ON THE GENERAL STABLE RANK FOR PAIRS

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Recent works of M.A.Rieffel ([4],[5]) on the so-called "cancellation" property for projective modules over C^*-algebras give sufficient motivation to consider his theory of stable ranks as a fruitful instrument in the non-stable K-theory of C^*-algebras.

For example, in order to give a sufficient condition for a stably free module to be actually free, Rieffel used the general stable rank (gsr) of a C^*-algebra.

But if one wants to study the behaviour of this invariant in extensions, one finds that the inequality "$gsr(A) \leq \max(gsr(J), gsr(A/J))$" does not hold in the general case (see the example given in the paper). The situation is the same if "gsr" is replaced by "sr" (stable rank), but in this case Rieffel proved the inequality

$$sr(A) \leq \max(sr(J), sr(A/J), csr(A/J)).$$

In [3] we introduced the <u>general stable rank of a pair</u> ($gsr_J(A)$) and proved the following inequalities

$$gsr(A) \leq \max(gsr(J), gsr_J(A)) \qquad (1)$$

$$sr(A) \leq \max(sr(J), sr(A/J), gsr_J(A)) \qquad (2).$$

In this paper we give further properties for our invariant which show that it has a "sub-homological" behaviour.

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We begin by recalling some notations and definitions.

For a unital C^*-algebra A and an integer n one takes

$$Lg_n(A) = \{(a_1,\dots,a_n) \in A^n / \exists b_1,\dots,b_n \in A \text{ s.t. } b_1a_1+\dots+b_na_n=1\}$$

Considering $Lg_n(A)$ as a set of column vectors one has an action of the group $GL_n(A)$ (invertible matrices of $M_n(A)$) by left multiplication. Rieffel's definition for the <u>general stable rank</u> of A is

$gsr(A)=\min\{m\in N \ / \ GL_n(A)$ acts transtively on $Lg_n(A)$ for every $n\geq m\}$.

If the above set if empty one takes $gsr(A)=\infty$, and in the non-unital case one works in the algebra $\tilde{A}$ obtained by adjoining a unit.

To show that we cannot expect to have "$gsr(A)\leq \max(gsr(J),gsr(A/J))$" we give the following

<u>EXAMPLE</u> Consider $\mathcal{T}$ the Toeplitz C^*-algebra and the ideal $\mathcal{K}\subset\mathcal{T}$ of compact operators. One has $\mathcal{T}/\mathcal{K}\cong C(T)$ ([2]) and $gsr(\mathcal{K})=gsr(C(T))=1$ but $gsr(\mathcal{T})=2$ ($gsr(\mathcal{T})$ cannot be 1 since $\mathcal{T}$ contains a non-unitary isometry).

We now recall our definition for the <u>"gsr" of a pair</u> (with a little change of notation).

<u>DEFINITION</u> Suppose J is an ideal (by this we mean closed and two-sided ideal) of the unital C^*-algebra A. Then $gsr(A,J)$ is the least intger m such that for every $n\geq m$ the set $\{x\in GL_n(A/J) \ / \ \partial[x]=0\}$ acts transitively on $Lg_n(A/J)$. Again if no such m exists one takes $gsr(A,J)=\infty$ and in the non-unital case one proceeds as before. $\partial: K_1(A/J)\longrightarrow K_0(J)$ is the "index" homomorphism in K-theory ([1], [6]). In fact our invariant gives some information about ∂ namely we have

<u>LEMMA 1</u> Suppose $gsr(A,J)\leq n<\infty$. Then the map

$GL_{n-1}(A/J)\ni x\longmapsto \partial[x]\in \mathrm{Im}\,\partial$ is onto.

