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ISSN 0250 3638

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by

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PREPRINT SERIES IN MATHEMATICS
No. 4/1988

BUCURESTI

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January 1988

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A TIME DEPENDENT PROOF OF THE ASYMPTOTIC COMPLETENESS FOR DIFFERENTIAL OPERATORS OF PRINCIPAL TYPE AND FOR SHORT RANGE POTENTIALS

M. Pascu

In this paper we shall give a time dependent proof of the asymptotic completeness of wave operators in conditions which are close to those in [5].

We shall denote with P a real function belonging to $C^{\infty}(\mathbb{R}^n)$. This function and its derivatives have a polynomial growth. We suppose that there exist two open cones Ω_1 and Ω_2, $\Omega_1 \cup \Omega_2 = \mathbb{R}^n \setminus \{0\}$ and

(i) $(\exists)\ \delta > 0$, $(\exists) R_o > 0$, $(\exists) c > 0$, $(\exists) C_\alpha > 0$ $(|\alpha| \geq 2)$ such that

$$|P'(\xi)| \geq c|\xi|^{\delta}, \quad \xi \in \Omega_1, \quad |\xi| \geq R_o,$$

$$|P^{(\alpha)}(\xi)| \leq C_\alpha |P'(\xi)|\, |\xi|^{-\delta(|\alpha|-1)}, \quad \xi \in \Omega_1, \quad |\xi| \geq R_o, \quad |\alpha| \geq 2,$$

(ii) $|P(\xi)| \to \infty$ when $|\xi| \to \infty$, $\xi \in \Omega_2$.

We also assume that if $C_v = \{P(\xi);\ P'(\xi) = 0\}$, then

(iii) $\overline{C}_v$ is at most a countable set.

(This condition ensures the absence of singular continuous spectra (σ_{sc}) of the operators we shall consider below.)

The unperturbed (or free) Hamiltonian is given by $H_o = F^{-1}P(\cdot)F$, where $P(\cdot)$ is the self-adjoint operator of multiplication with $P(\xi)$ in $L^2(\mathbb{R}^n_\xi)$ and F is the Fourier transform

$$(Ff)(\xi) = \int e^{-i(x,\xi)} f(x)\, dx, \quad f \in \mathcal{S}(\mathbb{R}^n).$$

The operator H_o is a self-adjoint opertor in $L^2(\mathbb{R}^n)$ without singular continuous spectrum.

Now let V be a symmetric H_o -compact operator (i.e. $V(H_o + i)^{-1}$ is compact). Then it is well known ([11]) that $H = H_o + V$ is a self-adjoint operator in $L^2(\mathbb{R}^n)$ and his domain of definition equals $\mathrm{dom}(H_o)$. We suppose in addition that V is a short range perturbation of H_o :

(iv) $\|(H_o + i)^{-1} V \theta(\cdot/r)\| \in L^1(R_+ ; dr)$ for one (and hence for every) function $\theta \in C^\infty(\mathbb{R}^n; \mathbb{R})$, $\theta(x) = 0$ for $|x| \leq 1/2$, $\theta(x) = 1$ for $|x| \geq 1$.

We shall prove the next theorem:

Theorem. If the assumptions (i)-(iv) are satisfied, then:

(a) there exist the wave operators $W_\pm = \text{s-}\lim_{t \to \pm\infty} e^{itH} e^{-itH_o} P_{ac}(H_o)$, where $P_{ac}(H_o)$ is the orthogonal projection on the subspace of absolute continuity of H_o;

(b) $\mathrm{Ran}\, W_\pm = \mathcal{H}_{ac}(H)$ (Ran A denotes the range of the operator A);

(c) $\sigma_{sc}(H) = \emptyset$;

(d) the eigenvalues of H which are not in $\overline{C_V}$ are of finite multiplicity and they can accumulate only in the points of $\overline{C_V}$.

We shall give a time-dependent proof of this theorem. In [5] , L. Hörmander proved the existence and asymptotic completeness of the wave operators in the case of some short range perturbations of a differential operator with constant coefficients which satisfies:

(1) $\lim (|P(\xi)| + |P'(\xi)|) = \infty$, when $|\xi| \to \infty$

and

(2) $|P^{(\alpha)}(\xi)| \leq C_\alpha(|P(\xi)| + |P'(\xi)| + 1)$, $|\alpha| \geq 2$.

Hörmander called these operators simply characteristic. But his proof was based on stationary (time-independent) methods.

The first time-dependent proof of the asymptotic completeness is due to V. Enss ([3]), for the case $H_o = -\Delta$ (i.e. $P(\xi) = |\xi|^2$). Simon, in [13] , generalized the result of Enss to the case $P(\xi) \to \infty$ when $|\xi| \to \infty$. In [8] and [9] Muthuramalingam proved the asymptotic completeness by a time-dependent method for a class of simply characteristic operators with short range local perturbations.

His construction of the operators which separate the outgoing and incoming parts of a state is similar to that in [4] .

In this paper we construct these operators as pseudodifferential operators, like in [7] . Our proof was also inspired by the papers of Muthuramalingam [10] and Iftimie [6] .

Let us prove first the existence of the wave operators. Since $e^{itH}\, e^{-itH_0}$ are uniformly continuous with respect to t, it is sufficient to prove that there exist the limits

$$\lim_{t\to\pm\infty} e^{itH}\, e^{-itH_0} f$$

for f in a dense subset of $\mathcal{H}_{ac}(H_0)$. Accordingly to the appendix we can take this set to be equal to

$$\{f \in \mathcal{S}\ ;\ \hat{f} \in C_0^\infty(\{\xi;\ P'(\xi) \neq 0\})\}.$$

Such a function f is in dom(H_0). If $\psi^\nu \in C_0^\infty(R)$, $0 \leq \psi^\nu \leq 1$, $\psi^\nu(\lambda) = 1$ for $|\lambda| \leq \nu$, then

$$\|(1 - \psi^\nu(H))e^{-itH_0}f\| \leq \|(1 - \psi^\nu(H))(H + i)^{-1}\|\ \|(H + i)(H_0 + i)^{-1}\|\ \|(H_0 + i)f\|$$

converges to zero uniformly with respect to t in R. So the first assertion of the theorem is true if there exist

$$\lim_{t\to\pm\infty} \psi(H)e^{itH}\, e^{-itH_0} f,$$

for f like before and for every $\psi \in C_0^\infty(R)$.

