient deformation theoretic framework. The most natural and suitable for our purposes may be formulated in terms of bigraded Lie algebras of derivations ([17], see also [8],[9]); in particular it will allow us to treat simultaneously the graded algebra and Lie algebra cases in a perfect Eckman-Hilton dual manner. We shall next translate the properties of (co)formality and spherical generation from geometry to algebra and formulate several general results related to the characterization of the corresponding intrinsic properties, in the classical form of rigidity theorems. As we have already mentioned the general ideas are (implicitly) present in the literature; consequently our proofs will be merely sketchy.

Let Z_*^* be a bigraded k-vector space and denote by $A_*^* = FZ$ the free (bi)-graded commutative or graded Lie algebra (with respect to the upper degree) on Z; the bigrading on A is multiplicatively induced by the one on Z. We shall suppose moreover that

(I) either $Z_*^{\leq 0} = 0$ or $Z_*^{\geq 0} = 0$

(II) $\dim_k Z_*^n < \infty$, for any n.

Also let d_1 be a graded algebra derivation of A which is bihomogeneous of bidegree (1,1). Denote by $\mathrm{Der}^i_j(A)$ the k-vector space of bihomogeneous graded algebra derivations of A of bidegree (i,j) ($i,j \in \mathbb{Z}$). Der^*_* naturally becomes a bigraded Lie algebra, with bracket given by the graded commutator of derivations. Finally suppose that $[d_1,d_1] = 0$ (or equivalently $d_1^2 = 0$). Define a bigraded subalgebra $E^*_* \subset \mathrm{Der}^*_*$ by $E^i_j = \mathrm{Der}^i_{i+j}$ ($i,j \geqslant 0$, $i+j > 0$). Notice that $d_1 \in E^1_0$ and that $D = [d_1,(.)]$ turns E^*_* into a cochain complex, which splits as $E^*_* = \bigoplus E^*_m$. The resulting cohomology will be denoted by $H^*_*(E)$; $H^1_{\geqslant 1}$ will play a key role in our deformation theory.

We now describe the two main examples coming from geometry. If B^* is a graded algebra H^* as in the introduction then (FZ,d_1) is essentially Quillen's $\mathcal{L}$ construction on the dual coalgebra $\# H$ with zero differential ([16], see also [21]). To be more precise set $W^* = \#\Sigma^{-1}\overline{H}$ and $Z^*_* = Z^*_1 = W^{-*}$. Notice that the differential d_1 is quadratic when restricted to the free generators and is essentially constructed as the dual of the algebra multiplication of H. If B^* is a graded Lie algebra L^* as in the introduction then (FZ,d_1) is essentially the Koszul-Quillen construction $\mathcal{C}$ of the cochain algebra on L with trivial differential ([16], see also [21]). To be more precise set $Z^*_1 = \#\Sigma L^*$ and $Z^*_* = Z^*_1$, then notice that d_1 will again be quadratic, being essentially the dual of the Lie multiplication on L. The corresponding classical cohomologies will be denoted by $H^*_*(B,B)$ (but recall the reindexing conventions from the introduction !). We note that we have in fact a bigraded isomorphism $H^*_*(E) = H^*_*(B,B)$; for B = H this easily follows from [20], [17]; for B = L it can be proved by similar (even easier) methods, namely by a tedious but straightforward direct computation, which we omit.

The underlying deformation theory has $d_1+E^1_{\geqslant 1}$ as affine support; the perturbations will be denoted by $p = \sum_{n\geqslant 2} p_n$, $p_n \in \mathrm{Der}^1_n = E^1_{n-1}$. The variety of structure constants will be denoted by M ($M \subset d_1+E^1_{\geqslant 1}$); it is given by the deformation equation ([13], [17]) $[d_1+p,d_1+p] = 0$. It contains the distinguished point d_1. Consider next the group Aut of graded algebra automorphisms of A^*

and the subgroup G of (lower) filtration nondecreasing elements $g \in \mathrm{Aut}$ (i.e. $g = \sum_{n \geqslant 0} g_n$, $g_n(A_i^*) \subset A_{i+n}^*$, $\forall n,i$) with the property that g_0 commutes with d_1. The normal subgroup G_1 of G is defined by the condition $g_0 = \mathrm{id}$. These are both linear algebraic groups (infinite dimensional in general, as well as the affine variety M), the later being unipotent. If $q \in E_n^0$ $(n > 0)$ then $\exp(q) = \sum_{m \geqslant 0} (1/m!)q^m$ makes sense and it is an element of G_1. Finally G morphically acts on E and on M, by conjugation. For details, see [17], also [8], [9]. Also notice that if $\dim_k Z < \infty$ (for example if $\dim_k B < \infty$) then everything happens in the realm of honest finite-dimensional algebraic geometry (as in [13]) and is moreover defined over k. There is however a word of caution: usually one also considers the component $E_0^0 = \{q \in \mathrm{Der}_0^0; [d_1, q] = 0\}$. We choose not to include it here, because it does not affect at all the picture of M as a G_1-space, in which we are primarily interested, for topological reasons which will soon appear; on the other hand it also does not affect $H^1_{\geqslant 1}(E)$, i.e. the most important part of linearized deformation theory.

When $k = \mathbb{Q}$ and the deformation theory comes (as explained before) from $B = H$ (resp. L), the connection with topology is provided by the following results ([7], resp. [1]): the set of rational homotopy types X with $H^*(X;\mathbb{Q}) = H^*$ (resp. $\pi_*\Omega(X) \otimes \mathbb{Q} = L^*$) is in bijection with the orbit space M/G; a space X is formal (coformal) if and only if its orbit equals $G.d_1 = G_1.d_1$. At this point is worth mentioning that the equality $G.d_1 = G_1.d_1$ is valid in general. This is due to the easily seen fact that G is generated by G_1 together with the stabilizer of d_1.

Consequently we shall start in general with (FZ, d_1), construct a deformation theory for d_1 as explained above, and make the following obvious definitions: a point $m = d_1 + p \in M$ will be called <u>formal</u> if $G.m = G.d_1$ (d_1 will be called the <u>canonical formal point</u>); d_1 will be called <u>intrinsically formal</u> if $M = G.d_1$.

For any $n \geqslant 1$, consider the set of <u>integrable elements</u>

$$I_n^1 = \left\{ p_{n+1} \in \mathrm{Der}_{n+1}^1 = E_n^1 \; ; \; \begin{array}{c} [d_1 + p_{n+1} + p_{n+2} + \ldots, d_1 + p_{n+1} + p_{n+2} + \ldots] = 0 \\ \text{for some } p_{n+i} \in \mathrm{Der}_{n+i}^1,\ i \geqslant 2 \end{array} \right\}.$$

Expanding the bracket we deduce that $[d_1, I_n^1] = 0$. Consequently we may consider the set of <u>integrable</u> <u>cohomology</u> <u>classes</u>, denoted by $IH_n^1 \subset H_n^1(E)$, obtained as the image of I_n^1 by the natural projection of $Z_n^1(E)$ onto $H_n^1(E)$.

Here is our first characterization of intrinsic formality.

1.1. <u>Proposition</u>. d_1 is intrinsically formal if and only if $IH_n^1 = \{0\}$, for any $n \geqslant 1$.

