A SHORT EASY PROOF OF PONTRYAGIN'S MINIMUM PRINCIPLE

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1. INTRODUCTION

The aim of this paper is to give a short easy proof of Pontryagin's Minimum Principle for a fixed-time optimal control problem without end-point constraints using a Dynamic Programming approach.

The proof in this paper is considerably simpler and shorter than the proofs in [1] and [2] replacing the result which states that the value function is a (continuous) viscosity subsolution of the Hamilton-Jacobi-Bellman equation by two other results: the first one is well known and states that the value function is nondecreasing along admissible trajectories ([4],[6]) and the other, an easy corollary of the first one, is one of the differential inequalities in [11] (see also [7]) expressed in terms of the extreme contingential derivatives of the value function.

As shown in [11] and [7], these differential inequalities imply the fact that the value function is a special type of viscosity solution whenever it is continuous and imply also some other differential properties taken recently as generalizations of classical solutions of the Hamilton-Jacobi-Bellman equation. Moreover, the differential inequalities in [11] and [7] may provide sufficient optimality conditions ([10]) extending the so called "verification theorem" ([6]) to problems for which the value function is not differentiable.

For the sake of completeness we prove all the statements we need with the exception of the classical theorem on differentiability of solutions of Carathéodory differential equations with respect to initial data (which may be found in [8]) and the result concerning the monotonicity of the value function along admissible trajectories (which may be found in [4] and[6]).

To simplify the proof we consider only Mayer optimal control problems; for a Bolza problem it is more convenient to write it as a Mayer problem ([3],[4],[6], etc.) and to derive the Minimum Principle from that of the later problem.

Since the results concerning the monotonicity of the value function along admissible trajectories and the differential inequalities verified by the value function remain valid in infinit-dimensional spaces, the proof in this paper may be carried out for any control system on a Banach space for which a theorem on differentiability of solutions of the correponding differential equations with respect to initial data may be established; this is certainly the case for the optimal control problem in [1].

As a byproduct of our proof we obtain an improved variant of the results in [14] and [5] replacing Clarke's generalized gradient by the Fréchet superdifferential of the value function in the case the data of the problem are differentiable.

A very interesting open problem is to obtain a similar proof of Pontryagin's Minimum Principle for optimal control problems with end-point constraints; such a result will cover most of the theory of necessary optimality conditions in control theory.

2. NOTATIONS, DEFINITIONS AND PRELIMINARY RESULTS

We consider the problem of minimizing the functional $C(.)$ defined by:

$$C(u(.))=g(x(T;u(.))) \quad , \quad u(.)\in \mathcal{U}(t_o,x_o) \tag{2.1}$$

over a set $\mathcal{U}(t_o,x_o)$ of admissible controls which is a specified class of measurable mappings $u(.):[t_o,T]\longrightarrow U\subset \mathbb{R}^m$ for which the absolutely continuous solution, $x(.;u(.))$, of the initial value problem:

$$\frac{dx}{dt}=f(t,x,u(t)),\ x(t_o)=x_o \tag{2.2}$$

is defined on the interval $[t_o,T]$.

As usual in the proof of Pontryagin's Minimum Principle ([3],[4],[6], etc.) we assume that the data of the problem satisfy the following:

HYPOTHESIS 2.1. The function $g:\mathbb{R}^n\longrightarrow\mathbb{R}$ is of class C^1, D is an open subset of $\mathbb{R}\times\mathbb{R}^n$, $f:D\times U\longrightarrow\mathbb{R}^n$ is continuous, for any $u\in U$, $t\in pr_1D=\{t\in\mathbb{R};\ (\exists)\ x\in\mathbb{R}^n: (t,x)\in D\}$ the mapping $f(t,.,u)$ is differentiable and $(t,x,u)\longmapsto D_2f(t,x,u)$ is continuous; we assume also that $T\in pr_1D$ and $(t_o,x_o)\in D$ are taken such that $t_o<T$.

