NORMED ALMOST LINEAR SPACES WITH BASES

by

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1. INTRODUCTION

In /2/ we introduced the concept of a normed almost linear space which generalizes the concept of a normed linear space. An example of a normed almost linear space which is not a normed linear space is the collection of all nonempty, bounded and convex subsets of a non-trivial normed linear space (see Example 6.1 in Sect. 6). Roughly speaking, a normed almost linear space X satisfies some of the axioms of a linear space but, in compensation, the norm satisfies besides all the axioms of an usual norm on a linear space, also an additional one which makes the framework productive. At first sight, it seems that no topology on X exists, since the norm does not generate even a semi-metric on X if we simply imitate the linear case.

The following two subsets of X play an important role: $V_X = \{x \in X : x+(-1x)=0\}$ and $W_X = \{x \in X : x=-1x\}$. Among the simplest classes of normed almost linear spaces are those of the form $X = V_X$ (when we recover the class of normed linear spaces), $X = W_X$ and $X = W_X+V_X$. Note that the above example of a normed almost linear space does not belong to any of these classes.

In /2/ we have also introduced the "dual" space X^* of a normed almost linear space X, where the functionals are no longer linear but "almost linear", which is also a normed almost linear space; when X is a normed linear space then X^* defined by us is the usual dual space of X. Though the proof is very simple, it is worth to mention here that we always have $X^* = W_{X^*}+V_{X^*}$ (see Proposition 3.3 in Sect. 3) and by Proposition 3.15 of /3/ and Corollary 3.4 of /5/ we have $X^* = V_{X^*}$ iff X is a normed linear space.

The main tool for the theory of normed almost linear spaces was given in /5/ where we showed that any normed almost linear space X can be "embedded" in a normed linear space E_X . Though the embedding mapping $\omega_X : X \longrightarrow E_X$ is not in general one-to-onr, it has enough properties to permit us the use of the techniques of normed linear spaces. As a consequence we can define a semi-metric ρ_X on X, which is a metric exactly when ω_X is one-to-one. Thus a topology on X always exists (which is not Hausdorff in general) and when X is a normed linear space then ρ_X is the metric generated by the norm.

In /3/ we have begun the study of the almost linear spaces X with bases, [illegible] In contrast to the linear case, an almost linear space can have no basis, even when it is a normed almost linear space. On the other hand, when an almost linear space has a basis, then it is a normed almost linear space for a certain norm (/3/).

In this paper we further our study on normed almost linear spaces, especially when they have bases. For a normed almost linear space X (not necessarily with a basis) we show (Theorem 3.5) that E_{X^*} (the normed linear space in which we embed X^*) is E_X^* equipped in general with another norm than that of E_X^* .

When a normed almost linear space X has a basis, it has some relevant properties. For such a space X we prove that ω_X is one-to-one, hence the topology generated by ρ_X is now Hausdorff (Theorem 4.1) and that its range $\omega_X(X)$ is closed in E_X (Theorem 4.6)

In the framework of normed almost linear spaces X with bases we can generalize some well-known results from the theory of linear or normed linear spaces. For example, the cardinality of two bases of X is the same (Corollary 4.4), we generalize the Riesz's Theorem (Theorem 5.1) and we prove that if dim X = n then dim X^* = n (Theorem 5.2). Concerning the last result we mention here Example 6.8, where X has a basis but X^* has no basis and also the fact that the assumption dim $X^* < \infty$ does not imply that X has a basis. There are also some other differences between the "linear" and "almost linear" case . For example, when dim $X < \infty$, the embedding mapping $Q: X \longrightarrow X^{**}$ can be not onto X^{**}. In this paper we also give sufficient conditions in order that some results from the linear case hold in our more general framework.

This paper contains 6 sections (including the introduction). In Sect. 2, besides notation, we give all definitions and results from our previous papers, necessary for an easy understanding of this paper. In Sect. 3 we establish some new results on normed almost linear spaces (not necessarily with bases) which will be used in the subsequent sections. Sections 4 and 5 are devoted to the normed almost linear spaces with bases, the latter being concerned with finite-dimensional spaces. In Sect. 6 we collect all examples (counter-examples) at which we shall refer in the text.

Finally, we draw attention that in /2/-/4/ we have worked with an equivalent definition of the norm and in /2/ and /4/ the last axiom of the norm is superfluous.

2. PRELIMINARIES

Besides notation, in this section we mainly recall some definitions and results from our previous papers. As in those papers, we assume that all spaces are over the real field R. We denote by R_+ the set $\{\lambda \in R : \lambda \geq 0\}$ and by N the set $\{1,2,3,\ldots\}$.

An almost linear space ([8]) is a set X together with two mappings $s: X \times X \longrightarrow X$ and $m: R \times X \longrightarrow X$ satisfying (L_1)-(L_8) below. We denote $s(x,y)$ by $x+y$ and $m(\lambda,x)$ by $\lambda \circ x$ (or λx). Let $x,y,z \in X$ and $\lambda, \mu \in R$. (L_1) $x+(y+z) = (x+y)+z$; (L_2) $x+y = y+x$; (L_3) There exists an element $0 \in X$ such that $x+0 = x$ for each $x \in X$; (L_4) $1 \circ x = x$; (L_5) $0 \circ x = 0$; (L_6) $\lambda \circ (x+y) = \lambda \circ x + \lambda \circ y$; (L_7) $\lambda \circ (\mu \circ x) = (\lambda \mu) \circ x$; (L_8) $(\lambda + \mu) \circ x = \lambda \circ x + \mu \circ x$ for $\lambda, \mu \in R_+$.

Let $V_X = \{x \in X : x+(-1 \circ x) = 0\}$ and $W_X = \{x \in X : x = -1 \circ x\}$ ([2]). These are almost linear subspaces of X (i.e., closed under addition and multiplication by reals) and V_X is a linear space. An almost linear space X is a linear space iff $X = V_X$, iff $W_X = \{0\}$.

In an almost linear space X we shall always use the notation $\lambda \circ x$ (in particular $-1 \circ x$) for $m(\lambda,x)$ (for $m(-1,x)$), the notation λx (in particular $-x$) being used only in a linear space.

A subset $\mathcal{B}$ of the almost linear space X is called a basis of X (/3/, Definition 2.1) if for each $x \in X \setminus \{0\}$ there exist unique sets $\{b_1,\ldots,b_n\} \subset \mathcal{B}$, $\{\lambda_1,\ldots,\lambda_n\} \subset R \setminus \{0\}$ (n depending on x) such that $x = \sum_{i=1}^{n} \lambda_i \circ b_i$, where $\lambda_i > 0$ for $b_i \notin V_X$.

In contrast to the case of a linear space, there exist almost linear spaces which have no basis (see examples in Sect. 6).

2.1. REMARK. (/3/). Let X be an almost linear space with a basis $\mathcal{B}$.

(i) The set $\{\lambda_b \circ b : b \in \mathcal{B}, \lambda_b \neq 0, \lambda_b > 0 \text{ for } b \notin V_X\}$ is also a basis of X.

(ii) The set $\mathcal{B} \cap V_X$ is a basis of V_X .

(iii) X satisfies the law of cancellation, i.e., the relations $x,y,z \in X$, $x+y=x+z$ imply that $y=z$.

2.2. THEOREM. (/3/, Theorem 2.8). <u>If the almost linear space X has a basis, then there exists a basis $\mathcal{B}$ of X with the property that whenever $b \in \mathcal{B} \setminus V_X$ then $-1 \circ b \in \mathcal{B}$.</u>

2.3. REMARK. Let $\mathcal{B}$ be a basis of the almost linear space X having the property from Theorem 2.2.

(i)(/3/). The set $\{ b+(-1\circ b) ; b \in \mathcal{B} \setminus V_X \}$ is a basis of W_X .

(ii) If $X = W_X + V_X$ then $\mathcal{B} = (\mathcal{B} \cap W_X) \cup (\mathcal{B} \cap V_X)$. Indeed, since for $b \in \mathcal{B}$ we have $b = w+v$, $w \in W_X$, $v \in V_X$, by (i) above and Remark 2.1 (ii) it follows that either $b = w$ or $b = v$. This result is no longer true if the basis $\mathcal{B}$ has not the property from Theorem 2.2 (see Example 6.5 (i)).

A <u>normed almost linear space</u> ([2]) is an almost linear space X together with a <u>norm</u> $|||\cdot||| : X \longrightarrow R$ satisfying (N_1)-(N_4) below. Let $x,y \in X$, $w \in W_X$ and $\lambda \in R$. (N_1) $|||x+y||| \leq |||x||| + |||y|||$; (N_2) $|||x||| = 0$ iff $x=0$; (N_3) $|||\lambda \circ x||| = |\lambda| \, |||x|||$; (N_4) $|||x||| \leq |||x+w|||$. Note that $|||x||| \geq 0$ for each $x \in X$. We denote by $B_X(0,\lambda)$ the set $\{ x \in X : |||x||| \leq \lambda \}$, $B_X = B_X(0,1)$ and $S_X = \{ x \in X : |||x||| = 1 \}$.

As we have already noted in the introduction, when X is an almost linear space with a basis, there exists a norm $|||\cdot|||$ on X such that X together with this norm is a normed almost linear space.

2.4. LEMMA.(/3/). <u>Let X be a normed almost linear space and let</u> $x,y \in X$, $w_i \in W_X$, $v_i \in V_X$, $i=1,2$. <u>We have</u>:

(i) <u>If</u> $x+y \in V_X$ then $x,y \in V_X$

(i) *If* $x+y \in V_X$ *then* $x, y \in V_X$.