Applying the Cook method we deduce that it is sufficient to prove that

$$\int_{\pm 1}^{\infty} \|\psi(H)Ve^{-itH_0} f\|\, dt < \infty .$$

Now let θ be a function like in assumption (iv). We have

$$\|\psi(H)Ve^{-itH_0} f\| \leq \|\psi(H)(H_0 + i)\|\ \|(H_0 + i)^{-1} V\theta(\cdot/(at))\|\ \|f\| +$$
$$+ \|\psi(H)V\|\ \|(1 - \theta(\cdot/(at)))e^{-itH_0} f\| .$$

The properties of V and the choice of θ ensure that the first term in the sum is in L^1. In order to prove that the second term is in L^1 too, we use the following estimation:

$$\left|e^{-itH_0} f(x)\right| \leq C(m)(1 + |x| + |t|)^{-m} \left\|(1 + |x|^m)f\right\|, \quad \frac{x}{t} \notin \mathcal{O},$$

for every $m \in \mathbb{N}$; here $\mathcal{O}$ is a neighbourhood of the set

$$\left\{P'(\xi);\ \xi \in \operatorname{supp} \hat{f}\right\}$$

(Lemma 1, pag. 128, [13]).

Hence if we take a sufficiently small,

$$\left\|(1 - \theta(\cdot/(at))e^{-itH_0} f\right\| \in L^1$$

and consequently the second term of the sum is integrable.

q.e.d.

We pass now to the proof of the other statements of the theorem. We distinguish two cases: the case when $\xi \in \Omega_1$ and the case when $\xi \in \Omega_2$.

We consider first the case when $\xi \in \Omega_1$. Let us denote

$$C_b^\infty = \left\{g: \mathbb{R} \to \mathbb{R};\ g \text{ is } C^\infty \text{ in } \mathbb{R} \text{ and } g^{(j)} \in L^\infty,\ (\forall) j\right\}$$

and

$$S^m_{\rho,0} = \Big\{a: \mathbb{R}^{3n} \to \mathbb{C};\ a \text{ is } C^\infty \text{ in } \mathbb{R}^{3n} \text{ and } (\forall)\ \alpha, \beta, \gamma \in \mathbb{N}^n,\ (\exists) C = C_{\alpha,\beta,\gamma}$$
$$\text{such that } \left|\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma a(x, y, \xi)\right| \leq C(1 + |\xi|)^{-\rho|\alpha|},\ (\forall) x, y, \xi \in \mathbb{R}^n\Big\}.$$

<u>Lemma 1</u>. If $P: \Omega_1 \to \mathbb{R}$ satisfies (i), then

(a) $(\exists)\ \varepsilon > 0$, $\rho > 0$ such that for every $g \in C_b^\infty(\mathbb{R})$, $g(\lambda) = 0$ for λ in a neighbourhood of 0 and for every function $\chi \in C^\infty(\mathbb{R}^n)$ which is positive homogeneous of order zero (for $|\xi| \geq 1$) and whose support is included in Ω_1 (for $|\xi| \geq 1$) we have that $(g \circ |P'|^\varepsilon)\chi \in S^0_{\rho,0}$;

(b) for every $\psi \in C^\infty(\mathbb{R})$ we havethat

$$\left|\partial_\xi^\alpha \psi\left(\frac{x \cdot P'(\xi)}{|x|\,|P'(\xi)|}\right)\right| \leq C\,|\xi|^{-\delta|\alpha|}, \quad \xi \in \Omega_1,\ |P'(\xi)| \geq c > 0,\ |x| > 0.$$

<u>Proof</u>. (a) Since $\chi \in S^0_{1,0}$ and $\operatorname{supp} \chi \subset \Omega_1$ (for $|\xi| \geq 1$), it is sufficient to estimate the derivatives $\partial_\xi^\alpha(g \circ |P'|^\varepsilon)$ on Ω_1 for $|\xi| \geq R_0$. Every expresion of this kind is a finite sum of terms as follows:

$$C_{\beta_1,\dots,\beta_k}\, \partial_\xi^{\beta_1}|P'(\xi)|^\varepsilon \dots \partial_\xi^{\beta_k}|P'(\xi)|^\varepsilon\, g^{(k)}(|P'(\xi)|^\varepsilon),$$

where $\beta_1 + \dots + \beta_k = \alpha$, $1 \leq k \leq |\alpha|$. Also it can be shown by induction that

$\partial_\xi^\beta |P'(\xi)|^\varepsilon$ is a finite sum of terms like

$$c_{\gamma_1,\dots,\gamma_l} P^{(\gamma_1)}(\xi)\dots P^{(\gamma_l)}(\xi)|P'(\xi)|^{\varepsilon-l},$$

where l is an even positive number, $2 \le l \le 2|\beta|$, $\sum_{i=1}^{l} |\gamma_i| = |\beta| + l$ and $\gamma_i \ne 0$, $(\forall)i$. Our assumptions imply that

$$\left|P^{(\gamma_1)}(\xi)\dots P^{(\gamma_l)}(\xi)\right| |P'(\xi)|^{\varepsilon-l} \le C|P'(\xi)|^{l} |\xi|^{-\delta\sum_{i=1}^{l}(|\gamma_i|-1)} |P'(\xi)|^{\delta-l} =$$
$$= C|\xi|^{-\delta|\beta|} |P'(\xi)|^\varepsilon$$

for $|\xi| \ge R_0$, $|\beta| \ge 1$.

Now since $P'(\xi)$ has polynomial growth there exist $\varepsilon > 0$ and $\rho > 0$ such that

$$|\xi|^{-\delta|\beta|} |P'(\xi)|^\varepsilon \le \text{const.} |\xi|^{-\rho|\beta|}, \quad |\xi| \ge 1, \quad \beta \ne 0.$$

Thus

$$\left|\partial_\xi^\alpha g(|P'(\xi)|^\varepsilon)\right| \le C|\xi|^{-\rho(|\beta_1|+\dots+|\beta_k|)} = C|\xi|^{-\rho|\alpha|}, \quad |\xi| \ge R_0.$$

This accomplishes the proof of the first statement of the lemma.