Proof. Suppose that d_1 is intrinsically formal and let $m = d_1 + p_{n+1} + p_{n+2} + \ldots \in M$, i.e. $[p_{n+1}] \in IH_n^1$. We know that there exists $g = 1 + g_1 + \ldots \in G_1$ such that $g.m = d_1$. We may then use the argument of [1],p.24 (see also [9],p.202) to modify g so as to have $g = 1 + g_n + g_{n+1} + \ldots$ and consequently $g_n \in Der_n^0 = E_n^0$. Expand $gm = d_1 g$ to $d_1 + p_{n+1} + g_n d_1 + \ldots = d_1 + d_1 g_n + \ldots$, hence $[p_{n+1}] = 0$ and $IH_n^1 = \{0\}$.

Conversely start with $m = d_1 + p_2 + p_3 + \ldots \in M$ and inductively suppose that $p_2 = p_3 = \ldots = p_n = 0$, hence that $p_{n+1} \in I_n^1$, and consequently $p_{n+1} = [d_1, q_n]$ for some $q_n \in E_n^0$; for the point $\exp(-q_n).m$ we then have that $p_2 = p_3 = \ldots$ $\ldots = p_{n+1} = 0$, and induction may go on. Finally note that the sequence $\exp(-q_n)\ldots\exp(-q_1)$ converges to a well-defined element $g \in G_1$ [8] and that $g.m = d_1$; therefore $M = G_1.d_1$ and d_1 is intrinsically formal.

If $k \subset K$ is a field extension we get $(FZ, d_1) \otimes_k K$ and a deformation theory for $d_1 \otimes K$.

1.2. <u>Corollary</u>. If $d_1 \otimes K$ is intrinsically formal then d_1 is intrinsically formal.

Proof. If $m = d_1 + p_{n+1} + p_{n+2} + \ldots \in M(k)$ then obviously $m \otimes K \in M(K)$ and the intrinsic formality of $d_1 \otimes K$ implies that $[p_{n+1} \otimes K] = 0$ in $H_n^1 \otimes K$, hence $[p_{n+1}] = 0$ in H_n^1, by linear algebra.

We should point out that the real point in our corollary is that we do not either impose additional finiteness conditions as in [12] or pass from the finite dimensional to the general case by a pro-algebraic groups argument as in [11], but rather give a direct convergence argument, to be compared with the one in [6]. However, our characterization has a serious drawback: it is nonlinear and hard to check.

We are going now to try a linearization. In order to be able to use some simple geometric arguments we shall suppose that $\dim_k Z < \infty$. We shall next consider the (Zariski) tangent spaces at d_1 to $G_1.d_1$ and M. We note that we have $B^1_{\geq 1}(E) \subset T_{d_1}(G_1.d_1)$ (which is in fact an equality, see [5]) and $T_{d_1}(M) \subset Z^1_{\geq 1}(E)$ [13]. We define the normal cohomology by $N^1_{\geq 1} = T_{d_1}(M)/B^1_{\geq 1}(E)$. It is a linear k-subspace of $H^1_{\geq 1}(E)$.

1.3. Proposition. Suppose that $\dim_k Z < \infty$. Then $d_1 \otimes K$ is intrinsically formal for any extension $k \subset K$ if and only if $N^1_{\geq 1} = 0$.

Proof. Let us assume first that $N^1_{\geq 1} = 0$ and $k = \bar{k}$. The same argument as in the proof of the rigidity theorem of [13] may then be used to deduce that $G_1.d_1 = G.d_1$ is (Zariski) open in M. We next invoke a homogeneity argument, as in [1]. Define a 1-parameter subgroup of G by letting $t \in k^*$ act on A^i_j as $t^{j-i}.\mathrm{id}$. It follows that $t.(d_1+p_2+p_3+\ldots) = d_1+tp_2+t^2p_3+\ldots$, hence $G.d_1 \subset \overline{G.m}$ (Zariski closure) for any $m \in M$. We infer that $M = G.d_1$ and d_1 is intrinsically formal. If $k \subset K$ is an arbitrary extension the condition $N^1_{\geq 1} = 0$ (which is independent on the ground field) gives that $d_1 \otimes \bar{K}$ is intrinsically formal, hence $d_1 \otimes K$ is intrinsically formal, by the previous corollary. For the converse implication it will plainly be enough to suppose that $k = \bar{k}$ and that d_1 is intrinsically formal and deduce that $N^1_{\geq 1} = 0$. To this end we shall consider the orbit map $f: G_1 \longrightarrow M$, given by $f(g) = g.d_1$. The intrinsic formality of d_1 and the unipotence of G_1 imply that it is a dominant morphism between irreducible varieties, therefore its differential is surjective on a Zariski open nonvoid subset of G_1 ([5]). Using a translation if necessary it follows that $d_{id}(f)$ is surjective and this readily implies that $T_{d_1}(M) \subset B^1_{\geq 1}(E)$, hence $N^1_{\geq 1} = 0$, as desired.

The above result suggests that the vanishing of $N^1_{\geq 1}$ is related to rationality properties of the variety of structure constants M. Our next rigidity result makes this guess a little more precise.

1.4. Proposition. Suppose that $\dim_k Z < \infty$.

(i) If $N^1_{\geqslant 1} = 0$ then the k-rational points of M are dense in $M(\bar{k})$.

(ii) If the k-rational points of M are dense in $M(\bar{k})$ then d_1 is intrinsically formal if and only if $N^1_{\geqslant 1} = 0$.

Proof. We shall treat (i) and (ii) simultaneously. Taking into account the preceding proposition it will be enough to assume that $N^1_{\geqslant 1} = 0$ in (i) and that d_1 is intrinsically formal in (ii). Both these assumptions imply that $M(k) = G_1(k).d_1$, at the level of k-rational points. This in turn implies that $G_1(\bar{k}).d_1 \subseteq \overline{M(k)}$. To see this we look at the orbit map $f:G_1 \longrightarrow M$ we have already considered in the preceding proof. We have $G_1(\bar{k}).d_1 = f(G_1(\bar{k})) = f(\overline{G_1(k)})$ (G_1 is an affine space being a unipotent group) $\subseteq \overline{f(G_1(k))} = \overline{G_1(k).d_1} = \overline{M(k)}$. But if $N^1_{\geqslant 1} = 0$ we know that $M(\bar{k}) = G_1(\bar{k}).d_1$ and our claim in (i) follows. On the other hand the assumptions made in (ii) imply that $M(\bar{k}) = \overline{M(k)} = \overline{f(G_1(k))}$, hence the orbit morphism is dominant. We may then deduce that $N^1_{\geqslant 1} = 0$ as in the previous proof, and we are done.

1.5. Remarks. Consider the following finite complex [1]

$$X = S^5_c \vee S^5_d \vee S^{14} \vee S^{23}_a \vee S^{23}_b \cup_\omega e^{28}$$

obtained from a wedge of spheres by attaching a 28-cell along $\omega = [S^{14},S^{14}]+[S^{23}_a,S^5_c]+[S^{23}_b,S^5_d]$, and a modified version, namely

$$Y = S^5_c \vee S^5_d \vee S^{14}_x \vee S^{14}_y \vee S^{23}_a \vee S^{23}_b \cup_\eta e^{28}$$

where $\eta = [S^{14}_x,S^{14}_x]+[S^{14}_y,S^{14}_y]+[S^{23}_a,S^5_c]+[S^{23}_b,S^5_d]$.