(ii) *If* $w_1+v_1 = w_2+v_2$ *then* $w_1 = w_2$ *and* $v_1 = v_2$.

Let X be an almost linear space. A functional $f: X \longrightarrow R$ is called an *almost linear functional* ([2]) if f is additive, positively homogeneous and the restriction $f|W_X \geqslant 0$. Let $X^{\#}$ be the set of all almost linear functionals on X. Define the addition in $X^{\#}$ by $(f_1+f_2)(x) = f_1(x) + f_2(x)$, $x \in X$ and the multiplication by reals $\lambda \in R$ by $(\lambda \circ f)(x) = f(\lambda \circ x)$, $x \in X$. The element $0 \in X^{\#}$ is the functional which is 0 at each $x \in X$. Then $X^{\#}$ is an almost linear space. When X is a normed almost linear space, for $f \in X^{\#}$ define $|||f||| = \sup \{|f(x)| ; x \in B_X\}$ and let $X^* = \{f \in X^{\#} ; |||f||| < \infty\}$. Then X^* is a normed almost linear space (/2/) called the *dual* space of X. The dual space X^* is $\neq \{0\}$ if $X \neq \{0\}$ since the following generalization of a corollary of Hahn-Banach Theorem holds:

2.5. THEOREM. (/5/, Corollary 3.4). *If* X *is a normed almost linear space, then for each* $x \in X$ *there exists* $f \in S_{X^*}$ *such that* $f(x) = |||x|||$.

Let X,Y be two almost linear spaces. A mapping $T: X \longrightarrow Y$ is called a *linear operator* if $T(\lambda_1 \circ x_1 + \lambda_2 \circ x_2) = \lambda_1 \circ T(x_1) + \lambda_2 \circ T(x_2)$, $x_i \in X$, $\lambda_i \in R$, $i=1,2$. When X and Y are normed almost linear spaces, a linear operator $T: X \longrightarrow Y$ is called a *linear isometry* if $|||T(x)||| = |||x|||$ for each $x \in X$. We draw attention that a linear isometry is not always one-to-one. For $A \subset X$ we denote by $T(A)$ the set $\{T(a) ; a \in A\}$.

2.6. REMARK. (/5/). If T is a linear isometry of X *onto* Y then $T(V_X) = V_Y$, $T(W_X) = W_Y$ and the restriction $T|V_X$ is one-to-one.

The main tool for the theory of normed almost linear spaces is the following result:

2.7. THEOREM. (/5/, Theorem 3.2). For any normed almost linear space $(X, |||\cdot|||)$ there exist a normed linear space $(E_X, \|\cdot\|_{E_X})$ and a mapping $\omega_X : X \to E_X$ with the following properties :

(i) $E_X = \omega_X(X) - \omega_X(X)$ and $\omega_X(X)$ can be organized as an almost linear space where the addition and the multiplication by non-negative reals are the same as in E_X .

(ii) For $z \in E_X$ we have

$$(2.1) \quad \|z\|_{E_X} = \inf \{ |||x||| + |||y||| ; x, y \in X, \ z = \omega_X(x) - \omega_X(y) \}$$

and $(\omega_X(X), \|\cdot\|_{E_X})$ is a normed almost linear space.

(iii) ω_X is a linear isometry of $(X, |||\cdot|||)$ onto $(\omega_X(X), \|\cdot\|_{E_X})$.

(iv) For the dual space X^* we have

$$(2.2) \quad S_{X^*} = \{ \tilde{f}\omega_X : \tilde{f} \in S_{E_X^*} , \ \tilde{f}|\omega_X(W_X) \geq 0 \}$$

2.8. REMARK. If on E_X there exists another norm $\|\cdot\|_1$ such that $\|\omega_X(x)\|_1 = |||x|||$ for each $x \in X$ then for each $z \in E_X$ we have $\|z\|_1 \leq \|z\|_{E_X}$. Indeed, if $z = \omega_X(x) - \omega_X(y)$, $x, y \in X$ then $\|z\|_1 \leq$ $\leq \|\omega_X(x)\|_1 + \|\omega_X(y)\|_1 = |||x||| + |||y|||$, whence by (2.1) we get $\|z\|_1 \leq \|z\|_{E_X}$.

2.9. COROLARRY.(/5/, Corollary 3.3). For any normed almost linear space $(X, |||\cdot|||)$ the function $\rho_X : X \times X \to R$ defined by

(2.3) $$\rho_X(x,y) = \| \omega_X(x) - \omega_X(y) \|_{E_X} \qquad (x,y \in X)$$

is a semi-metric on X and we have

(2.4) $$\rho_X(-1\circ x, -1\circ y) = \rho_X(x,y) \qquad (x,y \in X)$$

In a normed almost linear space X the semi-metric ρ_X generates a topology on X (which is not Hausdorff in general) and in the sequel any topological concept will be understood for this topology. Clearly ρ_X is a metric on X iff ω_X is one-to-one. Note that even when ρ_X is not a metric on X we can use sequences instead of nets. Moreover, for $v_1, v_2 \in V_X$ we have $\rho_X(v_1,v_2) = |||v_1 - v_2|||$.

2.10. LEMMA.(/6/, Lemma 6.1). A normed almost linear space $(X, |||\cdot|||)$ is complete iff $(E_X, \|\cdot\|_{E_X})$ is a Banach space and $\omega_X(X)$ is norm-closed in E_X.

The proof of the following lemma is contained in the proof of (/5/, Theorem 3.2 (iv), fact I).

2.11. LEMMA. Let $(X, |||\cdot|||)$ be a normed almost linear space and let $x,y \in X$. If $\omega_X(x) = \omega_X(y)$, then for each $\varepsilon > 0$ there exist $x_\varepsilon, y_\varepsilon, u_\varepsilon \in X$ such that $|||x_\varepsilon||| + |||y_\varepsilon||| < \varepsilon$ and $x + y_\varepsilon + u_\varepsilon = y + x_\varepsilon + u_\varepsilon$.

Let K be the following cone of E_X^*:

(2.5) $$K = \{ \tilde{f} \in E_X^* : \tilde{f} | \omega_X(W_X) \geq 0 \}$$

Except for (iv) below, the statements from the next corollary are particular cases of some results from (~~1~~6/). They can be also easily proved using Theorem 2.7 (iv) and the assertion (iv) follows since K is w^*-closed in E_X .

2.12. COROLLARY. Let X be a normed almost linear space and let K be the cone of E_X^* defined by (2.5). Then $(E_{X^*}, \|\cdot\|_{E_{X^*}})$ and ω_{X^*} from Theorem 2.7 for $(X^*, |||\cdot|||)$ are the following :

(i) $E_{X^*} = K-K$ equipped with the norm

$$\|\tilde{f}\|_{E_{X^*}} = \inf\{\|\tilde{f}_1\|_{E_X^*} + \|\tilde{f}_2\|_{E_X^*} ; \tilde{f}=\tilde{f}_1-\tilde{f}_2,\ \tilde{f}_1,\tilde{f}_2 \in K\}$$

(ii) For $f \in X^*$ we have $\omega_{X^*}(f) = \tilde{f} \in K$, $|||f||| = \|\tilde{f}\|_{E_X^*}$ and $f = \tilde{f}\omega_X$, where $\tilde{f}$ is obtained using (2.2). We also have $\omega_{X^*}(X^*) = K$.

(iii) ω_{X^*} is one-to-one.

(iv) $\omega_{X^*}(X^*)$ is closed in E_{X^*}.

As we shall show in Theorem 3.5, we have $E_{X^*} = E_X^*$ in (i). We draw attention that the norm on E_{X^*} given in (i) is in general not equal with the norm on E_X^* (see Example 6.3 (i)).

For a normed almost linear space X, let $Q_X : X \longrightarrow X^{**}$ be the mapping defined by

$$(Q_X(x))(f) = f(x) \qquad\qquad (f \in X^*)$$

It is straightforward to show (using also Theorem 2.5) that we have:

2.13. PROPOSITION. For any normed almost linear space X, the mapping $Q_X : X \longrightarrow X^{**}$ is a linear isometry of X onto the almost linear subspace $Q_X(X)$ of X^{**}.

For a normed almost linear space X, in the next sections we shall denote $\|\cdot\|_{E_X}$ by $\|\cdot\|$ when this will not lead to misunderstanding. We shall also denote by $\|\cdot\|$ the norm on the dual or bidual space of E_X .

3. MISCELLANEOUS RESULTS IN NORMED ALMOST LINEAR SPACES

In this section X will stand for an arbitrary normed almost linear space. In the sequel $(E_X; \|\cdot\|)$ and ω_X are given by Theorem 2.7.

3.1. PROPOSITION. <u>If</u> ω_X <u>is one-to-one then</u> W_X <u>is a closed subset of</u> X. <u>Consequently</u>, $W_{\omega_X(X)}$ <u>is</u> $\|\cdot\|$-<u>closed in</u> $\omega_X(X)$.

PROOF. Let $w_n \in W_X$, $n \in N$ and $x \in X$ be such that

$$\lim_{n \to \infty} \rho_X(w_n, x) = 0$$

By (2.4) we also have

$$\lim_{n \to \infty} \rho_X(w_n, -1 \circ x) = 0$$

Consequently, $\rho_X(x, -1 \circ x) = 0$ and since ω_X is one-to-one, it follows that $x = -1 \circ x$, i.e., $x \in W_X$.