The class $\mathcal{U}(t_o,x_o)$ of admissible controls may be either one of the sets $\mathcal{U}_p(t_o,x_o)$, $p\in[1,\infty]$, of measurable mappings $u(.):[t_o,T]\longrightarrow U$ for which the solution $x(.;u(.))$ of (2.2) exists on $[t_o,T]$ and the derivative $t\longmapsto x'(t;u(.))=f(t,x(t;u(.)),u(t))$ belongs to $\mathbb{L}^p([t_o,T];\mathbb{R}^n)$, either the set $\mathcal{U}_r(t_o,x_o)$ of regulated admissible controls (i.e. $u(.)$ has one-sided limits at each point hence a countable number of discontinuities, all of the first kind) or the set $\mathcal{U}_{cp}(t_o,x_o)$ of piecewise continuous admissible controls.

As pointed out in [4], [$\updownarrow$] and elsewhere, the infimum of the functional $C(.)$ in (2.1) may depend essentially on the class of admissible controls but the results to follow remain valid for any such class.

To insure the existence of local Carathéodory solutions of the problems in (2.2) one may require that for any admissible control, $u(.)\in \mathcal{U}(t_o,x_o)$ the vector field $f_{u(.)}(t,x)=f(t,x,u(t))$ is locally integrably bounded but our proof require this condition only along the optimal trajectory.

As it is well known, the Dynamic Programming method consists in using the <u>value function</u> of the optimal control problem, defined by:

$$V(t,x)=\begin{cases} g(x) & \text{if } \ t=T,\ (T,x)\in D \\ \inf\left\{C(u(.));\ u(.)\in \mathcal{U}(t,x)\right\} & \text{if } (t,x)\in D_T^o \end{cases} \tag{2.3}$$

$$D_T^o=\left\{(t,x)\in D;\ t<T\right\},\quad D_T=\left\{(t,x)\in D;\ t\leqslant T\right\}$$

to obtain either necessary or sufficient optimality conditions (for every $(t,x)\in D_T^o$ the set $\mathcal{U}(t,x)$ of admissible controls is defined in the same way as $\mathcal{U}(t_o,x_o)$).

The first preliminary result we need is the following:

PROPOSITION 2.2 ([4], Proposition 4.5.i;[6], Theorem IV.3.1)

<u>If</u> $(s,y)\in D_T^o$ <u>and</u> $u(.)\in \mathcal{U}(s,y)$ <u>then the function</u> $t\longmapsto V(t,x(t;u(.))$ <u>is nondecreasing on the interval</u> $[s,T]$.

The other preliminary result we need is a corollary of the classical theorem on differentiability of solutions of Carathéodory differential equations with respect to initial data.

A mapping $\tilde{f}(.,.):\tilde{D}\longrightarrow \mathbb{R}^n$ defined on the open subset $\tilde{D}\subset \subset\mathbb{R}\times\mathbb{R}^n$ is said to be a Carathéodory-C^1 vector field if it has the following properties:

(i) There exists a null subset $J_o\subset pr_1\tilde{D}$ such that $\tilde{f}(t,.)$ is differentiable and $D_2\tilde{f}(t,.)$ is continuous for any $t \in pr_1\tilde{D} \setminus J_o$;

(ii) the mappings $\tilde{f}(.,x)$, $D_2\tilde{f}(.,x)$ are measurable for any $x \in pr_2\tilde{D}$;

(iii) $\tilde{f}(.,.)$ and $D_2\tilde{f}(.,.)$ are locally integrably bounded in the sense that for any $(s,y) \in \tilde{D}$ there exist $r>0$ and the integrable functions $m(.)$, $M(.)$ such that:

$$|f(t,x)| \leq m(t), \quad \|D_2f(t,x)\| \leq M(t) \quad (\forall)\ (t,x) \in B_r(s,y) \subset \tilde{D} \tag{2.4}$$

where $|.|$ is a norm (usually the Euclidian one) on $\mathbb{R}^n$, $\|.\|$ is the corresponding operatorial norm on the space of linear mappings $L(\mathbb{R}^n,\mathbb{R}^n)$ and $B_r(s,y)$ is the ball of radius r centered at (s,y).

For any $(s,y) \in \tilde{D}$ we denote by $x^*(.;s,y):I(s,y) \longrightarrow \mathbb{R}^n$ the non-continuable Carathéodory (absolutely continuous) solution of the initial value problem:

$$\frac{dx}{dt} = \tilde{f}(t,x),\ x(s)=y \tag{2.5}$$

where $I(s,y) \subset \mathbb{R}$ is the open interval on which $x^*(.;s,y)$ is defined; the mapping $x^*(.;.,.)$ is said to be <u>the maximal flow of the vector field</u> $\tilde{f}(.,.)$.