Concerning the relationship between the rigidity results given by Proposition 1.1 on one hand, Propositions 1.3 and 1.4 on the other hand and usual rigidity theorems, we note that we have in general $\bigtimes_{n\geqslant 1} IH^1_n \subset N^1_{\geqslant 1} \subset H^1_{\geqslant 1}(E)$ but the inclusions are strict. For the deformation theory of $H^*(X;k)$ it can be computed that $\dim H^1_{\geqslant 1} = 2$ and that $M(k) = \{d_1\}$, for any k, hence $N^1_{\geqslant 1} = 0$; the reason is that in general the ideal generated by the components of the deformation equation is not radical, and this makes (in general) the obstruction $N^1_{\geqslant 1} = 0$ hard to check. Starting with $H^*(Y;k)$, it can be verified that this algebra is intrinsically formal for $k = \mathbb{Q}$ or $\mathbb{R}$, but no more if $k = \bar{k}$. This shows that the inclusion $\bigtimes_{n\geq 1} IH^1_n \subset N^1_{\geq 1}$ is in general strict and that the intrinsic formality property is not independent of the extension of scalars. All these indicate that the best linear approximation to intrinsic formality is the classical one, namely $H^1_{\geq 1}(E)$. Traditionally it is shown that $H^1_{\geq 1}(E)=0$ is also a necessary condition for intrinsic formality by additionally assuming that $H^2_{\geq 2}(E)=0$ (the role of this extra assumption being to provide the equality $\bigtimes_{n\geq 1} IH^1_n = N^1_{\geq 1} = H^1_{\geq 1}$; see the next section for another class of assumptions which do the same job). There are however examples, as simple as $H^*(S^3 \vee S^3 \vee S^{13})$, where $H^1_{\geq 1}(E) = 0$ but $H^2_{\geq 2}(E) \neq 0$, which show that this choice of extra assumption is not the most natural one - to be compared to the situation in our Proposition 1.4.

We move now to the properties of (co)spherical generation. For making the relevant definitions we are going to suppose that $Z_* = Z_1$ in our deformation theoretic framework (as in the main two examples coming from topology) and set $\bar{Z} = \mathrm{Ker}(d_1|Z)$.

1.6. <u>Definition</u>. $m = d_1 + p \in M$ will be said to be <u>spherically</u> <u>generated</u> if for any $x \in \bar{Z}^n$ there exists $y \in (FZ)^n_{\geq 2}$ such that $(d_1+p)(x+y) = 0$.

Geometrically (FZ, d_1+p) will represent a minimal differential graded Lie algebra or commutative algebra, and $(FZ)_{\geq 2}$ will be a graded (d_1+p)-stable ideal. Denoting by $h: H^*(FZ, d_1+p) \twoheadrightarrow Z^*$ the morphism naturally induced by the projection $FZ \twoheadrightarrow FZ/(FZ)_{\geq 2} = Z$, it is well-known that it essentially represents the Hurewicz morphism or its dual (modulo some reindexing of the degrees). On the other hand, it is also immediate to see that we always have the inclusion $\mathrm{Im}\, h \subset \bar{Z}$ and that the spherical generation property in our Definition 1.6 is equivalent to $\mathrm{Im}\, h = \bar{Z}$. Taking into account that when $B = H$ we can identify $\bar{Z}$ with the primitives of the dual coalgebra $\# H$ and when $B = L$ we can identify $\bar{Z}$ with $\# QL$, one sees that the property stated in Definition 1.6 is equivalent to the property of (co)spherical generation stated in the introduction.

Let us notice that in general if $g \in G$ and $m \in M$ then m is spherically generated iff $g.m$ is spherically generated (since g may be regarded as an isomorphism between (FZ,m) and $(FZ,g.m)$ commuting with h and d_1) and also that plainly if $\mathrm{Ker}(m|Z) \supset \mathrm{Ker}(d_1|Z)$ then m is spherically generated. We thus see that the formality implies the spherical generation. We point out that the converse does not hold in general (see e.g. [15]).

1.7. Definition. d_1 is said to be intrinsically spherically generated if any $m \in M$ is spherically generated.

Obviously the intrinsic formality implies the intrinsic spherical generation. However there are intrinsically spherically generated examples which are not intrinsically formal [15].

The following constructions turn out to be useful for the characterization of the intrinsic spherical generation. Set $\overline{E} = \{d \in E;\ d|\overline{Z} = 0\}$. This is a bigraded Lie subalgebra of E^*_* and $d_1 \in \overline{E}^*_*$, consequently $\overline{E}^*_*$ will also be a subcomplex of (E^*_*,D). Put $\overline{M} = M \cap (d_1 + \overline{E}^1_{\geq 1})$.

1.8. Proposition. d_1 is intrinsically spherically generated if and only if the natural map $\overline{M} \longrightarrow M/G_1$ is onto.

Proof. If for any $m \in M$ there exists $g \in G_1$ such that $g.m \in \overline{M}$, we know that $g.m|\overline{Z} = 0$, hence $g.m$ is spherically generated, therefore m will be spherically generated and d_1 will be intrinsically spherically generated. Conversely, given $m \in M$ we use its spherical generation property and construct an element $g \in G_1$ as follows: we write $Z = \overline{Z} \oplus C$ and define a graded algebra automorphism g of FZ by describing its restriction to the free algebra generators, namely $gx = x$, for $x \in C$, and $gx = x+y$, for $x \in \overline{Z}^n$, where $y \in (FZ)^n_{\geq 2}$ and $m(x+y) = 0$. Since $g_0 = \mathrm{id}$, $g \in G_1$. By construction $g^{-1}.m \in \overline{M}$, so the converse implication is also established.

1.9. Proposition. If $H^1_{\geq 1}(E/\overline{E}) = 0$ then d_1 is intrinsically spherically generated.

Proof. Start with $m \in M$, $m = d_1+p_2+\dots+p_n+p_{n+1}+\dots$ We will show that the G_1-orbit of M contains a point whose all perturbations $p_2, p_3, \dots$ lie in $\overline{E}$ and then use the preceding result to conclude that d_1 is intrinsically spherically generated. In what follows the idea of proof is taken from [8] (see also [17]). Assume inductively that $p_2,\dots,p_n \in \overline{E}$. Recall that $[m,m] = 0$, look at the homogeneous part of degree $n+2$ of this equality, remember that $\overline{E}$ is a Lie subalgebra of E and conclude that p_{n+1} represents a cycle of $(E/\overline{E})^1_n$, $n \geqslant 1$. Consequently there exists $q_n \in E^0_n = \mathrm{Der}^0_n(A^*_*)$ such that $p_{n+1} + [q_n, d_1] \in \overline{E}$. Up to degree $n+1$, we have the equality $\exp(q_n).m = d_1 + p_2 + \dots + p_n + p_{n+1} + [q_n, d_1]$, where $\exp(q_n) \in G_1$. Induction goes well on and finally passing to the limit as in the proof of Proposition 1.1 we find the desired element of G_1.