The assumption on ω_X to be one-to-one is essential (see Example 6.1 (i)).

3.2. PROPOSITION. <u>If</u> $V_{\omega_X(X)}$ <u>is closed in</u> E_X <u>then the linear subspace</u> E_1 <u>of</u> E_X <u>defined by</u> $E_1 = (W_{\omega_X(X)} + V_{\omega_X(X)}) - (W_{\omega_X(X)} + V_{\omega_X(X)})$

is closed in E_X .

PROOF. Let $\{z_n\}_{n=1}^{\infty} \subset E_1$ and $z \in E_X$ such that $\lim_{n\to\infty} \|z_n - z\| = 0$. Since for each $n \in N$, $z_n = w_n + v_n - \overline{w}_n$ for some $w_n, \overline{w}_n \in W_{\omega_X(X)}$, $v_n \in V_{\omega_X(X)}$ and $z = x_1 - x_2$, $x_i \in \omega_X(X)$, $i=1,2$, we have

(3.1) $$\lim_{n\to\infty} \|(w_n + v_n + x_2) - (\overline{w}_n + x_1)\| = 0$$

By (2.4) it follows that

(3.2) $$\lim_{n\to\infty} \|(w_n - v_n + (-1\circ x_2)) - (\overline{w}_n + (-1\circ x_1))\| = 0$$

By (3.1) and (3.2) we get

(3.3) $$\lim_{n\to\infty} \| 2w_n + x_2 + (-1\circ x_2) - (2\overline{w}_n + x_1 + (-1\circ x_1))\| = 0$$

Let $\widetilde{w}_i = (x_i + (-1\circ x_i))/2 \in W_{\omega_X(X)}$, $i=1,2$. By (3.3) we get $\lim_{n\to\infty} (w_n - \overline{w}_n) = \widetilde{w}_1 - \widetilde{w}_2$, whence using (3.1) it follows that $\lim_{n\to\infty} v_n = \widetilde{w}_2 - \widetilde{w}_1 + x_2 - x_1 \in V_{\omega_X(X)}$, since $V_{\omega_X(X)}$ is closed in E_X . Let $v = \widetilde{w}_2 - \widetilde{w}_1 + x_2 - x_1 \in V_{\omega_X(X)}$. We get $z = x_1 - x_2 = (\widetilde{w}_1 + v) - \widetilde{w}_2 \in E_1$, which completes the proof.

When $V_{\omega_X(X)}$ is not closed in E_X , the conclusion of the above proposition is no longer true (see Example 6.4).

The next result shows that the dual space X^* has a particular form in the class of normed almost linear spaces.

3.3. PROPOSITION. <u>We have</u> $X^* = W_{X^*} + V_{X^*}$, <u>for any normed almost linear space</u> X .

PROOF. Let $f \in X^*$ and define the following two functionals on X ;

$$f_1(x) = (f(x)+f(-1\circ x))/2 \qquad (x \in X)$$

$$f_2(x) = (f(x)-f(-1\circ x))/2 \qquad (x \in X)$$

Then $f_1, f_2 \in X^*$. Clearly $f_1 \in W_{X^*}$, $f_2 \in V_{X^*}$ and we have $f = f_1+f_2$.

Let us note here that in view of Lemma 2.4 (ii), this decomposition of f as $f = f_1+f_2$, $f_1 \in W_{X^*}$, $f_2 \in V_{X^*}$ is unique.

It is immediate that for $X=W_X$ we have $X^* = W_{X^*}$. The converse is not always true (see Example 6.1 (ii)), but we can prove ;

3.4. PROPOSITION. <u>If</u> ω_X <u>is one-to-one and</u> $X^* = W_{X^*}$ <u>then</u> $X = W_X$.

PROOF. Let $x_o \in X \setminus W_X$. Then $\omega_X(x_o) \notin \mathrm{cl}\ (\omega_X(W_X))$ (the closure of $\omega_X(W_X)$ in E_X), since otherwise, by Remark 2.6 and Proposition 3.1, $\omega_X(x_o) \in \omega_X(W_X)$, whence $\omega_X(x_o) = \omega_X(w_o)$, for some $w_o \in W_X$. By the assumption on ω_X it follows $x_o = w_o \in W_X$, a contradiction. Since $\mathrm{cl}\ (\omega_X(W_X))$ is a closed convex cone of E_X and $\omega_X(x_o) \notin \mathrm{cl}\ (\omega_X(W_X))$, by the separation theorem, there exists $\tilde{f} \in S_{E_X^*}$ such that $\tilde{f}(\omega_X(x_o)) < \inf f(\mathrm{cl}\ (\omega_X(W_X)) = 0$. Since $\tilde{f}|\omega_X(W_X) \geqslant 0$, by Theorem 2.7 (iv), $f = \tilde{f}\omega_X \in S_{X^*}$ and we have $f(x_o) = \tilde{f}(\omega_X(x_o)) < 0$. By hypothesis, $f \in W_{X^*}$, whence $f(x_o) = f(-1\circ x_o$ Let $w = x_o + (-1\circ x_o) \in W_X$. We have $f(w) = 2f(x_o) < 0$, a contradiction. Consequently $X = W_X$, which completes the proof.

For the definition of K in the next result, see (2.5).

THEOREM 3.5. <u>We have</u> $K-K = E_X^*$, <u>i.e., the linear space</u> E_{X^*} <u>equals</u> E_X^* .

PROOF. Let $\tilde{f}_o \in E_X^* \setminus (K-K)$, $\|\tilde{f}_o\| = 1$. Then $K \cap (\tilde{f}_o+K) = \emptyset$ and so $\{K \cap B_{E_X^*}\} \cap (\tilde{f}_o+K) = \emptyset$. Since K is w^*-closed in E_X^* and $K \cap B_{E_X^*}$ is w^*-compact in E_X^* , by the separation theorem there exists $z \in E_X$, $\|z\| = 1$ and $\alpha \in R$ such that

$$(3.4) \qquad \sup\{\tilde{f}(z) : \tilde{f} \in K \cap B_{E_X^*}\} < \alpha < \inf\{\tilde{f}_o(z)+\tilde{f}(z) : \tilde{f} \in K\}$$

Clearly, we have $0 < \alpha < \tilde{f}_o(z)$. If $z \in cl\ (\omega_X(W_x))$ then there exists $w \in W_X$, $|||w||| = 1$ such that

$$(3.5) \qquad \| z - \omega_X(w)\| < \tilde{f}_o(z) - \alpha$$

By Theorem 2.5 there exists $f \in S_{X^*}$ such that $f(w) = |||w||| = 1$. Then by Corollary 2.12 (ii), $\tilde{f} = \omega_{X^*}(f) \in K$, $\|\tilde{f}\| = 1$ and $\tilde{f}(\omega_X(w)) = 1$. By (3.5) we get $|\tilde{f}(z)-1| < \tilde{f}_o(z) - \alpha$ and so $\tilde{f}(z) > 1-\tilde{f}_o(z)+\alpha \geqslant \alpha$, which contradicts the left hand inequality in (3.4) since $\tilde{f} \in K \cap B_{E_X^*}$. Consequently $z \notin cl\ (\omega_X(W_X))$. By the separation theorem there exists $\tilde{f} \in S_{E_X^*}$ such that $\tilde{f}(z) < \inf \tilde{f}(cl\ (\omega_X(W_X))) = 0$. Then $\tilde{f}|\omega_X(W_X) \geqslant 0$ and so $\tilde{f} \in K$. Since $\tilde{f}(z) < 0$, this contradicts the right hand inequality in (3.4). Consequently $\tilde{f}_o \in K-K$, which completes the proof.

3.6. PROPOSITION. <u>If</u> $\dim E_X < \infty$ <u>and</u> ω_X <u>is one-to-one then</u> $X^* = X^\#$.

PROOF. Let $f \in X^\#$ and for $z \in E_X$, $z = \omega_X(x_1) - \omega_X(x_2)$,

$x_i \in X$, $i=1,2$, define $\tilde{f}(z) = f(x_1)-f(x_2)$. Then $\tilde{f}$ does not depend on the representation of z, since if $z = \omega_X(x_1)-\omega_X(x_2) = \omega_X(x_3)-\omega_X(x_4)$, $x_i \in X$, $1 \leq i \leq 4$, then $\omega_X(x_1+x_4) = \omega_X(x_2+x_3)$ and by our assumption on ω_X we get $x_1+x_4 = x_2+x_3$. Hence $f(x_1)+f(x_4) = f(x_2)+f(x_3)$, i.e., $\tilde{f}(z) = f(x_1)-f(x_2) = f(x_3)-f(x_4)$ and so $\tilde{f}$ is well defined. Using the properties of ω_X given by Theorem 2.7, it is easy to show that $\tilde{f} \in E_X^{\#} = E_X^{*}$. Consequently for each $x \in X$ we have $|f(x)| = |\tilde{f}(\omega_X(x))| \leq \|\tilde{f}\| \|\omega_X(x)\| = \|\tilde{f}\| |||x|||$, which proves that $f \in X^*$.

The assumption on ω_X to be one-to-one is essential in the above result (see Example 6.2 (iii)).