THEOREM 2.3 ([8], Ch. 13, Remark 18.4.16) <u>If</u> $x^*(.;.,.)$ <u>is the maximal flow of the Carathéodory-C^1 vector field</u> $\tilde{f}(.,.)$ <u>then for any</u> $(s,y) \in \tilde{D}$, $t \in I(s,y)$, <u>the mapping</u> $x^*(t;s,.)$ <u>is differentiable at</u> y, $D_3x^*(.;.,.)$ <u>is continuous and</u> $t \mapsto D_3x^*(t;s,y)$ <u>is the unique absolutely continuous matrix-valued solution of the variational equation:</u>

$$\frac{dZ}{dt} = D_2\tilde{f}(t,x^*(t;s,y))Z \tag{2.6}$$

<u>satisfying the initial condition:</u>

$$D_3x^*(s;s,y)=I_n = \text{the identity matrix} \tag{2.7}$$

We actually need the following corollary of this theorem (see also Lemma 2 in [2]):

COROLLARY 2.4. *If the hypotheses of Theorem 2.3 are satisfied,* $(t_o,x_o)\in\widetilde{D}$, $T\in I(t_o,x_o)$, $t_o<T$ *and* $\widetilde{x}(.)=x^*(.;t_o,x_o)$ *then the mapping* $t\longmapsto D_3x^*(T;t,\widetilde{x}(t))$ *is the unique absolutely continuous matrix-valued solution of the problem:*

$$\frac{dZ}{dt}=-Z.D_2\widetilde{f}(t,\widetilde{x}(t)),\quad Z(T)=I_n\quad (Z(t_o)=D_3x^*(T;t_o,x_o)\ )\qquad (2.8)$$

and if $J_1\subset[t_o,T]$ *is the null subset such that:*

$$\widetilde{x}'(t)=\widetilde{f}(t,\widetilde{x}(t))\quad (\forall)\ t\in[t_o,T]\setminus J_1\qquad (2.9)$$

then $x^*(T;.,\widetilde{x}(t))$ *is differentiable at* $t\in[t_o,T]\setminus J_1$ *and:*

$$D_2x^*(T;t,\widetilde{x}(t))=-D_3x^*(T;t,\widetilde{x}(t)).\widetilde{f}(t,\widetilde{x}(t))\quad (\forall)t\in[t_o,T]\setminus J_1\qquad (2.10)$$

Moreover, for any $t\in[t_o,T]\setminus J_1$, *the mapping* $x_T^*(.,.)=$ $=x^*(T;.,.)$ *is differentiable at* $(t,\widetilde{x}(t))$ (its derivative being given by: $Dx_T^*(t,\widetilde{x}(t))=(D_2x^*(T;t,\widetilde{x}(t)),D_3x^*(T;t,\widetilde{x}(t)))$).

Proof. From the uniqueness of the solutions $x^*(.;s,y)$ we infer that: $x^*(s;t,x^*(t;s,y))=y$ $(\forall)$ $(s,y)\in\widetilde{D}$, $t\in I(s,y)$ hence taking $(s,y)=(T,\widetilde{x}(T))$ we obtain:

$$x^*(T;t,\widetilde{x}(t))=\widetilde{x}(T),\quad D_3x^*(T;t,\widetilde{x}(t))=(D_3x^*(t;T,\widetilde{x}(T)))^{-1}\qquad (2.11)$$

and therefore, since at any point $t\in[t_o,T]$ at which $D_3x^*(.;T,\widetilde{x}(T))$ is differentiable one has: $\frac{d}{dt}(D_3x^*(t;T,\widetilde{x}(T)))^{-1}=$ $=-(D_3x^*(t;T,\widetilde{x}(T)))^{-1}.\frac{d}{dt}D_3x^*(t;T,\widetilde{x}(T)).(D_3x^*(t;T,\widetilde{x}(T)))^{-1}$, from (2.11) and Theorem 2.3 it follows that $t\longmapsto D_3x^*(T;t,\widetilde{x}(t))$ is the unique absolutely continuous solution of (2.8).