1.10. Remarks. $H^*_*(E/\overline{E})$ is computable directly from the initial data (FZ, d_1) as

$$H^i_j(E/\overline{E}) = \mathrm{Hom}^i(H^*_1(FZ,d_1), H^*_{i+j+1}(FZ,d_1)), \ \forall\, i,j,$$

where Hom^i denotes the k-linear homomorphisms which are homogeneous of upper degree i. This may be seen as follows. Up to reindexing the degrees (E^*_*, D) is nothing else but the bigraded complex of the derivations of the bigraded (free) differential algebra (FZ, d_1) into itself. Going back to the definition of $\overline{E}$ it is immediate to see that the bigraded complex $((E/\overline{E})^*_*, D)$ is given by the bihomogeneous derivations of the bigraded (free) differential algebra $(F\overline{Z}, 0)$, which is a bigraded differential subalgebra of (FZ, d_1), into (FZ, d_1), which is considered as a bigraded differential module over $(F\overline{Z}, 0)$ in the natural way. This immediately gives that $H^*_*(E/\overline{E}) = \mathrm{Hom}^*_*(\overline{Z}, H(FZ, d_1))$. Our stated formula follows at once, by noticing that $\overline{Z}^* = H^*_1(FZ, d_1)$ and by properly reindexing. It is also worth mentioning that the space $\mathrm{Hom}^1(H^*_1(FZ,d_1), H^*_{\geqslant 2}(FZ,d_1))$ naturally appears as the target space of an obstruction theory for the spherical generation property.

When dealing with the deformation theory of a graded algebra B^* ($B^* = H^*$ or L^*) the method of proving that $H^*_*(E) = H^*_*(B,B)$ also gives that $H^*_*(E/\overline{E}) = H^*_*(B, QB)$ (by simply observing that $\overline{Z} = \# QB$, modulo a shift of the degrees). We also mention that one has $H^*_*(FZ, d_1) = H^*_*(B, k)$, where k is considered as a trivial B-module. Finally we remark that the vanishing of

$H^1_{\geqslant 1}(E/\bar{E})$ is not in general a necessary condition for intrinsic spherical generation (see however the next section). This may be seen by inspecting the already mentioned Félix example; in this example $H^*(X;k)$ is intrinsically formal (hence spherically generated) for any k, but $\mathrm{Hom}^1(H^*_1(FZ,d_1),H^*_{\geqslant 2}(FZ,d_1)) \neq 0$ ([1], p.40-41; the computation of $H^*_*(FZ,d_1)$ is carried out in terms of Sullivan models, but the translation to Quillen models can be easily done, see e.g. [21]).

Our next two results explore the gap between the properties of formality and spherical generation.

1.11. Proposition. Assume that $H^1_{\geqslant 1}(\bar{E}) \longrightarrow H^1_{\geqslant 1}(E)$ is zero. Then the formal points of M coincide with the spherically generated points of M.

Proof. Let $m = d_1+p_2+\dots+p_n+p_{n+1}+\dots$ be a spherically generated point of M. We have to show that the G_1-orbit of m contains a point with $p_2=p_3=\dots=0$. We inductively assume that we have found a point in $G_1.m$ which has $p_2=\dots=p_n=0$ (and of course which is again spherically generated). We are going to use this last property to deduce that $p_{n+1} \in \bar{E}^1_n \bmod B^1_n(E)$; since we already know that $p_{n+1} \in Z^1_n(E)$ (use $[m,m]=0$) the assumption that $H^1_{\geqslant 1}(\bar{E}) \longrightarrow H^1_{\geqslant 1}(E)$ is zero forces $p_{n+1} = 0 \bmod B^1_n(E)$. The rest of the proof follows exactly as in Proposition 1.1. Spherical generation implies, for any $x \in \bar{Z}^h$, the existence of $y = y_2+y_3+\dots$, $y_i \in (FZ)^h_i$, with the property that $(d_1+p_{n+1}+\dots)(x+y_2+y_3+\dots) = 0$. Looking at the homogeneous component of degree n+2, we see that $d_1y_{n+1}+p_{n+1}x = 0$. Making linearly our choice for y, we find an upper degree zero linear map $q_n:\bar{Z} \longrightarrow (FZ)_{n+1}$ with the property that $p_{n+1}|\bar{Z} + d_1q_n = 0$. Extend q_n to a derivation $q_n \in E^0_n$ and notice that $p_{n+1} + [d_1,q_n] \in \bar{E}^1_n$ as claimed. Our proof is thus completed.

1.12. Remark. Let B^* be a graded algebra as in the introduction and consider the associated deformation theory, as explained before. As we have already mentioned, it is not hard to identify $H^*_*(E)$ with $H^*_*(B,B)$. In fact one may identify the map $H^*_*(\bar{E}) \longrightarrow H^*_*(E)$ appearing in the above proposition with the map $H^*_*(B,DB) \longrightarrow H^*_*(B,B)$ naturally induced by the inclusion of the B-submodule of decomposables, $DB \hookrightarrow B$. Thus Proposition 1.11 fully gives the proof of Proposition D of the introduction. It also leads to the following rigidity theorem.

1.13. Proposition. Suppose that d_1 is intrinsically spherically generated. If $H^1_{\geq 1}(\bar{E}) \to H^1_{\geq 1}(E)$ is zero then d_1 is intrinsically formal. Suppose in addition that $H^2_{\geq 2}(\bar{E}) = 0$. Then d_1 is intrinsically formal if and only if $H^1_{\geq 1}(\bar{E}) \to H^1_{\geq 1}(E)$ is zero.

Proof. Assuming that d_1 is intrinsically spherically generated and $H^1_{\geq 1}(\bar{E}) \to H^1_{\geq 1}(E)$ is zero we infer that d_1 is intrinsically formal by using Proposition 1.11. Conversely assume that d_1 is intrinsically formal and $H^2_{\geq 2}(E) = 0$. Start with a D-cocycle $p_n \in Z^1_n(\bar{E})$, $n \geq 1$. We will use the assumption that $H^2_{\geq 2}(\bar{E}) = 0$ to show that p_n is integrable; to be more precise we will show the existence of $p_{n+1}, \ldots$, $(p_{n+i} \in \bar{E}^1_{n+i})$ with the property that $d_1 + p_n + p_{n+1} + \ldots \in M$. From this fact it may be deduced, exactly as in the proof of Proposition 1.1, that $p_n \in B^1_n(E)$ (d_1 being intrinsically formal). As far as our integrability claim is concerned, set $m = m_0 + m_1 + \ldots$, $m_i \in \bar{E}^1_i$ and $m_o = d_1$. The condition $[m,m] = 0$ is equivalent with the following set of homogeneous conditions:

$$(*_i) \qquad -2[d_1, m_i] = \sum_{j=1}^{i-1} [m_j, m_{i-j}], \quad i \geq 1.$$

Inductively suppose that $m_1, \ldots, m_{i-1}$ have been constructed $(i > n)$ with the property that $(*_1), \ldots, (*_{i-1})$ hold and $m_1 = \ldots = m_{n-1} = 0$, $m_n = p_n$. The right hand side of the equality $(*_i)$ is then an element $m(i) \in \bar{E}^2_{\geq 2}$. If we show that it is in fact a D-cocycle then $H^2_{\geq 2}(\bar{E}) = 0$ implies that $(*_i)$ may be solved for m_i, and we are done. Set $m' = m_0 + m_1 + \ldots + m_{i-1}$; then $[m', [m', m']] = 0$, by the Jacobi identity. Looking at the homogeneous component of degree i of this equality, recall that $[m', m']$ has trivial homogeneous components in degrees $< i$ (by induction) and that the homogeneous component of degree i of $[m', m']$ equals $m(i)$, and consequently infer that $[d_1, m(i)] = 0$. This was all we needed to continue the induction. The proof of the proposition is complete.