The proof of the second statement proceeds analogously.

q.e.d.

In the sequel we must split every state of the (physical) system in its outgoing and incoming parts. This partition is performed by means of two pseudodifferential operators. In order to define these operators, first we shall choose some appropriate functions:

1) $g \in C^\infty([0,\infty))$, $0 \le g \le 1$, $g(\lambda) = 1$ for $\lambda \ge c$, $g(\lambda) = 0$ for $\lambda \le c/2$, where $c > 0$ is a fixed constant;

2) $\theta \in C^\infty(\mathbb{R}^n)$, $0 \le \theta \le 1$, $\theta(x) = 1$ for $|x| \ge 2$, $\theta(x) = 0$ for $|x| \le 1$;

3) $\chi \in C^\infty(\mathbb{R}^n)$, positive homogeneous of order zero for $|\xi| \ge 1$, $\operatorname{supp} \chi \subset \Omega_1$;

4) $\psi_\pm \in C^\infty(\mathbb{R})$, $0 \le \psi_\pm \le 1$, $\psi_+(\lambda) + \psi_-(\lambda) = 1$ for $\lambda \in [-1, 1]$, $\psi_+(\lambda) = 1$ for $\lambda \in [\sigma_0, 1]$, $\psi_-(\lambda) = 1$ for $\lambda \in [-1, -\sigma_0]$, where $\sigma_0 \in (0, 1)$ is fixed.

Let ε and ρ be like in Lemma 1, (a). For $r \geqslant 1$ we define

$$a_{r,\pm}(x,\xi) = \chi(\xi) g(|P'(\xi)|^{\varepsilon}) \psi_{\pm}\left(\frac{x \cdot P'(\xi)}{|x|\,|P'(\xi)|} \right) \theta\left(\frac{x}{r}\right);$$

since $\rho < \delta$, we deduce from Lemma 1 that $a_{r,\pm} \in S^{0}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n)$, uniformly with respect to $r \geqslant 1$.

We can now define the pseudodifferential operators mentioned before:

$$P_{r,\pm} f(x) = (2\pi)^{-n} \iint e^{i(x-y,\xi)} a_{r,\pm}(y,\xi) f(y)\, dy\, d\xi \ , \quad f \in \mathcal{S} .$$

The integral is an oscillatory integral and its value coincides with the value of the iterated integral -we integrate first with respect to y. These operators are continuous operators defined in $\mathcal{S}$, the space of rapidly decreasing functions, with values in $\mathcal{S}$; their formal adjoints are

$$P^{*}_{r,\pm} f(x) = (2\pi)^{-n} \iint e^{-i(x-y,\xi)} a_{r,\pm}(x,\xi) f(y)\, dy\, d\xi =$$

$$= (2\pi)^{-n} \int e^{i(x,\xi)} a_{r,\pm}(x,\xi) \hat{f}(\xi)\, d\xi .$$

It is known that $P_{r,\pm}$ can be extended to continuous operators in $L^2(\mathbb{R}^n)$, and their norms depend on a finite number of derivatives of $a_{r,\pm}$ ([1]).

Let $f \in \mathcal{S}$. The free evolution of its outgoing and incoming parts is

$$f_{\pm}(t) = e^{-itH_0} P_{r,\pm} f ,$$

where e^{-itH_0} is the unique continuous extension to $L^2(\mathbb{R}^n)$ of the operator

$$e^{-itH_0} g(x) = (2\pi)^{-n} \int e^{i(x,\xi) - itP(\xi)} \hat{g}(\xi)\, d\xi \ , \quad g \in \mathcal{S} .$$

Thus, $E_{r,\pm}(t) = e^{-itH_0} P_{r,\pm}$ are continuous operators in L^2 and

$$E_{r,\pm}(t) f(x) = (2\pi)^{-n} \int e^{i(x,\xi) - itP(\xi)} \widehat{P_{r,\pm} f}(\xi)\, d\xi =$$

$$= (2\pi)^{-n} \int e^{i(x,\xi) - itP(\xi)} \left(\int e^{-i(y,\xi)} a_{r,\pm}(y,\xi) f(y)\, dy \right) d\xi =$$

$$= (2\pi)^{-n} \int \left(\int e^{i(x-y,\xi) - itP(\xi)} a_{r,\pm}(y,\xi) f(y)\, dy \right) d\xi \ , \quad f \in \mathcal{S} .$$

<u>Lemma 2</u>. The operators $E_{r,\pm}(t)$, $P_{r,\pm}$ have the following properties:

(i) $\sup_{t \gtrless 0,\, r \geqslant 1} \| E_{r,\pm}(t) \| < \infty$;

(ii) $P_{r,+} + P_{r,-} = \chi(D) g(|P'(D)|^{\varepsilon}) \theta(\cdot/r)$;

(iii) $\| P_{r,\pm} - P^{*}_{r,\pm} \| \leqslant C r^{-1}$, $r \geqslant 1$.

Proof. (i) is a consequence of the considerations which precede the lemma.

(ii) This formula is obtained by a straightforward calculation.

(iii) For f in $\mathcal{S}$, we have

$$(P_{r,\pm} - P^*_{r,\pm})f(x) = (2\pi)^{-n} \iint e^{i(x-y,\xi)} (a_{r,\pm}(y,\xi) - a_{r,\pm}(x,\xi))f(y)\, dy\, d\xi =$$

$$= (2\pi)^{-n} \iint e^{i(x-y,\xi)} \left(\int_0^1 \frac{d}{ds} a_{r,\pm}(sy + (1-s)x, \xi)\, ds\right) f(y)\, dy\, d\xi =$$

$$= (2\pi)^{-n} \iint e^{i(x-y,\xi)} (y - x)\cdot\left(\int_0^1 (\partial_x a_{r,\pm})(sy + (1-s)x, \xi)\, ds\right) f(y)\, dy\, d\xi =$$

$$= -i(2\pi)^{-n} \iint e^{i(x-y,\xi)} \left(\int_0^1 ((\partial_x \cdot \partial_\xi) a_{r,\pm})(sy + (1-s)x, \xi)\, ds\right) f(y)\, dy\, d\xi .$$