To prove that $x^*(T;.,\widetilde{x}(t))$ is differentiable at $t\in[t_o,T]\setminus J_1$ we note that from (2.11) it follows: $x^*(T;t+r,\widetilde{x}(t+r))=x^*(T;t,\widetilde{x}(t))=$ $=\widetilde{x}(T)$ $(\forall)$ $t,t+r\in I(t_o,x_o)$, from (2.9) it follows that: $(\widetilde{x}(t+r)-\widetilde{x}(t))/r\longrightarrow\widetilde{f}(t,\widetilde{x}(t))$ as $r\longrightarrow 0$ and therefore, since $D_3x^*(.;.,.)$ is continuous, we have: $(x^*(T;t+r,\widetilde{x}(t))-x^*(T;t,\widetilde{x}(t)))/r$ $=-(x^*(T;t+r,\widetilde{x}(t+r))-x^*(T;t+r,\widetilde{x}(t)))/r=$

$$= -\int_0^1 D_3x^*(T;t+r,\tilde{x}(t)+s(\tilde{x}(t+r)-\tilde{x}(t))).((\tilde{x}(t+r)-\tilde{x}(t))/r)ds \longrightarrow$$

$\longrightarrow -D_3x^*(T;t,\tilde{x}(t)).\tilde{f}(t,\tilde{x}(t))$ as $r \to 0$ and (2.10) is proved.

The last statement is a well known result according to which $x^*_T(.,.)$ is differentiable at $(t,\tilde{x}(t))$ if the partial derivatives $D_1x^*_T(t,\tilde{x}(t))$, $D_2x^*_T(t,\tilde{x}(t))$ exist and $D_2x^*_T(s,y)$ exists in a neighbourhood of $(t,\tilde{x}(t))$ and is continuous at $(t,\tilde{x}(t))$.

The differential inequality we shall prove in the next section involves the <u>upper-right contingential derivative</u> of a function $F(.):X\subset\mathbb{R}^n \longrightarrow \bar{\mathbb{R}}=[-\infty,+\infty]$ at a point x in the effective domain, $\mathrm{dom}\, F(.)=\{x\in X;\ F(x)\in R\}$, in a direction $\bar{x}\in R^n$:

$$\bar{D}^+_K F(x;\bar{x}) = \limsup_{\substack{(s,y)\to(0+,\bar{x})\\ x+sy\in X}} (F(x+sy)-F(x))/s \qquad (2.12)$$

which is of interest only at directions $\bar{x}$ in the right-contingent cone at x of X defined by: $K^+_x X=\{\bar{x}\in R^n;(\exists)\ (s_k,y_k)\longrightarrow(0+,\bar{x}),\ x+s_k y_k\in X\ (\forall)\ k\in\mathbb{N}\}$ since using the convention $\sup \emptyset=-\infty$, from (2.12) it follows: $\bar{D}^+_K F(x;\bar{x})=-\infty\ (\forall)\ \bar{x}\in R^n\setminus K^+_x X$.

The differential inequalities in [11] and [7] involve also the other extreme contingential derivatives of $F(.)$ at x:

$$\underline{D}^+_K F(x;\bar{x}) = \liminf_{\substack{(s,y)\to(0+,\bar{x})\\ x+sy\in X}} (F(x+sy)-F(x))/s \qquad (2.13)$$

$$\bar{D}^-_K F(x;\bar{x})=-\underline{D}^+_K F(x;-\bar{x}),\quad \underline{D}^-_K F(x;\bar{x})=-\bar{D}^+_K F(x;-\bar{x})$$

which extend to functions of vector variables the well known Dini derivatives and are natural generalizations of the classical (Fréchet) derivative since $F(.)$ is differentiable at a point $x\in \mathrm{Int\, dom}\, F(.)$ iff $F^+_K(x;\bar{x})=\bar{D}^+_K F(x;\bar{x})=\underline{D}^+_K F(x;\bar{x})\in R$ exists for any $\bar{x}\in R^n$ and $F^+_K(x;.)$ is linear; for other properties of these generalized derivatives we refer to [13],

[9], [12]

3. THE PROOF OF PONTRYAGIN'S MINIMUM PRINCIPLE

We take the following corollary of Proposition 2.2 as part of our proof of Pontryagin's Minimum Principle:

LEMMA 3.1 ([11],[7]). *If* $V(.,.):D_T \longrightarrow \bar{R}$ *is the value function in* (2.3) *of the problem* (2.1)-(2.2) *then its upper right contingential derivatives satisfy the inequality*:

$$\bar{D}_K^+ V((t,x);(1,f(t,x,u))) \geqslant 0 \quad (\forall)\ u \in U,\ (t,x) \in D_T^o \cap \mathrm{dom}\ V(.,.) \quad (3.1)$$

Proof. Let $(t_o,x_o) \in D_T^o \cap \mathrm{dom}\ V(.,.)$ hence such that $t_o < T$, $(t_o,x_o) \in D$ and $V(t_o,x_o) \in R$, let $u_o \in U$ and let $x_o(.):[t_o,t_1] \longrightarrow R^n$; $t_1 < T$, be the unique C^1-solution of the initial value problem: $x'=f(t,x,u_o)$, $x(t_o)=x_o$; since $(x_o(t_o+s)-x_o)/s \longrightarrow x_o'(t_o)= =f(t_o,x_o,u_o)$ as $s \longrightarrow 0+$, from (2.12) it follows:

$$\bar{D}_K^+ V((t_o,x_o);(1,f(t_o,x_o,u_o))) \geqslant \limsup_{s \longrightarrow 0+} (V(t_o+s,x_o(t_o+s))-V(t_o,x_o))/s \quad (3.2)$$

Obviously only one of the following two possibilities may occur: 1) $\mathcal{U}(t,x_o(t))=\emptyset$ $(\forall)$ $t \in (t_o,t_1]$; 2) there exists $t_2 \in \in (t_o,t_1]$ such that $\mathcal{U}(t_2,x_o(t_2)) \neq \emptyset$. In the first case from (2.3) it follows that $V(t,x_o(t))=+\infty$ $(\forall)$ $t \in (t_o,t_1]$ and therefore, from (3.2) it follows that $\bar{D}_K^+ V((t_o,x_o);(1,f(t_o,x_o,u_o)))=+\infty$ and (3.1) is verified.

In the second case there exists an admissible control, $u_2(.) \in \mathcal{U}(t_2,x_o(t_2))$ which we use to define an admissible control with respect to (t_o,x_o) as follows:

$$u(t)=\begin{cases} u_o & \text{if } t \in [t_o,t_2) \\ u_2(t) & \text{if } t \in [t_2,T] \end{cases}$$

The admissible trajectory $x(.;u(.))$ corresponding to $u(.)$ is obviously given by: $x(t;u(.))=x_o(t)$ if $t \in [t_o,t_2]$ and $x(t;u(.))=x(t;u_2(.))$ if $t \in [t_2,T]$; on the other hand, from Proposition 2.2 it follows that the function $t \longmapsto V(t,x(t;u(.)))$

is nondecreasing hence $V(t,x_0(t)) \geqslant V(t_0,x_0)$ $(\forall)$ $t\in[t_0,t_2]$ and therefore from (3.2) it follows (3.1) and Lemma 3.1 is proved

THEOREM 3.2. Let the data $g(.)$, $f(.,.,.)$ of the problem (2.1)-(2.2) satisfy Hypothesis 2.1, let $(t_0,x_0)\in D_T^o$, let $\tilde{u}(.)\in\mathcal{U}(t_0,x_0)$ be an optimal control such that the vector field $\tilde{f}(.,.)$ defined by:

$$\tilde{f}(t,x)=f(t,x,\tilde{u}(t)),\ (t,x)\in D_T=\{(t,x)\in D;\ t\leqslant T\} \tag{3.3}$$

has the properties in (2.4) at each point $(s,y)\in D_T$ and let $\tilde{x}(.)=x(.;\tilde{u}(.))$ be the corresponding optimal trajectory.

Then there exists an absolutely continuous mapping $p(.):[t_0,T]\longrightarrow\mathbb{R}^n$ satisfying the following conditions:

$$p'(t)=-D_2H(t,\tilde{x}(t),\tilde{u}(t),p(t)) \text{ a.e. on } [t_0,T] \tag{3.4}$$

where the Hamiltonian H is defined by:

$$H(t,x,u,p)=\langle p,f(t,x,u)\rangle\ ,\ (t,x)\in D,\ u\in U,\ p\in R^n \tag{3.5}$$

$$p(T)=Dg(\tilde{x}(T)) \tag{3.6}$$

$$H(t,\tilde{x}(t),\tilde{u}(t),p(t))=\tilde{H}(t,\tilde{x}(t),p(t))=\min_{u\in U} H(t,\tilde{x}(t),u,p(t)) \text{ a.e.} \tag{3.7}$$