1.14. Remark. The proof of the usual rigidity theorem "d_1 is intrinsically formal if and only if $H^1_{\geq 1}(E) = 0$, in the presence of the condition $H^2_{\geq 2}(E) = 0$"

is a particular case of the above proof obtained just by setting $\overline{E} = E$.

1.15. Example: wedges of spheres and products of Eilenberg-MacLane spaces. We will briefly examine the simplest case occuring in our deformation theory, namely $d_1 = 0$. Topologically this means algebras $B^* = H^*$ or L^* with trivial multiplication (e.g. the cohomology of wedges of spheres or the loop homotopy of products of Eilenberg-MacLane spaces). Let us observe that in this particular case ($d_1 = 0$) all the properties we have considere , namely the intrinsic formality, the intrinsic spherical generation, $H^1_{\geqslant 1}(E) = 0$, $H^1_{\geqslant 1}(E/\overline{E}) = 0$, are equivalent, being in fact equivalent to $E^1_{\geqslant 1} = 0$. It will plainly suffice to see that the intrinsic spherical generation implies the intrinsic formality (this follows immediately from Proposition 1.8, since in our case $\overline{M} = \{d_1\}$) and that in turn the intrinsic formality implies that $E^1_{\geqslant 1} = 0$. Indeed, supposing that $E^1_{\geqslant 1} \neq 0$ we can find a nonzero linear map $q_n : Z^m \longrightarrow (FZ)^{m+1}_{n+2}$, $n \geqslant 1$. We extend it to a derivation $q_n \in E^1_n$ by setting $q_n | Z^h = 0$ for $h \neq m$. The fact that $Z_* = Z_1$ readily implies that $[q_n, q_n] = 0$, for degree reasons. Therefore $q_n \in M$, $q_n \neq d_1$ and consequently $d_1 = 0$ is not intrinsically formal, since in our case $G.d_1 = d_1$. We finally mention that the condition $E^1_{\geqslant 1} = 0$ may be explicitely translated into an arithmetic condition involving the sequence of the degrees of the elements of a k-basis of Z^*.

2. SKELETAL PROPERTIES

This section is mainly devoted to the proof of theorems A,B and C of the introduction. We are going first to translate their statements into the language of deformation theory. To this end, it will be convenient to reformulate the conditions we have imposed on (FZ, d_1) in the previous section, in the following way: Z^*_* will be a bigraded k-vector space with the properties:

(I) $Z^{\leq 0}_* = 0$

(II) $\dim_k Z^m_* < \infty$, for any m

(III) $Z_* = Z_1$.

FZ will denote the free graded (by upper degree) Lie algebra or commutative algebra on Z^* and d_1 will be a graded (with respect to the upper degree) algebra derivation of FZ, which is homogeneous of degree 1 with respect to the lower graduation and homogeneous of degree ± 1 (more precisely deg $d_1 =$ $= -1$ in the free Lie and deg $d_1 = +1$ in the free commutative algebra case) with respect to the upper graduation, and which has the property that $[d_1, d_1] = 0$. (This is consistent both with our conditions in Section 1 and the situation coming from the deformation theory for an algebra B^*, $B^* = H^*$ or L^*. If $Z^{<0} \neq 0$ in our old setting then just relabel Z^n as Z^{-n} and notice that positive upper degree derivations become negative, and so on.)

We shall denote by $Z^{\leqslant n}$ the bigraded subspace of Z^*_* given by $Z^{\leqslant n} = \bigoplus_{m \leqslant n} Z^m$. Property (I) implies that $F(Z^{\leqslant n})$ will be a bigraded differential subalgebra of FZ, which will obviously satisfy all the conditions imposed on (FZ, d_1). We shall denote it by $(FZ, d_1)(n)$ and make the convention to label by $(.)(n)$ all its associated objects.

2.1. <u>Definition</u>. The bigraded differential algebra $(FZ, d_1)(n) =$ $= (F(Z^{\leqslant n}), d_1 | F(Z^{\leqslant n}))$ will be called the <u>n-skeleton</u> of (FZ, d_1). We will say that d_1 is <u>skeletally intrinsically formal</u> (<u>spherically generated</u>) if $d_1(n)$ is intrinsically formal (spherically generated), for any n.

Notice that if B^* is a graded algebra and $B(n)$ is its n-skeleton (as defined in the introduction) then the deformation theory of $B(n)$ is given by $(FZ, d_1)(n)$, modulo an obvious shift of the dimension n.

Finally we shall make one more assumption on (FZ, d_1)

(IV) if deg $d_1 = +1$ then $Z^1 = 0$.

(it is a harmless usual 1-connectivity assumption; anyway it holds for the examples comming from topology and will help us to treat the cases $B^* = H^*$ and $B^* = L^*$ simultaneously).

The key inductive step in what follows is provided by the following exact sequence of bigraded differential spaces, which relates the deformation theories of $d_1(n)$ and $d_1(n+1)$:

$$(1)\qquad 0 \longrightarrow K^*_*(n) \longrightarrow E^*_*(n+1) \xrightarrow{r} E^*_*(n) \longrightarrow 0.$$

Here r is defined by restricting the derivations of $F(Z^{\leq n+1})$ to $F(Z^{\leq n})$ (if deg $d_1 = -1$, remember that we are dealing with nonpositive upper degree derivations, so the restriction is obviously possible; if deg $d_1 = +1$, use $Z^1 = 0$ and a little counting degrees argument to see that the restriction to $F(Z^{\leq n})$ leaves this subalgebra invariant). The restriction map r is clearly onto, takes $d_1(n+1)$ to $d_1(n)$ and plainly is a Lie algebra map, hence also a chain map.

When things come from an algebra B it is not hard to see that the map induced at the cohomological level, $H^*_*K(n) \to H^*_*E(n+1)$ coincides, if $B^* = H^*$, with the map $H^*_*(B(n+2),B^{n+2}(n+2)) \to H^*_*(B(n+2),B(n+2))$, and if $B^* = L^*$ with the map $H^*_*(B(n),B^n(n)) \to H^*_*(B(n),B(n))$, where both these last two maps are naturally induced by the inclusion of the top dimensional homogeneous component of $B(m)$, $B^m(m) \hookrightarrow B(m)$, viewed as a $B(m)$-module map.

With these preliminaries, Theorem A will be a consequence of the following

2.2. <u>Proposition</u>. The assertions below are equivalent

(i) d_1 is skeletally intrinsically formal

(ii) $H^1_{\geq 1}E(n) = 0$, for any n

(iii) The map $H^1_{\geq 1}K(n) \to H^1_{\geq 1}E(n+1)$ is zero, for any n.