At the last step we have integrated by parts in the oscillatory integral. We must estimate the derivatives of the new amplitude

$$\int_0^1 \partial_x\, \partial_\xi\, a_{r,\pm}(sy + (1-s)x, \xi)\, ds .$$

We have

$$\partial_{x_i} a_{r,\pm}(x,\xi) = r^{-1} (\partial_{x_i} \theta)\left(\frac{x}{r}\right) \psi_\pm\left(\frac{x\cdot P'(\xi)}{|x|\,|P'(\xi)|}\right) g(|P'(\xi)|^\varepsilon)\chi(\xi) +$$

$$+ \frac{1}{|x|} \theta\left(\frac{x}{r}\right) \psi'_\pm\left(\frac{x\cdot P'(\xi)}{|x|\,|P'(\xi)|}\right)\left(\frac{P^{(j)}(\xi)}{|P'(\xi)|} - \frac{x_i}{|x|^2}\, \frac{x\cdot P'(\xi)}{|P'(\xi)|}\right) g(|P'(\xi)|^\varepsilon)\chi(\xi).$$

On supp θ , $|x| \geq r$. Thus

$$\left|\partial_x a_{r,\pm}(x,\xi)\right| \leq Cr^{-1}.$$

The other derivatives of $a_{r,\pm}$ satisfy the same estimate (the constant C may be different). The Calderon-Vaillancourt theorem accomplishes the proof of the lemma.

q.e.d.

Lemma 3. Let φ be in $C_0^\infty(\mathbb{R}^n)$, $0 \leq \varphi \leq 1$, $\varphi(x) = 1$ for $|x| \leq 1/2$, $\varphi(x) = 0$, $|x| \geq 1$. Then there exists a constant b (which depends on c from the definition of g) such that for every $k > 0$, there exists $C = C(k) > 0$ with

$$\left\|\varphi\left(\frac{x}{b(r + |t|)}\right) E_{r,\pm}(t)\right\| \leq C(1 + r + |t|)^{-k},\quad t \gtrless 0.$$

(the sign + corresponds to $t > 0$.)

Proof. We shall apply the following lemma (belonging to Schur):

Let K be a continuous function defined in $\mathbb{R}^n \times \mathbb{R}^n$ and such that

$\sup_y \int |K(x,y)|\, dx \leq C$, $\sup_x \int |K(x,y)|\, dy \leq C$. Then the integral operator with kernel K is a continuous operator in $L^2(\mathbb{R}^n)$ with norm $\leq C$. We want to estimate the norm of the integral operator with kernel

$$K_b(x, y; r, t) = (2\pi)^{-n} \int e^{i(x-y,\xi) - itP(\xi)} \varphi\left(\frac{x}{b(r+|t|)}\right) a_{r,\pm}(y,\xi)\, d\xi$$

(the fact that we multiply with $\varphi\left(\frac{x}{b(r+|t|)}\right)$ permits us to consider this integral as an oscillatory integral, if b is small enough).

We shall consider only the case of the sign + , $t > 0$. On the support of the integrand we have

$$|y| \geqslant r,\ |P'(\xi)| > (c/2)^{1/\varepsilon} =: c_1 > 0,\ y \cdot P'(\xi) \geqslant -\sigma_0 |y|\,|P'(\xi)|.$$

Thus

$$|y + tP'(\xi)|^2 \geqslant (1-\sigma_0)(r^2 + t^2 c_1^2);$$

hence, if b is small enough,

$$|y + tP'(\xi)| \geqslant \sqrt{\frac{1-\sigma_0}{2}}\,(r + c_1 t) \geqslant 2b(1 + r + t),\ r \geqslant 1,\ t > 0.$$

Since $|x| \leqslant b(r+t)$ on the support of the integrand, we have that

$$|x - y - tP'(\xi)| \geqslant b(1 + r + t)$$

on this support.

Let L be the following differential operator:

$$L = -i\,\frac{x - y - tP'(\xi)}{|x - y - tP'(\xi)|^2} \cdot \nabla_\xi .$$

Then

$$L e^{i(x-y,\xi) - itP(\xi)} = e^{i(x-y,\xi) - itP(\xi)} .$$

So we can define K_b as an oscillatory integral:

$$K_b(x, y; r, t) = (2\pi)^{-n} \int e^{i(x-y,\xi) - itP(\xi)} \varphi\left(\frac{x}{b(r+|t|)}\right) ({}^tL)^N a_{r,+}(y,\xi)\, d\xi .$$

In order to apply the lemma of Schur, we remark that the following inequalities are also satisfied on the support of the integrand:

$$|y + tP'(\xi)| \geqslant (2b/c_1)(1 + |y| + t|P'(\xi)|),$$

$$|x - y - tP'(\xi)| \geqslant (b/c_1)(1 + |y| + t|P'(\xi)|) \geqslant$$

$$\geqslant (b/(2c_1))(1 + |x| + |y| + t|P'(\xi)|) \geqslant$$

$$\geqslant b_1(1 + |x| + |y| + t|\xi|^{\delta}) ,$$

where b_1 is a positive constant. Taking into account these inequalities, the condition imposed on P and the fact that $a_{r,+}(y, \xi) \in S^0_{\rho,0}(R^n \times R^n)$ we deduce that for every k there exists N such that

$$\left|({}^tL(x, y, \xi, \partial_\xi))^N a_{r,+}(y, \xi)\right| \leqslant C(1 + |x|)^{-n-1} (1 + |y|)^{-n-1} (1 + |\xi|)^{-n-1} (1 + r + t)^{-k} .$$

Consequently

$$\left|K_b(x, y; r, t)\right| \leqslant C(1 + |x|)^{-n-1} (1 + |y|)^{-n-1} (1 + r + t)^{-k} .$$

Now in order to accomplish the proof it is sufficient to apply the lemma of Schur.

q.e.d.

Lemma 4. Let g, θ and χ be as before. Then,

$$\lim_{r\to\infty} \sup_{t \gtrless 0} \left\|(1 - W_\pm W_\pm^*) \psi(H) \chi(D) g(|P'(D)|^{\varepsilon}) \theta(\tfrac{x}{r}) e^{-itH} f\right\| = 0$$

for every $\psi \in C_0^\infty(R \setminus \overline{C}_v)$ and for every f in L^2.