Proof. We note first that choosing an arbitrary $u_0\in U$ and a sufficiently small $r>0$ we may extend the vector field $\tilde{f}(.,.)$ in (3.3) to an open subset $\tilde{D}=\{(t,x)\in D;\ t\in(t_0-r,T+r)\}$ of $\mathbb{R}\times\mathbb{R}^n$ as follows:

$$\tilde{f}(t,x)=\begin{cases} f(t,x,\tilde{u}(t)) & \text{if } (t,x)\in D_T \\ f(t,x,u_0) & \text{if } (t,x)\in D,\ t\in(t_0-r,t_0)\cup(T,T+r)\end{cases} \tag{3.8}$$

so that $\tilde{f}(.,.)$ is a Carathéodory-C^1 vector field satisfying the hypotheses of Theorem 2.3.

Following the idea of the proof in [2] we consider the maximal flow, $x^*(.;.,.)$, of the vector field $\tilde{f}(.,.)$ in (3.8) and define the function $\tilde{V}(.,.):\tilde{D}_T \longrightarrow \mathbb{R}$ as follows:

$$\tilde{V}(s,y)=g(x^*(T;s,y)),\ (s,y)\in\tilde{D}_T=\{(s,y)\in\tilde{D};[s,T]\subset I(s,y)\} \quad (3.9)$$

where $I(s,y)$ is the open interval on which $x^*(.;s,y)$ is defined.

From the theorems on continuity of solutions with respect to initial data (see [8], Ch.12 and Remark 18.4.15) it follows that $(t,\tilde{x}(t))\in \mathrm{Int}\,\tilde{D}_T$ $(\forall)$ $t\in[t_o,T]$ and since $x^*(.;t_o,x_o)=$ $=\tilde{x}(.)$ the relations in (2.11) hold; moreover, since for any $(s,y)\in\tilde{D}_T\cap D_T^o$ (for which $s\in[t_o,T)$) the restriction map $\tilde{u}(.)|[s,T]$ is an admissible control with respect to the initial point (s,y) (the corresponding admissible trajectory being $x^*(.;s,y)|[s,T]$) from (2.3) it follows:

$$\begin{aligned}&\tilde{V}(t,\tilde{x}(t))=g(\tilde{x}(T))=V(t,\tilde{x}(t)) \quad (\forall)\ t\in[t_o,T]\\ &\tilde{V}(s,y)\geqslant V(s,y) \quad (\forall)\ (s,y)\in\tilde{D}_T\cap D_T^o\end{aligned} \quad (3.10)$$

where $V(.,.)$ is the value function in (2.3).

Since $g(.)$ is assumed to be continuously differentiable and since according to Corollary 2.4 the mapping $t\longmapsto D_3x^*(T,t,\tilde{x}(t))$ is absolutely continuous, the mapping $p(.)$ defined by:

$$p(t)=D_2V(t,\tilde{x}(t))=Dg(\tilde{x}(T)).D_3x^*(T;t,\tilde{x}(t)),\ t\in[t_o,T] \quad (3.11)$$

is absolutely continuous and moreover, from (2.8) it follows that at any point $t\in[t_o,T]$ at which $D_3x^*(T;.,\tilde{x}(.))$ is differentiable we have: $p'(t)=Dg(\tilde{x}(T)).(-D_3x^*(T;t,\tilde{x}(t)).D_2\tilde{f}(t,\tilde{x}(t)))$ $=-D_2H(t,\tilde{x}(t),\tilde{u}(t),p(t))$ and (3.4) is proved; from (2.8) and (3.11) it follows that $p(T)=Dg(\tilde{x}(T))$ and (3.6) is also proved.