PROOF. Take $m>n-1$ and $y\in GL_m(A/J)$. Take y^m the last column of y. Of course $y^m\in Lg_m(A/J)$ and since $m\geq gsr(A,J)$, there exists $T\in GL_m(A/J)$ with $\partial[T]=0$ such that $Ty^m=(0,\dots,1)^t$ ("t" stands for transpose). Ty will have the form $Ty=\begin{pmatrix} x & 0 \\ a & 1 \end{pmatrix}$ with $a\in M_{1\times(m-1)}(A/J)$ and $x\in GL_{m-1}(A/J)$. It is clear that there is a continuous path in $GL_m(A/J)$ joining Ty with $\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$ and so $[Ty]_{K_1(A/J)}=[x]_{K_1(A/J)}$. Of course $\partial[x]=\partial[Ty]=$ $=\partial[T]+\partial[y]=\partial[y]$. Consequently we found $x\in GL_{m-1}(A/J)$ such that

$\partial[y] = \partial[x]$. If we still have $m-1 > n-1$ we continue this procedure. After $m-n+1$ steps we obtain an element of $GL_{n-1}(A/J)$ with same index as y.

REMARKS Since $K_0(\{0\}) = \{0\}$ we have $gsr(A) = gsr(A, \{0\})$. Also one easily gets

$$gsr(A/J) \leq gsr(A,J) \qquad (3).$$

To prove one of the main results we have to recall one fact from [3]

THEOREM 1 Suppose $x \in GL_n(A/J)$ and $n \geqslant gsr(J)-1$. Then the following conditions are equivalent:

(i) there exists a lifting $X \in GL_n(A)$ for x

(ii) $\partial[x] = 0$.

Now we can prove that inequalities (1) and (3) hold in a more general setting, namely

THEOREM 2 Suppose K and J are ideals of A such that $K \subset J$. Then

(i) $gsr(A/K, J/K) \leq gsr(A,J)$

(ii) $gsr(A,K) \leqslant \max(gsr(A,J), gsr(J,K))$.

PROOF. There is no difficulty to show that we can suppose A unital. In the following diagram of C^*-algebras and $*$-homomorphisms

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \xhookrightarrow{i_1} & J & \xrightarrow{\pi_1} & J/K & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle i_3} & & \downarrow{\scriptstyle i_4} & & \\
0 & \longrightarrow & K & \xhookrightarrow{i_2} & A & \xrightarrow{\pi_2} & A/K & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i_1} & & \| & & \downarrow{\scriptstyle \pi_4} & & \\
0 & \longrightarrow & J & \xhookrightarrow{i_3} & A & \xrightarrow{\pi_3} & A/J & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} & & \| & & \\
0 & \longrightarrow & J/K & \xrightarrow{i_4} & A/K & \xrightarrow{\pi_4} & A/J & \longrightarrow & 0
\end{array}$$

the four rows are exact and all the squares are commutative. If we denote by $\partial_1, \partial_2, \partial_3, \partial_4$ the "index" homomorphisms corresponding to each row, by naturality of the exact sequence of K-theory ([1],[6]), we obtain the following diagram of groups and homomorphisms in which each square is commutative:

$$\begin{array}{ccccccc} K(J/K) & \xrightarrow{i_{4*}} & K(A/K) & \xrightarrow{\pi_{4*}} & K(A/J) & = & K(A/J) \\ \downarrow \partial_1 & & \downarrow \partial_2 & & \downarrow \partial_3 & & \downarrow \partial_4 \\ K_0(K) & = & K_0(K) & \xrightarrow{i_{1*}} & K_0(J) & \xrightarrow{\pi_{1*}} & K_0(J/K) \end{array}$$