Proof. Accordingly to Lemma 2, we have that

$$\chi(D) g(|P'(D)|^{\varepsilon}) \theta(\tfrac{x}{r}) e^{-itH} f = P_{r,+} e^{-itH} f + P_{r,-} e^{-itH} f .$$

So it is sufficient to prove that

$$\lim_{r\to\infty} \sup_{t \gtrless 0} \left\|(1 - W_\pm W_\pm^*) \psi(H) P_{r,+} e^{-itH} f\right\| = 0 \tag{1}$$

and that

$$\lim_{r\to\infty} \sup_{t \gtrless 0} \left\|(1 - W_\pm W_\pm^*) \psi(H) P_{r,-} e^{-itH} f\right\| = 0 . \tag{2}$$

(Here $t > 0$ corresponds to W_+ and $t < 0$ to W_-.)

We shall consider only the case $t > 0$. The relation (1) will hold if

$$\lim_{r\to\infty} \left\|(1 - W_+ W_+^*) \psi(H) P_{r,+}\right\| = 0. \tag{3}$$

First let us prove that

$$\lim_{r\to\infty} \left\|\psi(H)(1 - W_+) P_{r,+}\right\| = 0. \tag{4}$$

Because $\mathrm{dom}(H) = \mathrm{dom}(H_o) = \mathrm{dom}(V)$ and V is a short range potential we have

$$\psi(H)(1 - W_+) = -\int_0^\infty \psi(H) \frac{d}{dt}(e^{itH} e^{-itH_o})\, dt = -i\int_0^\infty e^{itH}\, \psi(H)Ve^{-itH_o}\, dt$$

(the integral is strongly convergent). Therefore

$$\|\psi(H)(1 - W_+)P_{r,+}\| \leq \int_0^\infty \|\psi(H)Ve^{-itH_o}\, P_{r,+}\|\, dt.$$

But $e^{-itH_o}\, P_{r,+} = E_{r,+}(t)$. Hence it is sufficient to prove that

$$\lim_{r\to\infty} \int_0^\infty \|\psi(H)VE_{r,+}(t)\|\, dt = 0. \tag{5}$$

If φ and b are like in Lemma 3, then

$$\|\psi(H)VE_{r,+}(t)\| \leq \|\psi(H)(H_o + i)\|\, \|(H_o + i)^{-1}\, V(1 - \varphi(\frac{x}{b(r+t)}))\|\, \|E_{r,+}(t)\| + $$
$$+ \|\psi(H)V\|\, \|\varphi(\frac{x}{b(r+t)})E_{r,+}(t)\|.$$

The operators $E_{r,+}(t)$ are uniformly bounded with respect to t and

$$\int_0^\infty \|(H_o + i)^{-1}\, V(1 - \varphi(\frac{x}{b(r+t)}))\|\, dt \leq \int_{br}^\infty \|(H_o + i)^{-1}\, V(1 - \varphi(\frac{x}{t}))\| \cdot$$
$$\cdot b^{-1}\, dt \to 0$$

when $r \to \infty$, accordingly to the assumption (iv). The estimation of the second term follows from Lemma 3. So we obtain the relation (5) and consequently the relation (4).

In order to prove (3) we remark that since $\psi(H)W_+ = W_+\psi(H_o)$ the following equalities hold:

$$0 = \lim_{r\to\infty} \|\psi(H)P_{r,+} - W_+\psi(H_o)P_{r,+}\| =$$
$$= \lim_{r\to\infty} \|W_+W_+^*\psi(H)P_{r,+} - W_+W_+^*W_+\psi(H_o)P_{r,+}\| =$$
$$= \lim_{r\to\infty} \|W_+W_+^*\psi(H)P_{r,+} - W_+\psi(H_o)P_{r,+}\| \geq$$
$$\geq \lim_{r\to\infty} \|(1 - W_+W_+^*)\psi(H)P_{r,+}\| - \lim_{r\to\infty} \|\psi(H)P_{r,+} - W_+\psi(H_o)P_{r,+}\| =$$
$$= \lim_{r\to\infty} \|(1 - W_+W_+^*)\psi(H)P_{r,+}\|.$$

At the third step we have used the fact that $\mathrm{Ran}\ \psi(H_o) \subset \mathcal{H}_{ac}(H_o) = \mathrm{Ran}\ W_+^*W_+$.

This ends the proof of the relation (3).

We turn now to the proof of the relation (2). It is sufficient to show that

$$\lim_{r\to\infty} \sup_{t>0} \| P_{r,-} e^{-itH} f \| = 0 .$$

Let us remark first that due to the Lemma 3

$$\lim_{r\to\infty} \sup_{t<0} \| E_{r,-}(t)^{*} \langle x \rangle^{-1-\varepsilon} \| = 0$$

and therefore

$$\lim_{r\to\infty} \sup_{t<0} \| P^{*}_{r,-} e^{-itH_0} f \| = 0 \tag{6}$$

for every $f \in \mathrm{dom}(\langle x \rangle^{1+\varepsilon})$, hence for f in a dense subset of $L^2(\mathbb{R}^n)$. But $P^{*}_{r,-} e^{-itH_0}$ is a family of uniform continuous operators. Thus the relation (6) will hold for every f in $L^2(\mathbb{R}^n)$.

Now let φ be in $C_0^\infty(\mathbb{R})$. In the proof of relation (4) we have used only that $\psi \in C_0^\infty(\mathbb{R})$. So it can be shown in a similar way that

$$\lim_{r\to\infty} \sup_{t<0} \| \varphi(H)(1 - e^{itH} e^{-itH_0}) P_{r,-} \| = 0 .$$

Performing the substitution $t \to -t$ and passing to the conjugate operators we obtain that

$$0 = \lim_{r\to\infty} \sup_{t>0} \| P^{*}_{r,-} \varphi(H) - P^{*}_{r,-} e^{-itH_0} \varphi(H) e^{itH} \| =$$

$$= \lim_{r\to\infty} \sup_{t>0} \| P^{*}_{r,-} e^{-itH} \varphi(H) - P^{*}_{r,-} e^{-itH_0} \varphi(H) \| .$$

Using the relation (6) we have that

$$\lim_{r\to\infty} \sup_{t>0} \| P^{*}_{r,-} e^{-itH} \varphi(H) f \| = 0 , \ (\forall) f \in L^2(\mathbb{R}^n).$$

Since $\varphi \in C_0^\infty(\mathbb{R})$ is arbitrary and $\| P_{r,-} - P^{*}_{r,-} \| \leq C r^{-1}$ (Lemma 2) the proof is complete.