To prove (3.7) we note first that from the definition in (2.12) of the upper-right contingential derivative and from the relations in (3.10) it follows:

$$\overline{D}_K^+\tilde{V}((t,\tilde{x}(t));(\bar{t},\bar{x}))\geqslant\overline{D}_K^+V((t,\tilde{x}(t));(\bar{t},\bar{x}))\ (\forall)t\in[t_o,T],(\bar{t},\bar{x})\in\mathbb{R}\times\mathbb{R}^n \quad (3.12)$$

hence from Lemma 3.1 it follows:

$$\bar{D}^{+}_{K}\,\tilde{V}((t,\tilde{x}(t));(1,f(t,\tilde{x}(t),u)))\geq 0 \quad (\forall)\ u\in U,\ t\in[t_o,T) \qquad (3.13)$$

On the other hand, from Corollary 2.4 and from (3.9) it follows that if $J_1\subset[t_o,T]$ is the null subset at which (2.9) is verified then for any $t\in[t_o,T]\setminus J_1$ the function $\tilde{V}(.,\tilde{x}(t))$ is differentiable and from (2.10), (3.11) and (3.5) it follows:

$D_1\tilde{V}(t,\tilde{x}(t))=Dg(\tilde{x}(T)).D_2x^*(T;t,\tilde{x}(t))=-Dg(\tilde{x}(T)).D_3x^*(T;t,\tilde{x}(t)).$

$.\tilde{f}(t,\tilde{x}(t))=-H(t,\tilde{x}(t),\tilde{u}(t),p(t))\quad(\forall)\ t\in[t_o,T]\setminus J_1$. Moreover, since $g(.)$ is assumed to be of class C^1, the partial derivative $D_2\tilde{V}(.,.)$ is continuous hence $\tilde{V}(.,.)$ is differentiable at $(t,\tilde{x}(t))$ for any $t\in[t_o,T]\setminus J_1$ and its derivative is given by:

$$D\tilde{V}(t,\tilde{x}(t))=(-H(t,\tilde{x}(t),\tilde{u}(t),p(t)),p(t)) \quad (\forall)\ t\in[t_o,T]\setminus J_1 \qquad (3.14)$$

Since from the definition in (2.12) of the upper-right contingential derivative it follows that at any such point

$\bar{D}^{+}_{K}\,\tilde{V}((t,\tilde{x}(t));(\bar{t},\bar{x}))=DV(t,\tilde{x}(t)).(\bar{t},\bar{x})=-H(t,\tilde{x}(t),\tilde{u}(t),p(t)).\bar{t}+$

$+<p(t),\bar{x}>\quad(\forall)\ t\in[t_o,T]\setminus J_1,\ (\bar{t},\bar{x})\in\mathbb{R}\times\mathbb{R}^n$, from (3.13) we obtain: $-H(t,x(t),u(t),p(t))+<p(t),f(t,x(t),u)>\ \geq 0\ (\forall)\ u\in U$, $t\in[t_o,T)\setminus J_1$ and the theorem is completely proved.

REMARK 3.3. We recall that the viscosity solutions of Hamilton-Jacobi equations ([1],[2],[7],[11], ↕ etc.) are defined in terms of the <u>Fréchet semidifferentials</u> of real-valued functions $F(.):X\subset\mathbb{R}^n\longrightarrow\mathbb{R}$ defined as follows:

$$\begin{aligned}\bar{\partial}F(x)&=\{p\in\mathbb{R}^n;\ \limsup_{h\to 0}(F(x+h)-F(x)-<p,h>)/|h|\leq 0\}\\ \underline{\partial}F(x)&=\{p\in\mathbb{R}^n;\ \liminf_{h\to 0}(F(x+h)-F(x)-<p,h>)/|h|\geq 0\}\end{aligned} \qquad (3.15)$$

As it is proved in [12], Theorem 5.2, the Fréchet semidifferentials may equivalently be defined using the extreme contingential derivatives in (2.12) and (2.13) as follows:

$$\overline{\partial} F(x)=\{p\in \mathbb{R}^n;\ \overline{D}_K^+F(x;\bar{x})\leq \ <p,\bar{x}>\ \leq \underline{D}_K^-F(x;\bar{x})\ (\forall)\ \bar{x}\in \mathbb{R}^n\}$$
$$\underline{\partial} F(x)=\{p\in \mathbb{R}^n;\ \overline{D}_K^-F(x;\bar{x})\leq \ <p,\bar{x}>\ \leq \underline{D}_K^+F(x;\bar{x})\ (\forall)\ \bar{x}\in \mathbb{R}^n\} \quad (3.16)$$