3.7. PROPOSITION. <u>The mapping</u> ω_X <u>is one-to-one iff</u> $Q_X : X \longrightarrow X^{**}$ <u>is one-to-one</u>.

PROOF. Suppose ω_X one-to-one and let $x,y \in X$ such that $Q_X(x) = Q_X(y)$. Let $z = \omega_X(x)-\omega_X(y) \in E_X$. We claim that $z \in \mathrm{cl}\,(\omega_X(X))$. Indeed, if $z \notin \mathrm{cl}\,(\omega_X(X))$, by the separation theorem there exists $\tilde{f} \in S_{E_X^*}$ such that $\tilde{f}(z) < \inf \tilde{f}(\mathrm{cl}\,(\omega_X(X)) = 0$. In particular, $\tilde{f}\,|\omega_X(W_X)) \geq 0$ whence $\tilde{f} \in K$. By Corollary 2.12 (ii), $f = \tilde{f}\omega_X \in X^*$ and so we have $\tilde{f}(z) = \tilde{f}(\omega_X(x)-\omega_X(y)) = \tilde{f}(\omega_X(x))-\tilde{f}(\omega_X(y)) = f(x)-f(y) < 0$, i.e., $Q_X(x) \neq Q_X(y)$, a contradiction. Consequently $z \in \mathrm{cl}\,(\omega_X(X))$ and so there exists a sequence $\{x_n\} \subset X$ such that $\lim_{n\to\infty} \|\omega_X(x_n)-z\| = 0$. Let $f_n \in S_{X^*}$, $f_n(x_n) = |||x_n|||$, $n \in N$, be given by Theorem 2.5 and let $\tilde{f}_n = \omega_{X^*}(f_n) \in K$. Then $f_n = \tilde{f}_n\omega_X$ (by Corollary 2.12 (ii)). By $Q_X(x) = Q_X(y)$, for each $n \in N$ we get that $f_n(x) = f_n(y)$, hence $\tilde{f}_n(\omega_X(x)) = \tilde{f}_n(\omega_X(y))$, whence $f_n(z) = 0$. We have

$$\|\omega_X(x_n)\| = |||x_n||| = f_n(x_n) = \tilde{f}_n(\omega_X(x_n)) = \tilde{f}_n(\omega_X(x_n)-z) \leq \|\omega_X(x_n)-z\| \longrightarrow 0$$

Thus, $\|z\| = \lim_{n\to\infty}\|\omega_X(x_n)\| = 0$ and so $\omega_X(x) = \omega_X(y)$. Since ω_X is one-to-one, we get $x=y$.

Conversely, suppose Q_X one-to-one and let $x,y\in X$ such that $\omega_X(x) = \omega_X(y)$. Let $f\in X^*$ and let $\tilde{f} = \omega_{X^*}(f)\in K$. Then $f = \tilde{f}\omega_X$ and we have $f(x) = \tilde{f}(\omega_X(x)) = \tilde{f}(\omega_X(y)) = f(y)$. Consequently $Q_X(x) = Q_X(y)$ and since Q_X is one-to-one, we get $x=y$, which completes the proof.

3.8. THEOREM. Let $X = W_X$ be such that $\omega_X(X)$ is closed in E_X and $\dim E_X < \infty$. Then Q_X is onto X^{**}.

PROOF. Using twice Theorem 3.5 we get that the linear space $E_{X^{**}}$ equals E_X^{**}. Let $F\in X^{**}\setminus\{0\}$ and let $\tilde{F} = \omega_{X^{**}}(F)\in E_X^{**}\setminus\{0\}$. Then $F = \tilde{F}\omega_{X^*}$ and $\tilde{F} = Q_{E_X}(z)$ for some $z\in E_X\setminus\{0\}$. If $z\in\omega_X(X)$ then $z = \omega_X(x)$ for some $x\in X$. For each $f\in X^*$ we have $F(f) = \tilde{F}(\omega_{X^*}(f)) = \omega_{X^*}(f)(\omega_X(x)) = f(x)$, i.e., $F = Q_X(x)$. We show now that the case $z\notin\omega_X(X)$ is not possible, which will complete the proof. Suppose $z\notin\omega_X(X) = \mathrm{cl}\,(\omega_X(X))$. Then there exists $\tilde{f}_o\in E_X^*\setminus\{0\}$ such that $\tilde{f}_o(z) < \inf \tilde{f}_o(\omega_X(X)) = 0$. Since $\tilde{f}_o|\omega_X(X)\geq 0$, it follows that $f_o = \tilde{f}_o\omega_X\in X^*$. Suppose $z = \omega_X(x)-\omega_X(y)$ for some $x,y\in X$. Then

$$(3.6)\qquad \tilde{f}_o(z) = \tilde{f}_o(\omega_X(x))-\tilde{f}_o(\omega_X(y)) = f_o(x)-f_o(y) < 0$$

On the other hand since $X = W_X$, we have $X^* = W_{X^*}$ and so $f_o\in W_{X^*}$. Consequently, $0\leq F(f_o) = \tilde{F}(\omega_{X^*}(f_o)) = \omega_{X^*}(f_o)(z) = \omega_{X^*}(f_o)(\omega_X(x))-\omega_{X^*}(f_o)(\omega_X(y)) = f_o(x)-f_o(y)$, which contradicts (3.6).

If all the assumptions in Theorem 3.8 hold, except for $X = W_X$ then the conclusion is not always true (see Example 6.5 (ii)).

4. NORMED ALMOST LINEAR SPACES WITH BASES

The normed almost linear spaces X with bases have some relevant properties and this section is devoted to them. In the proofs of Theorem 4.1 and Lemma 4.5 we shall use the following notation. If X has a basis $\mathcal{B}$, for $x \in X$ and $b \in \mathcal{B}$ we denote by $x(b)$ the element of X defined as follows. Suppose $x = \sum_{i=1}^{n} \lambda_i \circ b_i$, $b_i \in \mathcal{B}, \lambda_i \neq 0, \lambda_i > 0$ for $b_i \notin V_X$ be the unique decomposition of x by the elements of $\mathcal{B}$. We set:

$$x(b) = \begin{cases} x & \text{if } b \notin \{b_1,\dots,b_n\} \\ \sum_{\substack{i=1 \\ i \neq i_o}}^{n} \lambda_i \circ b_i & \text{if } b = b_{i_o} \end{cases}$$

Consequently, if $b \in \mathcal{B} \setminus V_X$ then $x = \lambda \circ b + x(b)$, $\lambda \in R_+$.

4.1. THEOREM. <u>If the normed almost linear space</u> X <u>has a basis</u> $\mathcal{B}$ <u>then</u> ω_X <u>is one-to-one</u>.

PROOF. Let $x_1, x_2 \in X$, $x_1 \neq x_2$ such that $\omega_X(x_1) = \omega_X(x_2)$. Then $x_i = \sum_{j=1}^{k} \alpha_{ij} \circ b_j + \sum_{j=k+1}^{n} \alpha_{ij} \circ b_j$, where for $1 \leq j \leq k$ we have $b_j \in \mathcal{B} \setminus V_X$, $\alpha_{ij} \geq 0$, $i=1,2$ and for $k+1 \leq j \leq n$ we have $b_j \in \mathcal{B} \cap V_X$, $\alpha_{ij} \in R$, $i=1,2$. Since $x_1 \neq x_2$, $\omega_X(x_1) = \omega_X(x_2)$ and $\omega_X|V_X$ is one-to-one (by Remark 2.6), it follows that there exists an index j_o, $1 \leq j_o \leq k$ such that $\alpha_{1j_o} \neq \alpha_{2j_o}$. Without loss of generality we can suppose $j_o=1$ and $\alpha_{11} > \alpha_{21}$. Since $\omega_X(x_1) = \omega_X(x_2)$, by Lemma 2.11 and Remark 2.1 (iii), for each $n \in N$ there exist $y_n, u_n \in X$ such that:

(4.1) $x_1+u_n = x_2+y_n$

(4.2) $|||y_n||| + |||u_n||| < \frac{1}{n}$

By (4.1) we get

$$\alpha_{11}\circ b_1+x_1(b_1)+\lambda_n\circ b_1+u_n(b_1) = \alpha_{21}\circ b_1+x_2(b_1)+\mu_n\circ b_1+y_n(b_1)$$

where $\lambda_n, \mu_n \in R_+$, and so

(4.3) $\mu_n > \alpha_{11}-\alpha_{21} > 0 \qquad (n \in N)$

Using (4.2) we get

$$\mu_n |||b_1+(-1\circ b_1)||| \leq |||\mu_n\circ(b_1+(-1\circ b_1))+y_n(b_1)+(-1\circ y_n(b_1))||| = \\ = |||y_n+(-1\circ y_n)||| < \frac{2}{n} \qquad (n \in N)$$

which contradicts (4.3) since $b_1 \notin V_X$. Consequently $x_1=x_2$, which completes the proof.

An immediate consequence of this result is the following:

4.2. REMARK. If X has a basis $\mathcal{B}$ then $\omega_X(\mathcal{B}) = \{\omega_X(b);\ b\in\mathcal{B}\}$ is a basis of the almost linear space $\omega_X(X)$. In Example 6.2 (i), X has no basis but $\omega_X(X)$ has a basis. Clearly, when ω_X is one-to-one then X has a basis iff $\omega_X(X)$ has a basis.

4.3. LEMMA. <u>Let</u> X <u>be a normed almost linear space. If</u> $\omega_X(X)$ <u>has a basis</u> $\mathcal{B}$, <u>then</u> $\mathcal{B}$ <u>is a basis of</u> E_X .