To prove (i) take $n \geqslant gsr(A,J)$ and $(a_1,\ldots,a_n) \in Lg_n((A/K)/(J/K)) \simeq Lg_n(A/J)$. From the definition there exists $S \in GL_n(A/J)$ such that $\partial_3[S]=0$ and $S(a_1,\ldots,a_n)^t=(1,0,\ldots,0)^t$. But $\partial_4[S]=\pi_{1*}\circ\partial_3[S]=0$ which shows that $\{S \in GL_n(A/J) \,/\, \partial_4[S]=0\}$ acts transitively on $Lg_n(A/J)$ and so we obtain (i). To prove (ii) let $n \geqslant \max(gsr(A,J), gsr(J,K))$ and $(a_1,\ldots,a_n) \in Lg_n(A/K)$. Consider $b=(\pi_4(a_1),\ldots,\pi_4(a_n))^t$. Clearly $b \in Lg_n(A/J)$ and, since $n \geqslant gsr(A,J)$, there exists $T \in GL_n(A/J)$ such that $\partial_3[T]=0$ and $Tb=(1,0,\ldots,0)^t$. Again we have $\partial_4[T]=0$ and, since $n \geqslant gsr(J,K) \geqslant gsr(J/K)$, by Theorem I there exists $S \in GL_n(A/K)$ such that $\pi_4(S)=T$ (for a $*$-homomorphism we keep the same notation for its extension to matrices). But this means that $\pi_4(S(a_1,\ldots,a_n)^t)=(1,0,..,0)^t$, and we obtain $S(a_1,\ldots,a_n)^t \in Lg_n(\tilde{J}/K)$ (of course $\tilde{J} \subset A$ and so $\tilde{J}/K \subset A/K$). On the other hand $i_{1*}\circ\partial_2[S]=\partial_3\circ\pi_{4*}[S]=\partial_3[T]=0$. Exactness of the sequence of K-theory for the first row gives $\partial_2[S] \in \operatorname{Ker} i_{1*}=\operatorname{Im}\partial_1$ $(\partial_1 : K_1(J/K) \longrightarrow K_0(K))$. Since $n \geqslant gsr(J,K)$, by Lemma I, there exists $R \in GL_n(\tilde{J}/K)$ such that $\partial_1[R]=\partial_2[S]$. Take $Y=R^{-1}S$. Since $R^{-1} \in GL_n(\tilde{J}/K)$ and $S(a_1,\ldots,a_n)^t \in Lg_n(\tilde{J}/K)$, we conclude that $Y(a_1,\ldots,a_n)^t \in Lg_n(\tilde{J}/K)$. From $n \geqslant gsr(J,K)$ we get the existence of $Z \in GL_n(\tilde{J}/K)$ such that $\partial_1[Z]=0$ and $ZY(a_1,\ldots,a_n)^t=(1,0,..,0)^t$. We have $\partial_2[ZY]=\partial_2[ZR^{-1}S]=\partial_2\circ i_{4*}[Z]-\partial_2\circ i_{4*}[R]+\partial_2[S]=\partial_1[Z]-\partial_1[R]+\partial_2[S]=\partial_1[Z]=0$. This shows that the set $\{X \in GL_n(A/K) \,/\, \partial_2[X]=0\}$ acts transitively on $Lg_n(A/K)$, which proves (ii).

For the following result we introduce the next

<u>DEFINITION</u> Suppose J is an ideal in A and K is an ideal in B. By a <u>$*$-homomorphism of pairs</u> $\varphi:(A,J) \longrightarrow (B,K)$ we mean a $*$-homomorphism $\varphi : A \longrightarrow B$ such that $\varphi(J) \subset K$. Given two $*$-homomorphisms of pairs

$\varphi,\psi:(A,J) \longrightarrow (B,K)$, we say that they are <u>homotopic</u> if there exists $\Phi:[0,1]\times A \longrightarrow B$ such that $\Phi(0,.)=\varphi$, $\Phi(1,.)=\psi$ and

(i) for every $t\in[0,1]$ $\Phi(t,.):(A,J) \longrightarrow (B,K)$ is a $*$-homomorphism of pairs

(ii) for every a A the map $\Phi(.,a):[0,1] \longrightarrow B$ is continuous.

The pairs (A,J) and (B,K) are <u>homotopicaly equivalent</u> if there exist two $*$-homomorphisms of pairs $\varphi:(A,J) \longrightarrow (B,K)$ and $\psi:(B,K) \longrightarrow (A,J)$ such that $\psi\circ\varphi$ and Id_A are homotopic and also $\varphi\circ\psi$ and Id_B are homotopic.

The following result shows the homotopy invariance for "gsr"

<u>THEOREM 3</u> If the pairs (A,J) and (B,K) are homotopicaly equivalent then $gsr(A,J)=gsr(B,K)$.