On the other hand, a large number of interesting results have been obtained lately using Clarke's generalized gradient in problems involving locally Lipschitz functions:

$$\partial_C F(x)=\{p\in \mathbb{R}^n;\ <p,\bar{x}>\ \leq D_C^+F(x;\bar{x})\ (\forall)\ \bar{x}\in \mathbb{R}^n\} \quad (3.17)$$

where Clarke's generalized directional derivative, $\overline{D}_C^+F(x;\bar{x})$ is defined as follows:

$$\overline{D}_C^+F(x;\bar{x})= \limsup_{(z,s)\to(x,0+)} (F(z+s\bar{x})-F(z))/s \quad (3.18)$$

It is easy to see that if $F(.)$ is locally Lipschitz at $x\in \mathrm{Int}(X)$ then one has ([12]):

$$\overline{\partial} F(x)\subset \partial_C F(x), \qquad \underline{\partial} F(x)\subset \partial_C F(x) \quad (3.19)$$

The main result in [14] states, essentially, that if $g(.)$ and $f(.,.,.)$ defining the problem (2.1)-(2.2) are locally Lipschitz, $U\subset \mathbb{R}^m$ is compact and $\tilde{u}(.)\in \mathcal{U}(t_o,x_o)$ is optimal then there exists an absolutely continuous mapping $p(.)$ satisfying (3.7) and certain inclusions replacing the equalities in (3.4) and (3.6) and satisfying also the relation:

$$(-\tilde{H}(t,\tilde{x}(t),p(t)),p(t))\in \partial_C V(t,\tilde{x}(t)) \quad (\forall)\ t\in[t_o,T] \quad (3.20)$$

In the case $g(.)$ and $f(.,.,.)$ satisfy Hypothesis 2.1, from the relations (3.12) and (3.14) in the proof of Theorem 2.3 and from (3.16) it follows that the mapping $p(.)$ in Theorem 2.3 satisfies the condition:

$$(-\tilde{H}(t,\tilde{x}(t),p(t)),p(t))\in \overline{\partial} V(t,\tilde{x}(t)) \quad \text{a.e. on } [t_o,T] \quad (3.21)$$

Moreover, proceeding like in the proof of Corollary 2.4 it is easy to see that if $U\subset \mathbb{R}^m$ is compact then the extreme contingential derivatives of the functions $\tilde{V}(.,.)$ and $V(.,.)$ satisfy

the relations: $\bar{D}^+_K V((t,\tilde{x}(t));(\bar{t},\bar{x})) \leq D^+_K \tilde{V}((t,x(t));(\bar{t},\bar{x})) \leq$
$-\tilde{H}(t,\tilde{x}(t),p(t)).\bar{t} + \langle p(t),\bar{x}\rangle \leq \underline{D}^-_K \tilde{V}((t,\tilde{x}(t));(\bar{t},\bar{x})) \leq \underline{D}^-_K V((t,\tilde{x}(t));(\bar{t},\bar{x})$
$(\forall)$ $(\bar{t},\bar{x}) \in \mathbb{R}\times\mathbb{R}^n$, $t \in [t_o,T]$ hence from (3.16) it follows:

$$(-\tilde{H}(t,\tilde{x}(t),p(t)),p(t)) \in \bar{\partial} V(t,\tilde{x}(t)) \quad (\forall)\ t \in [t_o,T] \tag{3.22}$$

which implies the relation (3.20) obtained in [14] since in this case the value function, V(.,.), is locally Lipschitz.

Similarly, relations (3.10) and (3.11) imply the fact that the absolutely continuous mapping p(.) in Theorem 2.3 verifies:

$$p(t) \in \bar{\partial} V(t,.)(\tilde{x}(t)) \quad (\forall)\ t \in [t_o,T] \tag{3.23}$$

which is stronger than the relation:

$$p(t) \in \partial_C V(t,.)(\tilde{x}(t)) \quad \text{a.e. on } [t_o,T] \tag{3.24}$$

obtained in [5] , in the case V(t,.) is locally Lipschitz.

We note however that (3.22) and (3.23) have been obtained in the case g(.) and f(t,.,u) are of class C^1 while the corresponding relations, (3.20) and (3.24), in [14] and [5], respectively have been obtained in the case g(.) and f(.,.,.) are only locally Lipschitz.

An interesting open problem is to obtain results analogous to those in [14] and [5] expressed in terms of the Fréchet semidifferntials in (3.15).

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