PROOF. We show that $\mathcal{B}$ is a linearly independent system in

E_X, whence the result follows since $E_X = \omega_X(X) - \omega_X(X)$. Let $b_1, \dots, b_n \in \mathcal{B}$ and $\alpha_1, \dots, \alpha_n \in R$ such that $\sum_{i=1}^{n} \alpha_i b_i = 0$. Let $I_1 = \{ i ; 1 \leq i \leq n, \alpha_i > 0 \}$ and $I_2 = \{ i ; 1 \leq i \leq n, \alpha_i < 0 \}$ and we must show that both I_1 and I_2 are empty. Without loss of generality, we can suppose $I_1 \neq \emptyset$. If $I_2 = \emptyset$, then $\sum_{i \in I_1} \alpha_i \circ b_i = \sum_{i \in I_1} \alpha_i b_i = 0$ and since $\omega_X(X)$ is a normed almost linear space, by Lemma 2.4 (i) we get that for $i \in I_1$ we have $b_i \in V_{\omega_X(X)}$ whence $\{ b_i ; i \in I_1 \}$ is a linearly dependent system in the linear space $V_{\omega_X(X)}$, which contradicts Remark 2.1 (ii). Consequently $I_2 \neq \emptyset$ and so we have $\sum_{i \in I_1} \alpha_i b_i = \sum_{i \in I_2} |\alpha_i| b_i$ whence $\sum_{i \in I_1} \alpha_i \circ b_i = \sum_{i \in I_2} |\alpha_i| \circ b_i \in \omega_X(X)$. Since $I_1 \cap I_2 = \emptyset$ this relation shows that $\mathcal{B}$ is not a basis of $\omega_X(X)$. Consequently $I_1 = I_2 = \emptyset$, which completes the proof.

4.4. COROLLARY. <u>If $\mathcal{B}$ and $\mathcal{B}'$ are two bases of X, then</u> card $\mathcal{B}$ = card $\mathcal{B}'$.

PROOF. By Remark 4.2 and Lemma 4.3, both $\omega_X(\mathcal{B})$ and $\omega_X(\mathcal{B}')$ are bases of the linear space E_X. Hence (/1/) card $\omega_X(\mathcal{B}) =$ card $\omega_X(\mathcal{B}')$. By Theorem 4.1 the conclusion follows.

We shall prove now a technical lemma which will be used in the proof of Theorem 4.6 and Proposition 4.8.

4.5. LEMMA. <u>Let</u> Y <u>be an almost linear subspace of</u> $\omega_X(X)$ <u>such that</u> Y <u>has a basis</u> $\mathcal{B}$. <u>Let</u> $\{u_n\}$, $\{x_n\}$, $\{y_n\}$ <u>be sequences of</u> Y <u>and let</u> $x, y \in Y$. <u>If the following relations hold</u>:

(4.4) $$u_n+y_n+y = x_n+x \qquad (n\in N)$$

(4.5) $$\|x_n\| + \|y_n\| \longrightarrow 0$$

<u>then the element</u> $x-y$ <u>of</u> E_X <u>belongs to</u> Y.

PROOF. Clearly, if $y\in V_Y$ ($\subset V_{\omega_X(X)}$) then $x-y\in Y$. If $y\notin V_Y$ then $y = \sum_{i=1}^{k} \lambda_i \circ b_i+v$ where $k\geq 1$, $b_1,\ldots,b_k\in \mathcal{B}\setminus V_Y$, $\lambda_i > 0$, $1\leq i\leq k$ and $v\in V_Y$. For each i, $1\leq i\leq k$, we have

$$\begin{array}{ll} u_n = \mu_{ni}\circ b_i+u_n(b_i), & \mu_{ni}\in R_+,\ n\in N \\ x_n = \alpha_{ni}\circ b_i+x_n(b_i), & \alpha_{ni}\in R_+,\ n\in N \\ y_n = \beta_{ni}\circ b_i+y_n(b_i), & \beta_{ni}\in R_+,\ n\in N \\ x = \nu_i\circ b_i+x(b_i) & \nu_i\in R_+ \end{array}$$

Using (4.5) we get

$$\alpha_{ni}\|b_i+(-1\circ b_i)\| \leq \|\alpha_{ni}\circ(b_i+(-1\circ b_i))+x_n(b_i)+(-1\circ x_n(b_i))\| = $$
$$= \|x_n+(-1\circ x_n)\| \longrightarrow 0$$

whence, since $b_i\notin V_Y$, $1\leq i\leq k$, it follows $\lim_{n\to\infty}\alpha_{ni} = 0$. Similarly $\lim_{n\to\infty}\beta_{ni} = 0$. By (4.4) we get

$$\mu_{ni}+\beta_{ni}+\lambda_i = \alpha_{ni}+\nu_i \qquad (n\in N)$$

and so

$$\lim_{n\to\infty}\mu_{ni} = \nu_i-\lambda_i \geq 0 \qquad (1\leq i\leq k)$$

Consequently $x = \sum_{i=1}^{k}\nu_i\circ b_i+x_1$, where $\nu_i\geq\lambda_i > 0$ for $1\leq i\leq k$ and $x_1\in Y$. Then $x-y = \sum_{i=1}^{k}(\nu_i-\lambda_i)\circ b_i+x_1-v\in Y$, since $\nu_i-\lambda_i\geq 0$,

$1 \leq i \leq k$ and $v \in V_Y$.

4.6. THEOREM. *If $\omega_X(X)$ has a basis (in particular, if X has a basis) then $\omega_X(X)$ is norm-closed in E_X .*

PROOF. Let $\{u_n\}_{n=1}^{\infty} \subset \omega_X(X)$ and $z \in E_X$, say, $z = x-y$, $x, y \in \omega_X(X)$, such that $\lim_{n\to\infty} \|u_n - z\| = \lim_{n\to\infty} \|u_n+y-x\| = 0$. By Theorem 2.7, for each $n \in N$ there exist $x_n, y_n \in \omega_X(X)$ such that

$$u_n+y-x = x_n-y_n \qquad (n \in N)$$

$$\|x_n\| + \|y_n\| \leq \frac{1}{n} + \|u_n+y-x\| \qquad (n \in N)$$

By Lemma 4.5 it follows that $z = x-y \in \omega_X(X)$, which completes the proof.

4.7. COROLLARY. *Let X be a normed almost linear space with a basis. Then X is complete iff E_X is a Banach space.*

PROOF. Use Lemma 2.10 and Theorem 4.6.

In view of Theorems 4.1 and 4.6, one can ask whether the conditions ω_X one-to-one and $\omega_X(X)$ norm-closed in E_X are sufficient for X to have a basis. The answer is in the negative (see Example 6.6 (i)).

As follows by Proposition 3.1, the set $W_{\omega_X(X)}$ is always norm-closed in $\omega_X(X)$. Example 6.3 (ii) shows that $W_{\omega_X(X)}$ is not always closed in E_X . We have:

4.8. PROPOSITION. *If $W_{\omega_X(X)}$ has a basis, then $W_{\omega_X(X)}$*

<u>is norm-closed in</u> E_X .

PROOF. Let $\{w_n\}_{n=1}^{\infty} \subset W_{\omega_X(X)}$ and $z = x-y \in E_X$, $x,y \in \omega_X(X)$ be such that $\lim_{n\to\infty} \| w_n+y-x \| = 0$. By Theorem 2.7, for each $n \in N$ there exist $x_n, y_n \in \omega_X(X)$ such that

$$w_n+y+y_n = x+x_n \qquad (n \in N) \tag{4.6}$$

$$\|x_n\| + \|y_n\| \leq \|w_n+y-x\| + \frac{1}{n} \qquad (n \in N) \tag{4.7}$$

By (4.6) we get

$$w_n+(-1\circ y)+(-1\circ y_n) = -1\circ x+(-1\circ x_n) \qquad (n \in N) \tag{4.8}$$

If we add the relations (4.6) and (4.8) we get

$$2w_n+y+(-1\circ y)+y_n+(-1\circ y_n) = x+(-1\circ x)+x_n+(-1\circ x_n) \qquad (n \in N) \tag{4.9}$$

Let us put

$$w' = (x+(-1\circ x))/2 \in W_{\omega_X(X)}$$

$$w'' = (y+(-1\circ y))/2 \in W_{\omega_X(X)}$$

$$\overline{w}_n = (x_n+(-1\circ x_n))/2 \in W_{\omega_X(X)} \qquad (n \in N).$$

$$\widetilde{w}_n = (y_n+(-1\circ y_n))/2 \in W_{\omega_X(X)} \qquad (n \in N)$$

Then by (4.9) we have

$$w_n+w''+\widetilde{w}_n = w'+\overline{w}_n \qquad (n \in N) \tag{4.10}$$

and by (4.7) we get

$$(4.11) \qquad \|\bar{w}_n\| + \|\tilde{w}_n\| \leq \|x_n\| + \|y_n\| \leq \|w_n+y-x\| + \frac{1}{n} \qquad (n \in N)$$

By (4.10) and (4.11) using Lemma 4.5 we get $w'-w'' \in W_{\omega_X}(X)$. Using again (4.10) and (4.11) we obtain

$$z = \lim_{n\to\infty} w_n = w'-w'' \in W_{\omega_X}(X)$$

which completes the proof.

4.9. PROPOSITION. <u>If</u> $W_{\omega_X}(X)$ <u>has a basis then the set</u> $W_{\omega_X}(X)+V_{\omega_X}(X)$ <u>is</u> $\|\cdot\|_{E_X}$<u>-closed in the set</u> $E_1 = (W_{\omega_X}(X)+V_{\omega_X}(X))-(W_{\omega_X}(X)+V_{\omega_X}(X))$.