PROOF. Obviously we can suppose A and B unital and also that the $*$-homomorphisms φ and ψ in the definition are unit-preserving. By symmetry it suffices to prove only the inequality $gsr(A,J)\leq gsr(B,K)$. Let's denote by $\hat{\varphi}:A/J \longrightarrow B/K$ and $\hat{\psi}:B/K \longrightarrow A/J$ the $*$-homomorhisms induced by φ and ψ. Take $n\geq gsr(B,K)$ and $(a_1,\dots,a_n)\in Lg_n(A/J)$. Consider $(\hat{\varphi}(a_1),\dots,\hat{\varphi}(a_n))\in Lg_n(B/K)$. Since $n\geq gsr(B,K)$ there exists $T\in GL_n(B/K)$ such that $\partial_2[T]=0$ and $T(\hat{\varphi}(a_1),\dots,\hat{\varphi}(a_n))^t=(1,0,\dots,0)^t$. By naturality we have a commutative diagram of groups and homomorphisms

$$\begin{array}{ccc} K_1(B/K) & \xrightarrow{\hat{\psi}_*} & K_1(A/J) \\ \partial_2\downarrow & & \downarrow\partial_1 \\ K_0(K) & \xrightarrow[\psi_*]{} & K_0(J) \end{array}$$

Let $S=\hat{\psi}(T)\in GL_n(A/J)$. Since $\hat{\psi}$ induces isomorphism on K-theory we get $\partial_1[S]=0$. On the other hand $(\hat{\psi}\circ\hat{\varphi}(a_1),\dots,\hat{\psi}\circ\hat{\varphi}(a_n))$ can be obviously be joined by a continuous path in $Lg_n(A/J)$ with $(a_1,\dots,a_n)$ and, consequently (Corollary 8.5 of [4]) there exists $R\in GL_n^o(A/J)$ such that $(\hat{\psi}\circ\hat{\varphi}(a_1),\dots,\hat{\psi}\circ\hat{\varphi}(a_n))^t=R(a_1,\dots,a_n)^t$. ($GL_n^o(.)$ stands for the connected component of the identity in $GL_n(.)$). In particular we have $[R]_{K_1(A/J)}=0$. So we get $\partial_1[SR]=\partial_1[S]+\partial_1[R]=0$ and $SR(a_1,\dots,a_n)^t=$ $=(1,0,\dots,0)^t$ which means that $\{X\in GL_n(A/J)\ /\ \partial_1[X]=0\}$ acts transitively on $Lg_n(A/J)$.

For matrix algebras we have

<u>PROPSITION 1</u> $gsr(M_n(A),M_n(J))\leq\left\{\frac{gsr(A,J)-1}{n}\right\}+1$ ($\{t\}$ stands for the least integer greater than t).