<u>Lemma 5</u>. Let ψ_1, ψ_2 be in $C_0^\infty(\mathbb{R} \setminus \overline{C_V})$, f in $\mathcal{H}_c(H)$. Then

$$\lim_{T\to\infty} \frac{1}{T} \int_0^{\pm T} \| (1 - W_\pm W_\pm^{*}) \psi_1(H) \chi(D) \psi_2(H) e^{-itH} f \| \, dt = 0 .$$

Before starting the proof we state two results which will be used in this proof.

<u>Lemma 6</u> (RAGE theorem). Let A be a self-adjoint operator, f in $\mathcal{H}_c(A)$ and C a compact operator. Then

$$\lim_{T\to\infty} \frac{1}{T}\int_{-T}^{T} \|Ce^{-itA} f\| \, dt = 0.$$

(A proof of this result can be found in [12] .)

Lemma 7. If φ is in $C_0^\infty(\mathbb{R}^n)$ and H_0 is the self-adjoint operator corresponding to the convolution operator $P(D)$, where $P: \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfies the assumptions (i) and (ii), then $\varphi(\cdot)(H_0 + i)^{-1}$ is a compact operator.

Proof. The proof of this lemma can be derived from [2] . In that paper it was proved that if $h: R^n \longrightarrow R$ is a cvasidivergent function, then $\varphi(\cdot)(h(D) + i)^{-1}$ is a compact operator. A function h is said to be cvasidivergent if for every m, $\lim_{\nu\to\infty} |C_\nu \cap S_m| = 0$, where $S_m = \{x;\ |h(x)| \leq m\}$ and C_ν is the unit cube centred in $\nu \in Z^n$. We have denoted here with $|M|$ the Lebesgue measure of the measurable set M. In [2] it was also proved that if $h \in C^2(R^n; \mathbb{R})$ is such that

$$1 + |h(\xi)| + |h'(\xi)| \to \infty \quad \text{when} \quad |\xi| \to \infty$$

and

$$|\partial^\alpha h(\xi)| \leq C(1 + |h(\xi)|^2 + |h'(\xi)|^2)^{1/2}$$

for every ξ and for every α with $|\alpha| = 2$, then h is cvasidivergent. That proof is fit for this case too if we cover $\mathbb{R}^n$ with eventually smaller cubes $\tilde{C}_\nu$ with the property that $\tilde{C}_\nu \subset \Omega_1$ or $\tilde{C}_\nu \subset \Omega_2$. Hence P is a cvasidivergent function and consequently $\varphi(\cdot)(H_0 + i)^{-1}$ is a compact operator.

q.e.d.

Proof of Lemma 5. Let c be equal to $\inf\{|P'(\xi)|^\varepsilon\ ;\ P(\xi) \in \operatorname{supp} \psi_2\}$ and let g be in $C^\infty([0, \infty))$, $g(\lambda) = 1$ for $\lambda \geq c$, $g(\lambda) = 0$ for $\lambda \leq c/2$. Since $g(|P'(D)|^\varepsilon)\psi_2(H_0) = \psi_2(H_0)$ we have that

$$\psi_2(H) = (1 - g(|P'(D)|^\varepsilon))(\psi_2(H) - \psi_2(H_0)) + g(|P'(D)|^\varepsilon)\psi_2(H).$$

Now, $\psi_2(H) - \psi_2(H_0)$ is a compact operator since $(H + i)^{-1} - (H_0 + i)^{-1}$ is a compact one. Accordingly to RAGE theorem,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \|(\psi_2(H) - \psi_2(H_0))e^{-itH} f\| \, dt = 0.$$

Thus

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \|(1 - W_+W_+^*)\, \psi_1(H)\, \chi(D)(1 - g(|P'(D)|^{\varepsilon}))(\psi_2(H) - \psi_2(H_o))e^{-itH} f\| \, dt = 0 .$$

On the other hand, we can write

$$(1 - W_+W_+^*)\, \psi_1(H)\, \chi(D) g(|P'(D)|^{\varepsilon})\, \psi_2(H) e^{-itH} f =$$

$$= (1 - W_+W_+^*)\, \psi_1(H)\, \chi(D) g(|P'(D)|^{\varepsilon})\, \theta(\tfrac{\cdot}{r}) e^{-itH}\, \psi_2(H) f +$$

$$+ (1 - W_+W_+^*)\, \psi_1(H)\, \chi(D) g(|P'(D)|^{\varepsilon})(1 - \theta(\tfrac{\cdot}{r}))\, \psi_2(H) e^{-itH} f =$$

$$= f_{1,\, t,\, r} + f_{2,\, t,\, r}$$

By virtue of Lemma 4, $\|f_{1,\, t,\, r}\| \to 0$ when $r \to \infty$. The second term can be suitably estimated if we remark that

$$(1 - \theta(\tfrac{\cdot}{r}))\, \psi_2(H) = (1 - \theta(\tfrac{\cdot}{r}))(\psi_2(H) - \psi_2(H_o)) +$$

$$+ (1 - \theta(\tfrac{\cdot}{r}))(H_o + i)^{-1}\, (H_o + i)\, \psi_2(H_o)$$

is a compact operator (since $\psi_2(H) - \psi_2(H_o)$) and $(1 - \theta(\tfrac{\cdot}{r}))(H_o + i)^{-1}$ are compact ones) and if we apply again the RAGE theorem. Adding together all these facts we obtain the conclusion of the lemma.