PROOF. We shall denote as usual $\|\cdot\|_{E_X}$ by $\|\cdot\|$. Let $\{w_n\}_{n=1}^{\infty} \subset W_{\omega_X}(X)$, $\{v_n\}_{n=1}^{\infty} \subset V_{\omega_X}(X)$ and $z \in E_1$, say, $z=w+v-\bar{w}$, where $w,\bar{w} \in W_{\omega_X}(X)$ and $v \in V_{\omega_X}(X)$, such that

$$(4.12) \qquad \lim_{n\to\infty} \|w_n+v_n-z\| = \lim_{n\to\infty} \|w_n+v_n+\bar{w}-w-v\| = 0$$

By Corollary 2.9 we also have

$$(4.13) \qquad \lim_{n\to\infty} \|w_n-v_n+\bar{w}-w+v\| = 0$$

By (4.12) and (4.13) we get $\lim_{n\to\infty} \|w_n+\bar{w}-w\| = 0$, whence by Proposition 4.8 it follows that $w-\bar{w} \in W_{\omega_X}(X)$. Consequently $z = w-\bar{w}+v \in W_{\omega_X}(X)+V_{\omega_X}(X)$, which completes the proof.

Clearly, in view of Remark 2.3 (i) and Remark 4.2, the conclusions of Propositions 4.8 and 4.9 are always true in a normed almost linear space X with a basis.

An immediate consequence of Propositions 3.2 and 4.9 is the following:

4.10. COROLLARY. If $W_{\omega_X}(X)$ has a basis and $V_{\omega_X}(X)$ is closed in E_X, then $W_{\omega_X}(X)+V_{\omega_X}(X)$ is closed in E_X.

In particular, the conclusion of the above corollary holds when X has a basis and V_X is complete.

It is well known(/1/) that when X is a linear space, Y a linear subspace of X and $\mathcal{B}_o$ a basis of Y, then there exists a basis $\mathcal{B}$ of X such that $\mathcal{B}_o \subset \mathcal{B}$. This result is no longer true if we replace "linear" by "almost linear" even when X is a normed almost linear space with a basis (see Example 6.5 (iii)). A generalization of the above result is given in the next remark.

4.11. REMARK. Suppose X has a basis $\mathcal{B}$. If Y is a linear subspace of V_X and $\mathcal{B}_o$ is a basis of Y, then there exists a basis $\mathcal{B}'$ of X such that $\mathcal{B}_o \subset \mathcal{B}'$. The proof is obvious since $\mathcal{B}' = (\mathcal{B} \setminus V_X) \cup \mathcal{B}_1$, where $\mathcal{B}_1$ is a basis of V_X such that $\mathcal{B}_o \subset \mathcal{B}_1$.

4.12. REMARK. Suppose X has a basis $\mathcal{B}$.

(i) If Y is an almost linear subspace of X then Y has not necessarily a basis. This may happen even when Y is closed in X (see Example 6.6 (i)).

(ii) If $\mathcal{B}_1$ is a subset of $\mathcal{B}$ then $\mathcal{B}_1$ is not necessarily a basis of the almost linear subspace of X generated by $\mathcal{B}_1$ (see Example 6.5 (iv)).

5. FINITE-DIMENSIONAL NORMED ALMOST LINEAR SPACES

As in the case of a linear space, When X is a normed almost linear space with a basis $\mathcal{B}$ then card $\mathcal{B}$ is called the dimension of X and denoted by dim X. In view of Corollary 4.4 this is well defined. When $\dim X < \infty$ then X has some relevant properties as we shall see below. Firstly we show that the following generalization of Riesz's Theorem holds.

5.1. THEOREM. <u>Suppose</u> X <u>has a basis. We have</u> $\dim X < \infty$ <u>iff</u> B_X <u>is compact</u>.

PROOF. Suppose $\dim X < \infty$. By Remark 4.2 and Lemma 4.3 it follows that $\dim E_X < \infty$. Hence B_{E_X} is compact. By Theorem 4.6 it follows that $B_{\omega_X(X)}$ is compact, whence by Theorem 4.1, B_X is compact.

Conversely, suppose B_X compact. If we show that B_{E_X} is compact then $\dim E_X < \infty$ and since X has a basis, by Remark 4.2 and Lemma 4.3 it follows that $\dim X = \dim E_X < \infty$. Let $z_n \in B_{E_X}$, $n \in N$, and let $\varepsilon > 0$. Then by (2.1), for each $n \in N$ there exist $x_{1n}, x_{2n} \in X$ such that $z_n = \omega_X(x_{1n}) - \omega_X(x_{2n})$ and $|||x_{1n}||| + |||x_{2n}||| \leq 1+\varepsilon$. Since $B_X(0,1+\varepsilon)$ is compact, there exist subsequences $\{x_{in_k}\}_{k=1}^{\infty} \subset \{x_{in}\}_{n=1}^{\infty}$ and $x_i \in B_X(0,1+\varepsilon)$ such that $\lim_{k\to\infty} \rho_X(x_{in_k}, x_i) = 0$, $i=1,2$. Let $z = \omega_X(x_1) - \omega_X(x_2) \in E_X$. We have

$$\| z_{n_k} - z \| = \| \omega_X(x_{1n_k}) - \omega_X(x_{2n_k}) - \omega_X(x_1) + \omega_X(x_2) \| \leq$$

$$X(x_{1n_k}) - \quad (x_1) \quad + \quad X(x_{2n_k}) - \quad X(x_2) \quad =$$

$$\le \|\omega_X(x_{1n_k}) - \omega_X(x_1)\| + \|\omega_X(x_{2n_k}) - \omega_X(x_2)\| =$$
$$= \rho_X(x_{1n_k}, x_1) + \rho_X(x_{2n_k}, x_2) \longrightarrow 0 \quad \text{as } k \to \infty,$$

and so $\|z_{n_k} - z\| \to 0$ and $\|z\| \le 1$, which completes the proof.

The assumption on X to have a basis is essential for the implication B_X compact $\Rightarrow$ dim $X < \infty$ (see Example 6.6 (ii) ; note that in this example ω_X is one-to-one and $\omega_X(X)$ is closed in E_X).

THEOREM 5.2. _If_ dim X = n _then_ dim $X^* = n$.

PROOF. Let $\mathcal{B}$ be a basis of X, card $\mathcal{B} = n$. By Theorem 2.2 we can suppose that for $b \in \mathcal{B} \setminus V_X$ we have $-1 \circ b \in \mathcal{B}$. For each $b \in \mathcal{B}$ we shall define a functional $f_b \in X^*$ and we shall show that $\{f_b : b \in \mathcal{B}\}$ is a basis of X^*. By Theorem 4.1, Remark 4.2 and Lemma 4.3 we know that ω_X is one-to-one and dim $E_X = n$, whence by Proposition 3.6 it is enough to define f_b on $\mathcal{B}$ in such a way that $f_b(\bar{b} + (-1 \circ \bar{b})) \geqslant 0$ for each $\bar{b} \in \mathcal{B} \setminus V_X$ (since then by Remark 2.3 (ii), $f_b | W_X \geqslant 0$). For $b \in \mathcal{B}$ we define $f_b \in X^*$ in the following way. If $b \in \mathcal{B} \cap (W_X \cup V_X)$ and $\bar{b} \in \mathcal{B}$, define

$$f_b(\bar{b}) = \begin{cases} 1 & \text{if } \bar{b} = b \\ 0 & \text{if } \bar{b} \in \mathcal{B} \setminus \{b\} \end{cases}$$

If $\mathcal{B} \setminus \{W_X \cup V_X\} = \{b_1, -1 \circ b_1, \ldots, b_k, -1 \circ b_k\}$, $k \ge 1$, then for $1 \le i \le k$ define

$$f_{b_i}(b) = \begin{cases} 1 & \text{if } \bar{b} = b_i \text{ or } \bar{b} = -1 \circ b_i \\ 0 & \text{if } \bar{b} \in \mathcal{B} \setminus \{b_i, -1 \circ b_i\} \end{cases}$$

and

$$f_{-1\circ b_i}(\bar{b}) = \begin{cases} 1 & \text{if } \bar{b} = b_i \\ -1 & \text{if } \bar{b} = -1\circ b_i \\ 0 & \text{if } \bar{b} \in \mathcal{B} \setminus \{ b_i, -1\circ b_i \} \end{cases}$$

Clearly, for each $b \in \mathcal{B}$ we have $f_b(\bar{b}+(-1\circ\bar{b})) \geq 0$ for $\bar{b} \in \mathcal{B} \setminus V_X$. Note that for $b \in \{b_1,\dots,b_k\} \cup (\mathcal{B} \cap W_X)$ we have $f_b \in W_{X^*}$ and for $b \in \{-1\circ b_1,\dots,-1\circ b_k\} \cup (\mathcal{B} \cap V_X)$ we have $f_b \in V_{X^*}$. We show now that $\{f_b : b \in \mathcal{B}\}$ is a basis of X^*. We suppose that $k \geq 1$, $\mathcal{B} \cap W_X \neq \emptyset$ and $\mathcal{B} \cap V_X \neq \emptyset$, the other cases being simpler. Let $f \in X^*$ and let us put

$$\begin{array}{ll} \alpha_i = (f(b_i)+f(-1\circ b_i))/2 & 1 \leq i \leq k \\ \beta_b = f(b) & b \in \mathcal{B}\cap W_X \\ \gamma_i = (f(b_i)-f(-1\circ b_i))/2 & 1 \leq i \leq k \\ \delta_b = f(b) & b \in \mathcal{B}\cap V_X \end{array}$$

Since $f \in X^*$ we have $\alpha_i \geq 0$, $1 \leq i \leq k$ and $\beta_b \geq 0$ for $b \in \mathcal{B}\cap W_X$. It is easy to show that

$$f = \sum_{i=1}^{k} \alpha_i \circ f_{b_i} + \sum_{b \in \mathcal{B}\cap W_X} \beta_b \circ f_b + \sum_{i=1}^{k} \gamma_i \circ f_{-1\circ b_i} + \sum_{b \in \mathcal{B}\cap V_X} \delta_b \circ f_b$$

and that this representation is unique. This completes the proof since card $\{f_b : b \in \mathcal{B}\} = n$.