PROOF. Take $k\geq\left\{\frac{gsr(A,J)-1}{n}\right\}+1$, thar is $kn-n+1\geq gsr(A,J)$. Let $b\in Lg_k(M_n(A)/M_n(J))=Lg_k(M_n(A/J))$ considered as a left-invertible $kn\times n$ matrix. Of course $(b_{11},b_{21},\dots,b_{kn,1})^t\in Lg_{kn}(A/J)$ (it is the first column of b). Since $kn\geq gsr(A,J)+n-1\geq gsr(A,J)$ there exists $T_1\in GL_{kn}(A/J)$ such that $\partial[T_1]=0$ and $T_1(b_{11},b_{21},\dots,b_{kn,1})^t=(1,0,\dots,0)^t$. Let $a=T_1b\in Lg_k(M_n(A/J))$, and $c\in M_{n\times kn}(A/J)$ a left inverse for a, that is $ca=I_n$. Put $S_1=\begin{pmatrix}c_{11} & c_{12} & \dots & c_{1,kn}\\ 0 & & I_{kn-1} & \end{pmatrix}$ ($(c_{11},\dots,c_{1,kn})$ is the first row of c). Since the first column of a is $(1,0,\dots,0)^t$ we have $c_{11}=1$ and so $S_1\in GL_{kn}(A/J)$. Of course $[S_1]_{K_1(A/J)}=0$. On the other hand since $c_{11}a_{1p}+\dots+c_{1,kn}a_{kn,p}=0$ for every $p\neq 1$, if we let $X_1=S_1T_1$, we have $\partial[X_1]=0$ and $X_1b=\begin{pmatrix}1 & 0\\ 0 & b^1\end{pmatrix}$ with b^1 a left-invertible $(kn-1)\times(n-1)$ matrix. If we still have $kn-1\geq gsr(A,J)$ we continue this procedure and find $X_2\in GL_{kn-1}(A/J)$ with $\partial[X_2]=0$ such that $X_2b^1=\begin{pmatrix}1 & 0\\ 0 & b^2\end{pmatrix}$ with b^2 a left-invertible $(kn-2)\times(n-2)$ matrix, and so on. After n steps if we take $X=X_1\cdot\begin{pmatrix}1 & 0\\ 0 & X_2\end{pmatrix}\cdot\ldots\cdot\begin{pmatrix}I_{n-1} & 0\\ 0 & X_n\end{pmatrix}$ we obtain $\partial[X]=0$ and $Xb=\begin{pmatrix}I_n\\ 0\end{pmatrix}$ which ends the proof since $GL_k(M_n(A/J)=GL_{kn}(A/J)$ and the "index" homomorphism for the pair $(M_n(A),M_n(J))$ is exactly ∂.

For inductive limits we have

<u>THEOREM 4</u> Suppose $A=\varinjlim A_n$ and $J=\varinjlim J_n$ where $A_n\subset A_{n+1}$, $J_n=J\cap A_n$ and J_n is an ideal of A_n for every n. Then $gsr(A,J)\leq\liminf gsr(A_n,J_n)$.

PROOF. We may suppose $1_A\in A_n$ for every n. Take $k\geq\liminf gsr(A_n,J_n)$. Chooseing a subsequence we restrict to the situation $k\geq gsr(A_n,J_n)$ for every n. Let's denote by $\pi_n: A_n/J_n \longrightarrow A/J$ the $*$-homomorphisms induced by the embeddings $(A_n, J_n) \longrightarrow (A, J)$ and by $i_n : J_n \longrightarrow J$. Take $(a_1, \dots, a_k) \in Lg_k(A/J)$. Since $\bigcup_{n \in N} \pi_n(A_n/J_n)$ is dense in A/J there exists some n and $(b_1, \dots, b_k) \in Lg_k(\pi_n(A_n/J_n))$ close enough to $(a_1, \dots, a_k)$ such that there exists $R \in GL_k^0(A/J)$ for which $(a_1, \dots, a_k)^t = R(b_1, \dots, b_k)^t$ ((Corollary 8.5 of [4])). Under the assumptions about J, π_n will be injective and so there exists $(x_1, \dots, x_k) \in Lg_k(A_n/J_n)$ such that $(b_1, \dots, b_k) = (\pi_n(x_1), \dots, \pi_n(x_k))$. Inequality $k \geqslant gsr(A_n, J_n)$ gives the existence of an invertible $X \in GL_k(A_n/J_n)$ with $\partial_n[X] = 0$ such that $X(x_1, \dots, x_k)^t = (1, 0, \dots, 0)^t$. Take $T = R\pi_n(X)^{-1}$. The commutativity of the following diagram

$$\begin{array}{ccc} K_1(A_n/J_n) & \xrightarrow{\pi_{n*}} & K_1(A/J) \\ \partial_n \downarrow & & \downarrow \partial \\ K_0(J_n) & \xrightarrow{i_{n*}} & K_0(J) \end{array}$$

shows that $\partial[T] = 0$. Since $(a_1, \dots, a_k)^t = T(1, 0, \dots, 0)^t$ we conclude that $\{T \in GL_k(A/J) \ / \ \partial[T] = 0\}$ acts transitively on $Lg_k(A/J)$ which ends the proof.

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