We consider now the case $\xi \in \Omega_2$ and we shall prove that the conclusion of Lemma 5 remains true in this case too. First we introduce some notations. Let θ and $\psi_{\pm}$ be like in the case $\xi \in \Omega_1$, $S = P^{-1}(C_v) = \{\xi ;\ P'(\xi) = 0\}$, $\gamma \in C_o^{\infty}(R^n \setminus S)$. Then we put

$$b_{r,\pm}(x, \xi) = \gamma(\xi)\, \psi_{\pm}\Big(\frac{x \cdot P'(\xi)}{|x|\, |P'(\xi)|}\Big)\, \theta\big(\tfrac{x}{r}\big) ,$$

$$Q_{r,\pm} f(x) = (2\pi)^{-n} \iint e^{i(x-y,\, \xi)}\, b_{r,\pm}(y, \xi) f(y)\, dy\, d\xi \ , \quad f \in \mathcal{S} ,$$

$$F_{r,\pm}(t) f(x) = (2\pi)^{-n} \iint e^{i(x-y,\, \xi) - itP(\xi)}\, b_{r,\pm}(y, \xi) f(y)\, dy\, d\xi \ , \quad f \in \mathcal{S} .$$

Taking into account the fact that γ has compact support we can see that the conclusions of lemmas 2 and 3 remain true if $E_{r,\pm}(t)$ is substituted by $F_{r,\pm}(t)$. Hence the next result is true (analogous to Lemma 4).

<u>Lemma 4'</u>. Let ψ be in $C_o^{\infty}(R \setminus \overline{C}_v)$, θ and γ like before. Then

$$\lim_{r\to\infty} \sup_{t \gtrless 0} \|(1 - W_{\pm}W_{\pm}^*)\, \psi(H)\, \gamma(D)\, \theta(\tfrac{x}{r}) e^{-itH} f\| = 0 , \ (\forall) f \in L^2(R^n) .$$

In order to make sure that Lemma 5 remains valid in the case of a function

χ with the support included in Ω_2 , it is sufficient to replace the operator $\chi(D)g(|P'(D)|^{\varepsilon})$ in the proof of this lemma with the operator $\gamma(D) = \chi(D)\,\varphi(H_0)$, where $\varphi \in C_0^{\infty}(R \setminus \overline{C}_V)$, $\varphi(\lambda) = 1$ for λ in a neighbourhood of supp ψ_2. The assumption (ii) assures that γ is in $C_0^{\infty}(\mathbb{R}^n \setminus CS)$. Then the proof of Lemma 5 is repeated word for word.

Now let χ_1 and χ_2 be two smooth functions which are positive homogeneous of order zero for $|\xi| \geqslant 1$ and such that supp $\chi_i \subset \Omega_i$ for $|\xi| \geqslant 1$; also $\chi_1(\xi) + \chi_2(\xi) = 1$, $\xi \in \mathbb{R}^n$. Applying Lemma 5 to χ_1 and χ_2 and remarking that $\psi_1, \psi_2 \in C_0^{\infty}(R \setminus \overline{C}_V)$ are arbitrary we obtain that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^{\pm T} \left\| (1 - W_{\pm} W_{\pm}^{*}) e^{-itH} \psi(H) f \right\| dt = 0 \qquad (7)$$

for every f in $\mathcal{H}_c(H)$ and every ψ in $C_0^{\infty}(R \setminus \overline{C}_V)$.

<u>End of the proof of the theorem.</u> We shall consider again only the case of the sign + . It is known that Ran $W_+ \subset \mathcal{H}_{ac}(H)$. So if we prove that $\mathcal{H}_c(H) \subset$ Ran W_+, then Ran $W_+ = \mathcal{H}_{ac}(H)$ and $\mathcal{H}_{sc}(H) = \{0\}$. The set of those functions f in $\mathcal{H}_c(H)$ which satisfy $\psi(H)f = f$ for a function ψ in $C_0^{\infty}(\mathbb{R} \setminus \overline{C}_V)$ is a dense set in $\mathcal{H}_c(H)$. It is sufficient to prove that these functions are in Ran W_+. From the relation (7) results that there exists a sequence $t_k \to \infty$ such that

$$\left\| (1 - W_+ W_+^{*}) e^{-it_k H} f \right\| < \frac{1}{k} .$$

Since $W_+ W_+^{*} e^{-itH} = e^{-itH} W_+ W_+^{*}$ we obtain that

$$\left\| (1 - W_+ W_+^{*}) f \right\| < \frac{1}{k}$$

for every k. Thus $f = W_+ W_+^{*} f$ and consequently $f \in$ Ran W_+.

The last assertion of the theorem results from the fact that $(1 - W_+ W_+^{*})\,\psi(H)$ is a compact operator for every $\psi \in C_0^{\infty}(R \setminus \overline{C}_V)$. Remark that we have proved that $1 - W_+ W_+^{*} = 1 - W_- W_-^{*}$ = the orthogonal projection on the closed linear hull of the eigenvectors of H.

Let χ_1 and χ_2 be two homogeneous functions which have the properties listed above and let ψ_1 be a function in $C_0^{\infty}(R \setminus \overline{C}_V)$ such that $\psi_1(\lambda)\psi(\lambda) = \psi(\lambda)$ for every λ in R. Then

$$(1 - W_+W_+^*)\,\psi(H) = (1 - W_+W_+^*)\,\psi_1(H)\,\psi(H) =$$

$$= (1 - W_+W_+^*)\,\psi_1(H)(\chi_1(D) + \chi_2(D))\,\psi(H_o) + (1 - W_+W_+^*)\,\psi_1(H)(\psi(H) - \psi(H_o)).$$

The last term of the sum is a compact operator because $\psi(H) - \psi(H_o)$ is so.

Further we have

$$(1 - W_+W_+^*)\,\psi_1(H)\,\chi_1(D)\,\psi(H_o) = (1 - W_+W_+^*)\,\psi_1(H)\,\chi_1(D)g(|P'(D)|^{\varepsilon})\,\psi(H_o) =$$

$$= (1 - W_+W_+^*)\,\psi_1(H)\,\chi_1(D)g(|P'(D)|^{\varepsilon})(\theta(\tfrac{\cdot}{r}) + (1 - \theta(\tfrac{\cdot}{r})))\,\psi(H_o) =$$

$$= ((1 - W_+W_+^*)\,\psi_1(H)P_{r,+} + (1 - W_-W_-^*)\,\psi_1(H)P_{r,-} +$$

$$+ (1 - W_+W_+^*)\,\psi_1(H)g(|P'(D)|^{\varepsilon})(1 - \theta(\tfrac{\cdot}{r})))\,\psi(H_o)\,,$$

for a suitably choosen function g. But from the proof of Lemma 4 we have that

$$\lim_{r\to\infty} \left\| (1 - W_{\pm}W_{\pm}^*)\,\psi_1(H)P_{r,\pm} \right\| = 0.$$

On the other hand we know that $(1 - \theta(\frac{\cdot}{r})\,\psi(H_o)$ is a compact operator. Thus $(1 - W_+W_+^*)\,\psi_1(H)\,\chi_1(D)\,\psi(H_o)$ is a compact operator. Analogously it can be shown that $(1 - W_+W_+^*)\,\psi_1(H)\,\chi_2(D)\,\psi(H_o)$ is a compact operator.