The following more general question whether the condition X has a basis implies that X^* has a basis has in general a negative answer (see Example 6.8). Another question is whether the converse in Theorem 5.2 is true. Even when $X \equiv W_X$ the answer is in the negative as shown by Examples 6.2 (ii) and 6.3 (iii). In these examples we have either ω_X is not one-to-one or $\omega_X(X)$ is not closed in E_X, which by Theorems 4.1 and 4.6 are necessary condition for X to have a basis. So, the next question is whether the conditions dim $X^* = n$, ω_X one-to-one and $\omega_X(X)$ closed in E_X imply that dim $X = n$. The answer is also in the negative (see Example 6.7) It is in the affirmative when $X = W_X$ as we shall see below. For the proof of this result we need the following remark.

5.3. REMARK. If dim $X^* = n$ then dim $E_X = n$. Indeed, by Remark 4.2 and Lemma 4.3 it follows dim $E_{X^*} = n$, whence by Theorem 3.5 we get dim $E_X^* = n$, whence the conclusion follows.

5.4. PROPOSITION. *If* $X = W_X$ *is such that* ω_X *is one-to-one,* $\omega_X(X)$ *is closed in* E_X *and* dim $X^* = n$ *then* dim $X = n$.

PROOF. Since $X = W_X$ we have $X^{**} = W_{X^{**}}$. By the assumption dim $X^* = n$ and Theorem 5.2 we have dim $X^{**} = n$. Let $\{F_1, \ldots, F_n\}$ be a basis of X^{**}. By Remark 5.3 it follows that dim $E_X = n$, whence by Theorem 3.8 we get that Q_X is onto X^{**}. Let $\{b_1, \ldots, b_n\} \subset X = W_X$ be such that $F_i = Q_X(b_i)$, $i=1,\ldots,n$. We show that $\{b_1, \ldots, b_n\}$ is a basis of X. Let $x \in X$. Then $Q_X(x) = \sum_{i=1}^n \lambda_i \circ F_i$, $\lambda_i \geq 0$, and so by Proposition 2.13 we have $Q_X(x) = Q_X(\sum_{i=1}^n \lambda_i \circ b_i)$. By Proposition 3.7, Q_X is one-to-one and so $x = \sum_{i=1}^n \lambda_i \circ b_i$, $\lambda_i \geq 0$, $1 \leq i \leq n$. Suppose now that $x = \sum_{i=1}^n \lambda_i \circ b_i = \sum_{i=1}^n \mu_i \circ b_i$, $\lambda_i, \mu_i \in R_+$, $1 \leq i \leq n$. Then $Q_X(x) = \sum_{i=1}^n \lambda_i \circ F_i = \sum_{i=1}^n \mu_i \circ F_i$, whence $\lambda_i = \mu_i$, $1 \leq i \leq n$, i.e., $\{b_1, \ldots, b_n\}$ is a basis of X.

We conclude this section with the following generalization of the well known result from the theory of normed linear spaces that any finite dimensional normed linear space is complete.

5.5. PROPOSITION. If dim X = n then X is complete.

PROOF. By Remark 4.2 and Lemma 4.3 we get dim $E_X = n$, whence the conclusion follows by Corollary 4.7.

6. EXAMPLES

6.1. EXAMPLE. Let M be a normed linear space and let X be the collection of all nonempty, bounded and convex subsets A of M. For $A_1, A_2 \in X$ and $\lambda \in R$ define $A_1+A_2 = \{ a_1+a_2 ; a_1 \in A_1, a_2 \in A_2 \}$ and $\lambda \circ A_1 = \{ \lambda a_1 ; a_1 \in A_1 \}$. Let $0 \in X$ be the set $\{0\}$. Then X is an almost linear space and we have $W_X = \{ A \in X ; A = -1 \circ A \}$ (i.e., A is symmetric with respect to the origin of M) and $V_X = \{ \{x\} ; x \in M \}$. For $A \in X$ define $|||A||| = \sup_{a \in A} \|a\|$. Then X together with this norm is a normed almost linear space.

(i) Let M=R and X defined as above. Let

$Y = \{ [-\lambda, \lambda), (-\lambda, \lambda], (-\lambda ; \lambda), [-\lambda, \lambda] ; \lambda > 0 \} \cup \{0\}$.

Then Y is an almost linear subspace of X. We have $E_Y = R$ equipped with the usual norm and for $\lambda > 0$ let A_λ be any of the intervals $[-\lambda, \lambda), (-\lambda, \lambda], (-\lambda, \lambda), [-\lambda, \lambda]$. Then $\omega_Y(A_\lambda) = \lambda$ and $\omega_Y(\{0\}) = 0$. Clearly, ω_Y is not one-to-one. Let for $n \in N$, $w_n = (-1+\frac{1}{n}, 1-\frac{1}{n}) \in W_Y$ and $x = (-1, 1] \in Y \setminus W_Y$. We have $\lim_{n \to \infty} \rho(w_n, x) = \lim_{n \to \infty} \| \omega_Y(w_n) - \omega_Y(x) \| = \lim_{n \to \infty} | 1 - \frac{1}{n} - 1 | = 0$

(ii) Let Y be the normed almost linear space described in (i). We have $W_Y = \{ (-\lambda, \lambda), [-\lambda, \lambda] ; \lambda > 0 \} \cup \{0\}$, hence $Y \neq W_Y$. Since $\omega_Y(Y) = W_{\omega_Y(Y)}$ we have $Y^* = W_{Y^*}$. Here ω_Y is not one-to-one

6.2. EXAMPLE. Let $X = \{(\alpha,\beta)\in R^2 ;\ \beta > 0\} \cup \{(0,0)\}$. Define the addition as in R^2 and for $x = (\alpha,\beta)\in X$ and $\lambda \in R$ define $\lambda\circ x = (|\lambda|\alpha, |\lambda|\beta)$. Then X is an almost linear space such that $X = W_X$. For $(\alpha,\beta)\in X$ define $|||(\alpha,\beta)||| = \beta$. Then X is a normed almost linear space. We have $E_X = R$ equipped with the usual norm, $\omega_X((\alpha,\beta)) = \beta$, $(\alpha,\beta)\in X$ and $\omega_X(X) = R_+$. Here ω_X is not one-to-one but $\omega_X(X)$ is closed in E_X.

(i) The normed almost linear space $\omega_X(X)$ has a basis but X has no basis.

(ii) We have $X^* = \{\lambda\circ f_0 ;\ \lambda\in R\}$, where $f_0((\alpha,\beta)) = \beta$, $(\alpha,\beta)\in X$. Clearly $\{f_0\}$ is a basis of X^* and so $\dim X^* = 1$ but X has no basis.

(iii) Let $Y = \{(\alpha,\beta)\in X ;\ \alpha \geq 0\}$. Then Y is an almost linear subspace of the normed almost linear space X. Let $f: Y \to R$ be defined by $f((\alpha,\beta)) = \alpha+\beta$, $(\alpha,\beta)\in Y$. We have $f\in Y^{\#}\setminus Y^*$.

6.3. EXAMPLE. Let $X = \{(\alpha,\beta)\in R^2 ;\ \alpha \geq 0,\ \beta > 0\} \cup \{(0,0)\}$ We organize X as an almost linear space as in Example 6.2 and so we have $X=W_X$. For $x = (\alpha,\beta)\in X$ we define $|||x||| = \max\{\alpha,\beta\}$. Then X is a normed almost linear space.

(i) We have $E_X=R^2$, the unit ball of E_X is the hexagon which is the convex hull of the set $\{(1,0),(1,1),(0,1),(-1,0),(-1,-1),(0,-1)\}$ and $\omega_X((\alpha,\beta)) = (\alpha,\beta)$, $(\alpha,\beta)\in X$. We have $X^* = W_{X^*} = \{(\gamma,\delta)\in R^2 ;\ \gamma,\delta\in R_+\}$ and for $f = (\gamma,\delta)\in X^*$ we have $|||f||| = \gamma+\delta$. Here $E_{X^*} = R^2$ and $\|(\gamma,\delta)\|_{E_X} = \gamma+\delta$. Consequently the unit ball of E_{X^*} is a square while the unit ball of E_X is a hexagon, i.e., $\|\cdot\|_{E_{X^*}} \neq \|\cdot\|_{E_X^*}$.

(ii) The set $\omega_X(W_X)(=\omega_X(X))$ is not closed in E_X. Here ω_X is one-to-one.

(iii) Clearly, $X=W_X$ has no basis. On the other hand a basis of X^* is the set $\{(1,0),(0,1)\}$, i.e., $\dim X^* = 2$. As we have observed at (ii), ω_X is one-to-one but $\omega_X(X)$ is not closed in E_X.