Proof. Given Proposition 1.1, it will be enough to show that (iii)$\Rightarrow$(ii) and (i)$\Rightarrow$(ii). The proof of (iii)$\Rightarrow$(ii) goes by induction, uses the long exact cohomology sequence associated to (1) and starts by observing that one even has $E^1_{\geq 1}(n) = 0$ for $n \leq n_0$, where n_0 is the minimal degree of the nonzero homogeneous components of Z^*. The same remark may be used to start the induction in the proof of (i)$\Rightarrow$(ii). Suppose then that $H^1_{\geq 1}E(n) = 0$ and that $d_1(n+1)$ is intrinsically formal. We are going to prove that $H^1_{\geq 1}E(n+1) = \underset{j\geq 1}{\times} IH^1_j(n+1)$ and then use Proposition 1.1 again to conclude that $H^1_{\geq 1}(n+1) = 0$. To show $H^1_jE(n+1) \subset IH^1_j(n+1)$, for any $j \geq 1$, it will be enough to prove that any D-cocycle $\varphi_j \in Z^1_jK(n)$ is integrable when viewed as an element of $E^1_j(n+1)$ (use again the cohomology sequence of (1)). On the other hand, given

$q_j \in K^1_j(n)$, we know that it is given by a degree ± 1 linear map $Z^{\leqslant n+1} \longrightarrow (F(Z^{\leqslant n+1}))_{j+2}$, which is zero on $Z^{\leqslant n}$, by the definition of $K(n)$. Since plainly $q_j(Z^{n+1}) \subset F(Z^{\leqslant n})$, it follows that $q_j^2 = 0$ and that the conditions $[d_1+q_j, d_1+q_j] = 0$ and $[d_1, q_j] = 0$ are equivalent. Thus the elements of $Z^1_j K(n)$ are integrable in $E^1_j(n+1)$. Our proof is complete.

In what follows we will need a more precise comparison between the deformation theory of $d_1(n)$ and of $d_1(n+1)$. This will be accomplished by the use of the following commutative diagram of bigraded differential spaces, with exact rows and colums, which enlarges the short exact sequence (1)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{K}^*_*(n) & \longrightarrow & \overline{E}^*_*(n+1) & \xrightarrow{r} & \overline{E}^*_*(n) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K^*_*(n) & \longrightarrow & E^*_*(n+1) & \xrightarrow{r} & E^*_*(n) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tilde{K}^*_*(n) & \longrightarrow & (E/\overline{E})^*_*(n+1) & \xrightarrow{r} & (E/\overline{E})^*_*(n) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{2}$$

The middle row is just the exact sequence (1). The other exact rows are constructed by simply observing that $\overline{Z}(n+1) = \overline{Z}(n) \oplus \overline{Z}^{n+1}$ and consequently that the restriction map r sends the subcomplex $\overline{E}^*_*(n+1)$ onto the subcomplex $\overline{E}^*_*(n)$. The exactness of all collums readily follows.

Let us also remark that the morphisms induced in cohomology by the left half of the above diagram may be identified, when the diagram comes from the deformation theory of a graded algebra B^* ($B^* = H^*$ or $B^* = L^*$) with the diagram below

$$\begin{array}{ccc}
H^*_*(B(m), D^m B(m)) & \longrightarrow & H^*_*(B(m), DB(m)) \\
\downarrow & & \downarrow \\
H^*_*(B(m), B^m(m)) & \longrightarrow & H^*_*(B(m), B(m)) \\
\downarrow & & \downarrow \\
H^*_*(B(m), Q^m B(m)) & \longrightarrow & H^*_*(B(m), QB(m))
\end{array} \tag{3}$$

where the maps are induced by the various natural morphisms between the

$B(m)$-modules which are involved, as explained in the introduction, and m equals $n+1\pm 1$, where the plus (respectively minus) sign occurs in the case $B = H$ (respectively $B = L$).

The following proposition will imply Theorem B.

2.3. Proposition. The assertions below are equivalent

(i) d_1 is skeletally intrinsically spherically generated

(ii) $H^1_{\geqslant 1}(E/\bar{E})(n) = 0$, for any n

(iii) The map $H^1_{\geqslant 1}\tilde{K}(n) \longrightarrow H^1_{\geqslant 1}(E/\bar{E})(n+1)$ is zero, for any n

(iv) $H^1_{\geqslant 1}(E/\bar{E}) = 0$.

Proof. Given Proposition 1.9, it will be enough to show that (iii)$\Rightarrow$(ii), (i)$\Rightarrow$(ii) and (ii)$\Leftrightarrow$(iv). The proof of the first two implications goes paralelly to the one given in Proposition 2.2. The inductive proof of (iii)$\Rightarrow$(ii) starts as in Proposition 2.2 and continues with the aid of the long cohomology sequence of the bottom row of diagram (2). To prove (i)$\Rightarrow$(ii), inductively assume $H^1_{\geqslant 1}(E/\bar{E})(n) = 0$ and also that $d_1(n+1)$ is intrinsically spherically generated. It will suffice to show that $H^1_{\geqslant 1}\tilde{K}(n) \longrightarrow H^1_{\geqslant 1}(E/\bar{E})(n+1)$ is the zero map. To do this we first notice that the left column of the diagram (2) may be obtained from the exact sequence of k-vector spaces

$$0 \longrightarrow \bar{Z}^{n+1} \longrightarrow Z^{n+1} \longrightarrow Z^{n+1}/\bar{Z}^{n+1} \longrightarrow 0$$

by applying the functor $\mathrm{Hom}^*_*(\cdot,(F(Z^{\leqslant n+1}),d_1))$ and reindexing (this is a direct consequence of the definitions, plus the fact that $q(Z^{n+1}) \subset F(Z^{\leqslant n})$, for any $q \in E^*_*(n+1)$, which involves an easy degree argument and uses, if $\deg d_1 = 1$, the condition (IV)). As a consequence, we see that the long exact cohomology sequence of the left column of (2) splits into short exact sequences, in particular the map $H^1_{\geqslant 1}K(n) \longrightarrow H^1_{\geqslant 1}\tilde{K}(n)$ is onto. We will now make use of the assumption on the spherical generation property and deduce that $H^1_{\geqslant 1}\bar{K}(n) \longrightarrow H^1_{\geqslant 1}K(n)$ is also onto, thus finishing the proof of our implication. If $p_j \in \mathrm{Der}^{\pm 1}_j(n+1)$ $(j \geqslant 2)$ is a D-cocycle of $K^1_{\geqslant 1}(n)$, we know (as seen in the proof of Proposition 2.2) that $d_1+p_j \in M(n+1)$. Since $d_1(n+1)$