The proof of the theorem is now complete.

Remark. If P is a polynomial function, then using the Seidenberg-Tarski theorem we can see that the assumption (i) is equivalent with the following conditions (we suppose that $\Omega_1 = R^n \setminus \{0\}$):

$$\lim |P'(\xi)| = \infty\,, \text{ when } |\xi| \to \infty\,,$$

$$\lim \frac{P^{(\alpha)}(\xi)}{|P'(\xi)|} = 0, \text{ when } |\xi| \to \infty\,, (\forall)\,\alpha \text{ with } |\alpha| \geq 2.$$

A polynomial which is of principal type satisfies the assumption (i) in R^n.

Appendix

In this appendix we shall briefly describe the spectrum of H_0. Since H_0 is unitary equivalent to P (the operator of multiplication with $P(\xi)$) it is sufficient to consider the spectrum of this second operator. It is known that $\sigma(P) = \overline{\{P(\xi);\ \xi \in \mathbb{R}^n\}}$. Further we shall also use the following notations:

- $\sigma_{pp}(P)$ - the set of eigenvalues of P;
- $P_M(P)$ - the spectral projection associated to a measurable set $M \subset \mathbb{R}$;
- $A = \sigma(P) \setminus \overline{C}_v$.

A real number λ is in $\sigma_{pp}(P)$ iff $\{\xi;\ P(\xi) = \lambda\}$ has nonzero Lebesgue measure. This is natural because $P_{\{\lambda\}}$ is equal to the operator of multiplication with the characteristic function of the set $P^{-1}(\{\lambda\})$. We prove now that if $\overline{C}_v$ is at most countable then $P_A(P) = P_{ac}(P)$ and $\sigma_{sc}(P) = \emptyset$. Let us first show that $P_A(P) \subset P_{ac}(P)$. This statement is a consequence of the fact that if M is a measurable set contained in A and has Lebesgue measure zero, then $P^{-1}(M)$ has Lebesgue measure zero. But $P^{-1}(M)$ has Lebesgue measure zero iff every point $\xi_0 \in P^{-1}(M)$ has a neighbourhood U such that $U \cap P^{-1}(M)$ has Lebesgue measure zero. Supposing that $(\partial P/\partial \xi_1)(\xi_0) \neq 0$ it is sufficient to choose U such that the map

$$U \ni \xi \longrightarrow (P(\xi), \xi_2, \dots, \xi_n)$$

is a diffeomorfism on its range. Therefore $P_A(P) \subset P_{ac}(P)$.

Now if $f \in \mathcal{H}_c(P)$ then $(P_{\overline{C}_v}(P)f, f) = 0$, because $\overline{C}_v$ is at most countable. Thus $P_{ac}(P)P_{\overline{C}_v}(P) = 0$ and $P_{ac}(P) \subset P_A(P)$. We have also that $P_{sc}(P) \subset P_{\overline{C}_v}(P)$ and $P_{sc}(P)P_{\overline{C}_v}(P) = 0$. This implies that $P_{sc}(P) = 0$ and consequently $\sigma_{sc}(P) = \emptyset$.

A consequence of these remarks is the fact that

$$\{f \in \mathcal{S}\ ;\ \hat{f} \in C_0^\infty(\{\xi;\ P'(\xi) \neq 0\})\}$$

is a dense set in $\mathcal{H}_{ac}(H_0)$.

References

1. R. Coifman, Y. Meyer, Au delà des opérateurs pseudo-différentiels, Astérisque, 57 (1978).

2. E. B. Davies, Pl. Muthuramalingam, Trace properties of some highly anisotropic operators, J. London Math. Soc. (sec. series), 31, 1 (1985), 137-149.

3. V. Enss, Asymptotic completeness for quantum mechanical potential scattering, Comm. Math. Phys., 61 (1978), 285-291.

4. J. Ginibre, La méthode "dépendant du temps" dans le problème de la complétude asymptotique, preprint, Univ. Paris-Sud, 1980.

5. L. Hörmander, The Analysis of Linear Partial Differential Operators, vol. 2, Springer Verlag, Berlin-Heidelberg, 1983.

6. V. Iftimie, Sur la complétude asymptotique dans le cas "longue portée" et "dépendant du temps", Rev. Roum. Math. Pures Appl., 32, 5 (1987), 407-433.

7. H. Kitada, K. Yajima, A scattering theory for time dependent long-range potentials, Duke Math. J., 49 (1982), 341-375.

8. Pl. Muthuramalingam, A note on time-dependent scattering theory for $P_1^2 - P_2^2 + (1 + |Q|)^{-1-\varepsilon}$ and $P_1P_2 + (1 + |Q|)^{-1-\varepsilon}$ on $L^2(\mathbb{R}^n)$, Math. Z., 188, (1985), 339-348.

9. Pl. Muthuramalingam, A time dependent scattering theory for a class of simply characteristic operators with short range local potentials, J. London Math. Soc. 32, 2 (1985), 259-264.

10. Pl. Muthuramalingam, Enss' theory in long range scattering: second order hyperbolic and parabolic operators, Math. Nachr., 127 (1986), 145-163.

11. M. Reed, B. Simon, Methods of Modern Mathematical Physics, vol. 2, Academic Press, New York, 1975.

12. M. Reed, B. Simon, Methods of Modern Mathematical Physics, vol. 3, Academic Press, New York, 1979.

13. B. Simon, Phase space analysis of simple scattering systems: extensions of some work of Enss, Duke Math. J., 46, 1 (1979), 119-168.