is intrinsically spherically generated we may find k-linear maps $q_h:\overline{Z}^{n+1}\longrightarrow (F(Z^{\leq n+1}))^{n+1}_h$, $h\geq 1$, with the property that q_1 = inclusion and $(d_1+p_j)(q_1+q_2+\dots) = 0$ on $\overline{Z}^{n+1}$. Look at the homogeneous component of degree $j+1$ and deduce that $p_j|\overline{Z}^{n+1} + d_1q_j = 0$. Extend then q_j to Z^{n+1}, regard it as an element of $K^0_{j-1}(n)$ and conclude that $p_j + [d_1,q_j]\in\overline{K}^1_{j-1}(n)$, as desired. We are thus left with the proof of the equivalence (ii)$\Longleftrightarrow$(iv). We recall from the previous section (see 1.10) that we have $H^1_{\geq 1}(E/\overline{E}) =$ $= \mathrm{Hom}^{\pm 1}(\overline{Z},H^*_{>2}(FZ,d_1))$, and similarly for $H^1_{\geq 1}(E/\overline{E})(n)$. It follows that the vanishing of $H^1_{\geq 1}(E/\overline{E})$ is equivalent with $\mathrm{Hom}^{\pm 1}(\overline{Z}(n),H^*_{>2}(FZ,d_1)) = 0$, for any n. Observe that $\overline{Z}(n) = \overline{Z}^{\leq n}$ and consequently $\mathrm{Hom}^{\pm 1}(\overline{Z}(n),H^*_{>2}(FZ,d_1)) =$ $= \mathrm{Hom}^{\pm 1}(\overline{Z}(n),H^{\leq n\pm 1}_{>2}(FZ,d_1))$. If $\deg d_1 = -1$, it is immediate to see that the natural map $H^j_*(F(Z^{\leq n}),d_1)\longrightarrow H^j_*(FZ,d_1)$ is an isomorphism for any $j\leq n-1$. If $\deg d_1 = +1$, it is equally easy to see that $H^j_{>1}(F(Z^{\leq n}),d_1)\longrightarrow H^j_{>1}(FZ,d_1)$ is an isomorphism for any $j\leq n+1$. Therefore in both cases we have an isomorphism $\mathrm{Hom}^{\pm 1}(\overline{Z}(n),H^{\leq n\pm 1}_{>2}(FZ,d_1)) = \mathrm{Hom}^{\pm 1}(\overline{Z}(n),H^{\leq n\pm 1}_{>2}(F(Z^{\leq n}),d_1)) =$ $= \mathrm{Hom}^{\pm 1}(\overline{Z}(n),H^*_{>2}(F(Z^{\leq n}),d_1)) = H^1_{\geq 1}(E/\overline{E})(n)$, which finally gives the equivalence of (ii) and (iv) in our statement and ends the proof of Proposition 2.3.

2.4. Examples. We first come back to Example 1.15, namely to the case $d_1=0$. We assert that we can add the properties of skeletal intrinsic formality and of skeletal intrinsic spherical generation to the list of equivalent properties of d_1 given there. To see that both two skeletal properties are equivalent to $H^1_{\geq 1}(E/\overline{E}) = 0$, just note that all skeleta of (FZ,d_1) will still satisfy $d_1(n) = 0$, and then combine Example 1.15 and Proposition 2.3.

Assuming $\dim_k Z<\infty$, all generally valid implications between the four intrinsic properties we have considered are indicated below

$$\begin{tikzcd}
 & \text{skeletal intrinsic formality} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
\text{intrinsic formality} \arrow[dr, Rightarrow] & & \text{skeletal intrinsic spherical generation} \arrow[dl, Rightarrow] \\
 & \text{intrinsic spherical generation} &
\end{tikzcd}$$

Indeed, we may on one hand come back to the Félix example ([1]), which we have already discussed in 1.10, to see an example of a finite dimensional graded commutative k-algebra H^* which is intrinsically formal but not skeletally intrinsically spherically generated.

On the other hand one may use homogeneously generated graded algebras B^* (see [15] for the case B = H) to produce skeletally intrinsically spherically generated examples which are not intrinsically formal. We say that B^* is homogeneously generated if it is generated as an algebra by some homogeneous component B^m. Plainly all skeleta of B^* will share the same property. We claim that if B is homogeneously generated then it is skeletally intrinsically spherically generated. To see the vanishing of $\mathrm{Hom}^{\pm 1}(\overline{Z}^*, H^*_{\geqslant 2}(FZ, d_1))$, use the assumption on the homogeneous generation of B to deduce that $\overline{Z}$ equals the minimal degree non zero homogeneous component of Z^*.

We finally remark that the main difference between theorems A and B consists in the different skeletal behaviour of $H^*_*(B,B)$ and $H^*_*(B,QB)$. We shall indicate an example of a finite dimensional k-algebra B^* with the property that $H^1_{\geqslant 1}(B,B) = 0$ but not all $H^1_{\geqslant 1}(B(n),B(n))$ are zero. Set $B^* =$ $= H^*((S^5 \times (S^{14} \vee S^{23})) \vee S^5; k)$ ([1], p.26). Then $H^1_{\geqslant 1}(B,B) = 0$ (the computation of $H^1_{\geqslant 1}(E)$ is carried out in [1] in terms of Sullivan models, but this does not affect the result, see [17], [20], [21] or better try a direct - and easier - computation in terms of Quillen models). Observe next that $B(14) =$ $= H^*(S^5 \vee S^5 \vee S^{14}; k)$ is not intrinsically formal.

We finally take care of Theorem C.

2.5. Proposition. Suppose that d_1 is skeletally intrinsically spherically generated. Then the following assertions are equivalent

(i) d_1 is skeletally intrinsically formal.

(ii) The map $H^1_{\geqslant 1}(\overline{E})(n) \longrightarrow H^1_{\geqslant 1}(E)(n)$ is zero, for any n.

(iii) The map $H^1_{\geqslant 1}(\overline{K})(n) \longrightarrow H^1_{\geqslant 1}(E)(n+1)$ is zero, for any n.

Proof. Plan of the proof: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). Given Proposition 2.2, we only have to prove the last implication. This in turn will be an immediate consequence of Propositions 2.2 and 2.3, by a standard inductive argument which uses the various long exact cohomology sequences arising from the rows and columns of diagram (2). We may as usual inductively suppose that $H^1_{\geq 1}E(n) = 0$ and try to obtain $H^1_{\geq 1}E(n+1) = 0$ by showing that the map $H^1_{\geq 1}K(n) \longrightarrow H^1_{\geq 1}E(n+1)$ is zero. This will follow at once, if we are able to show that $H^1_{\geq 1}\overline{K}(n) \longrightarrow H^1_{\geq 1}K(n)$ is onto, from our hypotheses made in (iii). This last assertion will follow in turn from $H^1_{\geq 1}\widetilde{K}(n) = 0$. To obtain this vanishing property we will more generally show that the bottom row of diagram (2) is a split short exact sequence of bigraded differential spaces (and then use the skeletally intrinsic spherical generation assumption to deduce $H^1_{\geq 1}(E/\overline{E})(n+1) = 0$, hence $H^1_{\geq 1}\widetilde{K}(n) = 0$). As far as the splitting property is concerned, we have an obvious bigraded k-linear splitting s, $E^*_*(n+1) \underset{s}{\overset{r}{\rightleftarrows}} E^*_*(n)$, defined for $q \in E^*_*(n)$ by $sq|Z^{\leq n} = q$ and $sq|Z^{n+1} = 0$ (note however that it will not be in general compatible with the differentials). Since plainly $s\overline{E}^*_*(n) \subset \overline{E}^*_*(n+1)$, we will have an induced bigraded k-linear splitting, also denoted by s, $(E/\overline{E})^*_*(n+1) \underset{s}{\overset{r}{\rightleftarrows}} (E/\overline{E})^*_*(n)$. Pick $q \in E^*_*(n)$ and compute $r(s[d_1,q] - [d_1,sq]) = [d_1,q] - [d_1,rsq] = 0$ (since plainly r is a Lie algebra map); we also have $(s[d_1,q] - [d_1,sq])|\overline{Z}^{n+1} = 0$, by construction which shows that $s[d_1,q]-[d_1,sq] \in \overline{E}^*_*(n+1)$ and thus that the induced splitting on $E/\overline{E}$ is also a chain map, as desired, and concludes the proof of our proposition. We finally mention that it is the natural occurence of the condition "$H^1_{\geq 1}\overline{E} \longrightarrow H^1_{\geq 1}E$ is zero" in the study of the gap between intrinsic spherical generation and intrinsic formality which we really want to emphasize here (compare with Proposition 1.11)

2.6. <u>Example:</u> complete intersections. Let us say that d_1 is a <u>complete intersection</u> if $H^*_{>2}(FZ,d_1) = 0$. It follows then that d_1 is skeletally intrinsically formal. Indeed, we may show that $H^1_{\geq 1}K(n) = 0$, for any n, and then use Proposition 2.2(iii). As we have remarked before $H^1_{\geq 1}K(n) =$
$= \mathrm{Hom}^{\pm 1}(Z^{\leq n+1}, H^*_{>2}(F(Z^{\leq n+1}),d_1)) = \mathrm{Hom}^{\pm 1}(Z^{\leq n+1}, H^*_{>2}(FZ,d_1))$, see the proof

of Proposition 2.3, and we are done. As far as the terminology is concerned let us remark that when we are dealing with the deformation theory of a graded commutative algebra $B^* = H^*$ it coincides with the traditional one (as it may be seen by translating the condition on $H^*_{>2}(FZ,d_1)$ into the language of bigraded Halperin-Stasheff dga models [6], see [21] and also [2]). As classical examples we may quote $H^*(G/K;k)$, where $K \subset G$ is an equal rank pair of compact connected Lie groups. If $B^* = L^*$, the property of d_1 of being a complete intersection is equivalent to gl dim $L \leqslant 2$. This may be seen by observing that (FZ^*_*,d_1) is nothing else but the graded Koszul construction of the cochains on L, which may then be used (see [21]) to compute $\# \mathrm{Tor}^{UL}_{n,*}(k,k) = H^*_n(FZ,d_1)$, for any n. A nice class of geometric examples is provided by the m-dimensional compact closed manifolds M^m which are n-connected ($n \geqslant 1$) and of the dimension $m \leqslant 3n+2$. We assert that $\pi_*(\Omega M)\otimes k$ is a complete intersection, unless $H^*(M;k)$ is a truncated polynomial algebra of the form $k[x]/(x^3)$, with deg x = even. Indeed, set $M' = M \setminus \{point\}$ and notice that $M \simeq M' \cup e^m$. If $H^*(M;k)$ is not a monogenic algebra then we know (see [4]) that the attaching of the m-cell e^m gives rise to a "perfect murder" and consequently $\mathrm{Tor}^{UL'}_{\geqslant 3}(k,k) \longrightarrow \mathrm{Tor}^{UL}_{\geqslant 3}(k,k)$ is an isomorphism, where $UL' \to UL$ is the universal enveloping algebra map induced by the inclusion of M' into M by applying the $\pi_*\Omega(.)\otimes k$ functor. It will then be enough to show that gl dim $L' \leqslant 2$. We will actually show that L' is free; this in turn will be a consequence of the fact that $H^*(M';k)$ is both intrinsically formal and with trivial multiplication, and thus M' must be rationally equivalent to a wedge of spheres. Notice then first that the inclusion of M' into M induces an homology isomorphism up to dimension m-1 and second that $H_{\geqslant m}M' = 0$; it follows that H^*M' is intrinsically formal, being both n-connected (as soon as M is so) and trivial in dimensions $> 3n+1$ (as follows from our assumption $m \leqslant 3n+2$) - see [6]. On the other hand the homological n-connectivity of M' forces $H^+M'.H^+M'$ to be concentrated in degrees $\geqslant 2n+2$, while Poincaré duality on M implies that $H^{\geqslant m-n}M' = 0$;

recalling that $m \leqslant 3n+2$, we infer that $H^+M'.H^+M' = 0$, thus finishing the discussion of the cohomologically non-monogenic case. In the remaining cases it is immediate to see that either M is rationally equivalent to S^m, and $\mathfrak{A}_*(\Omega M)\otimes k$ is then plainly a complete intersection, or $H^*(M;k) = k[x]/(x^3)$, in which case it is equally easy to see that $\mathfrak{A}_*(\Omega M)\otimes k$ is finite dimensional abelian and not even intrinsically cospherically generated. We finally point out that with our hypotheses (the non cohomological monogenity excluded) $H^*(M;k)$ will always be also skeletally intrinsically formal (for the proper skeleta of H^*M just use again the criterion furnished by [6] and for H^*M itself add the results related to the presence of Poincaré duality of [18] or [10]).

References

[1] Y. Félix, Dénombrement des types de k-homotopie. Théorie de la déformation, Bull. Soc. Math. France, 108(3)(1980).

[2] Y. Félix and S. Halperin, Formal spaces with finite dimensional rational homotopy, TAMS 273(1982), 1-37.

[3] M. Gerstenhaber, On the deformation of rings and algebras, Ann. of Math. 79(1964), 59-104.

[4] S. Halperin and J.M. Lemaire, Suites inertes dans les algébres de Lie graduées, to appear Math. Scand.

[5] R. Hartshorne, Algebraic Geometry, Graduate texts in Math. 52 (Springer, Berlin, 1977).

[6] S. Halperin and J. Stasheff, Obstructions to homotopy equivalences, Advances in Math. 32(1979), 233-279.

[7] J.M. Lemaire and F. Sigrist, Dénombrement de types d'homotopie rationnelle, C.R. Acad. Sc. Paris 287(1978), 109-112.

[8] M. Markl, A deformation theory for bigraded algebras, preprint (unpublished).

[9] M. Markl, On the formality of products and wedges, Suppl. Rend. Circ. Mat. Palermo, Serie II, 16(1987), 199-206.

[10] T.J. Miller, On the formality of k-1 connected compact manifolds of dimension less than or equal to 4k-2, Illinois J. Math. 23(2)(1979), 253-258.

[11] J. Morgan, The algebraic topology of smooth algebraic varieties, Publ. IHES 48(1978), 137-204; correction Publ. IHES 64(1986), 185

[12] J. Neisendorfer and T. Miller, Formal and coformal spaces, Illinois J. Math. 22(4)(1978), 565-580.

[13] A. Nijenhuis and R.W. Richardson, Cohomology and deformations in graded Lie algebras, Bull. AMS 72(1966), 1-29.

[14] S. Papadima, Cell decompositions and dga models, Preprint INCREST 69(1982).

[15] S. Papadima, The cellular structure of formal homotopy types, J. Pure Appl. Alg. 35(1985), 171-184.

[16] D. Quillen, Rational homotopy theory, Ann. of Math. 90(1969), 205-295.

[17] M. Schlessinger and J. Stasheff, Deformation theory and rational homotopy type, to appear Publ. IHES.

[18] J. Stasheff, Rational Poincaré duality spaces, Illinois J. Math. 27(1983), 104-109.

[19] D. Sullivan, Infinitesimal computations in topology, Publ. IHES 47(1977), 269-331.

[20] D. Tanré, Cohomologie de Harrison et type d'homotopie rationnelle, in Algebra, algebraic topology and their interactions, Proc. Stockholm 1983, Lect. Notes in Math. 1183 (Springer, Berlin 1986), 361-370.

[21] D. Tanré, Homotopie rationnelle: Modèles de Chen, Quillen, Sullivan, Lect. Notes in Math. 1025 (Springer, Berlin 1983).