6.4. EXAMPLE. Let $X = \{(\alpha_i)\in \ell^1;\ \alpha_1 > 0\} \cup \{(\alpha_i)\in \ell^1; \alpha_1=0, \sum_{i=2}^{\infty} i\alpha_i = 0\}$. Define the addition as in ℓ^1 and for $x = (\alpha_i)\in X$ and $\lambda \in R$ define $\lambda \circ x = (|\lambda|\alpha_1, \lambda\alpha_2, \ldots, \lambda\alpha_n, \ldots)$. Then X is an almost linear space and we have $W_X = \{(\lambda,0,0,0,\ldots): \lambda\in R_+\}$ and $V_X = \{(\alpha_i)\in \ell^1;\ \alpha_1=0, \sum_{i=2}^{\infty} i\alpha_i=0\}$. For $x = (\alpha_i)\in X$, define $|||x||| = \sum_{i=1}^{\infty}|\alpha_i|$. Then X is a normed almost linear space. Clearly, we have $E_X = \ell^1$, $\omega_X(x) = x$, $x\in X$ and we show that $\|\cdot\|_{E_X}$ given by (2.1) is $\|\cdot\|_{\ell^1}$. Indeed, let $z = (\gamma_i)\in E_X$. By Remark 2.8 we have $\|z\|_{\ell^1} \le \|z\|_{E_X}$. For the other inequality we first observe that if $\gamma_1 > 0$ then $z\in X$ and so $\|z\|_{E_X} = |||z||| = \|z\|_{\ell^1}$. If $\gamma_1 < 0$, let $\varepsilon > 0$ and define $x = (\alpha_i)$, $y = (\beta_i)$ by $\alpha_1 = \varepsilon$, $\alpha_i = \gamma_i$, $2\le i<\infty$, $\beta_1 = \varepsilon - \gamma_1$, $\beta_i = 0$, $2 \le i < \infty$. We have $x,y\in X$ and $z = x-y$. Then $\|z\|_{E_X} \le |||x||| + |||y||| = \sum_{i=1}^{\infty}|\alpha_i| + \sum_{i=1}^{\infty}|\beta_i| = 2\varepsilon - \gamma_1 + \sum_{i=2}^{\infty}|\gamma_i| = 2\varepsilon + \sum_{i=1}^{\infty}|\gamma_i| = 2\varepsilon + \|z\|_{\ell^1}$. Since $\varepsilon > 0$ was arbitrary, it follows that $\|z\|_{E_X} \le \|z\|_{\ell^1}$. Consequently $\|z\|_{E_X} = \|z\|_{\ell^1}$.

For $n\in N$, let $v_n = (\alpha_{ni})\in V_X$ be defined by $\alpha_{n2} = 1/2$, $\alpha_{n,n+2} = -1/(n+2)$ and $\alpha_{ni} = 0$ for $i \ne 2, n+2$ and let $z = (0,1/2,0,0,0,\ldots)\in E_X\setminus V_X$. We have $\lim_{n\to\infty}\|v_n - z\| = 0$, i.e., V_X $(=V_{\omega_X(X)})$ is not closed in E_X. Since $z\notin E_1$ (see Proposition 3.2 for the definition of E_1) and $v_n\in E_1$ we have that E_1 is not closed in E_X. Let us note that since $z\notin X$, by Theorem 4.6, X has no basis.

6.5. EXAMPLE. Let $X = \{(\alpha,\beta)\in R^2;\ \beta\ge 0\}$. Define the

addition as in R^2

addition as in R^2 and for $x = (\alpha, \beta) \in X$, $\lambda \in R$ define $\lambda \circ x = (\lambda\alpha, |\lambda|\beta)$. Then X is an almost linear space and we have $V_X = \{(\lambda, 0); \lambda \in R\}$ and $W_X = \{(0, \lambda); \lambda \in R_+\}$, hence $X = W_X + V_X$. For $x = (\alpha, \beta) \in X$ define $|||x||| = |\alpha| + \beta$. Then X is a normed almost linear space. A basis of X is $\mathcal{B} = \{(1,0),(0,1)\}$. We have $E_X = R^2$, $\omega_X(x) = x$, $x \in X$ and for $z = (\gamma, \delta) \in E_X$ we have $\|z\| = |\gamma| + |\delta|$.

(i) A basis of X which does not satisfy the condition from Theorem 2.2 is the set $\mathcal{B} = \{(1,0),(1,1)\}$. We have $\mathcal{B} \cap W_X = \{0\}$ and $\mathcal{B} \cap V_X = \{(1,0)\}$.

(ii) Let $Y = \{(\alpha, \beta) \in X ; \beta \geq |\alpha|\}$. Then Y is an almost linear subspace of X where $W_Y = W_X$ and $V_Y = \{(0,0)\}$. We have $E_Y = R^2$ and $\omega_Y(y) = y$, $y \in Y$. The unit ball of E_Y is the hexagon which is the convex hull of the set $\{(1/2,1/2),(0,1),(-1/2,1/2),(-1/2,-1/2),(0,-1),(1/2,-1/2)\}$. We have $Y^* = \{(\alpha, \beta) \in R^2; \beta \geq 0\}$, $W_{Y^*} = \{(0, \lambda); \lambda \geq 0\}$, $V_Y = \{(\lambda, 0); \lambda \in R\}$ and $|||\cdot|||_{Y^*} = \|\cdot\|_{E_Y^*}|Y^*$. Straightforward (or using Theorem 3.5), it follows that $E_{Y^*} = E_Y^*$ and we also have $\|\cdot\|_{E_{Y^*}} = \|\cdot\|_{E_Y^*}$. The space $Y^{**} = \{(\alpha, \beta) \in R^2 ; \beta \geq 0\} \neq Q_Y(Y)$.

(iii) Let X and Y be as above. A basis of the almost linear subspace Y of X is $\mathcal{B}_0 = \{(1,1),(-1,1)\}$ but no basis $\mathcal{B}$ of X satisfies $\mathcal{B}_0 \subset \mathcal{B}$.

(iv) Let Y be the normed almost linear space defined in (ii) and let $\mathcal{B}_1 = \{(1,1)\}$. The almost linear subspace of Y generated by $\mathcal{B}_1$ is Y and $\mathcal{B}_1$ is not a basis of Y.

6.6. EXAMPLE. Let $X = \{(\alpha, \beta, \gamma) \in R^3 ; \alpha, \beta, \gamma \in R_+\}$. We define the addition as in R^3 and for $x = (\alpha, \beta, \gamma) \in X$ and $\lambda \in R$ we define $\lambda \circ x = (|\lambda|\alpha, |\lambda|\beta, |\lambda|\gamma)$. Then X is an almost linear space such that $X = W_X$. For $x = (\alpha, \beta, \gamma) \in X$ define $|||x||| =$

$= (\alpha^2+\beta^2+\gamma^2)^{1/2}$. Then X is a normed almost linear space. A basis of X is the set $\mathcal{B} = \{(1,0,0),(0,1,0),(0,0,1)\}$, i.e., dim X = 3.

(i) Let $A = \{(\alpha,\beta,\gamma)\in R^3 ; (\alpha-1)^2+(\beta-1)^2+(\gamma-1)^2 \leq 1/9\}$ Then $A \subset X$ and $Y = \{\lambda(\alpha,\beta,\gamma) ; (\alpha,\beta,\gamma)\in A, \lambda\in R_+\}$ is an almost linear subspace of X. We have $E_Y = R^3$ and $\omega_Y(y) = y$, $y\in Y$. The unit ball of E_Y is the convex hull in R^3 of the set $\{\pm y ; y\in Y, |||y||| = 1\}$. Here ω_Y is one-to-one and $\omega_Y(Y)$ is closed in E_Y . Clearly Y has no basis, though it is a closed almost linear subspace of X which has a basis.

(ii) Let Y be the normed almost linear space given in (i). The unit ball B_Y is compact and Y has no basis. As we observed at (ii), ω_Y is one-to-one and $\omega_Y(Y)$ is closed in E_Y .

6.7. EXAMPLE. Let $x_i\in R^3$, $1\leq i\leq 4$, where $x_1=(-1,1,2)$, $x_2=(-1,1,1)$, $x_3=(1,1,2)$, $x_4=(1,1,1)$ and let $X = \{\sum_{i=1}^4 \lambda_i x_i ; \lambda_i\in R_+ , 1\leq i\leq 4\}$. Define the addition in X as in R^3 and for $x = (\alpha,\beta,\gamma)\in X$ and $\lambda\in R$ define $\lambda\circ x = (\lambda\alpha, |\lambda|\beta, |\lambda|\gamma)$. Simple computations show that if $x = \sum_{i=1}^4 \lambda_i x_i$, $\lambda_i \geq 0$, then

$$\lambda\circ x = \begin{cases} \lambda(\lambda_1x_1+\lambda_2x_2+\lambda_3x_3+\lambda_4x_4) & \text{if } \lambda\geq 0 \\ |\lambda|(\lambda_3x_1+\lambda_4x_2+\lambda_1x_3+\lambda_2x_4) & \text{if } \lambda< 0 \end{cases}$$

i.e., $\lambda\circ x\in X$. Then X is an almost linear space. Note that $X \subset \{(\lambda_1,\lambda_2,\lambda_3)\in R^3 ; \lambda_2,\lambda_3\in R_+\}$. Clearly, $V_X = \{0\}$ and we show that we have

(6.1) $\quad W_X = \{(0,\mu,\nu)\in R^3 ; 0\leq\mu\leq\nu\leq 2\mu\}$

Let $w\in W_X$. Then $w = \sum_{i=1}^4 \lambda_i x_i$ for some $\lambda_i \geq 0$, $1\leq i\leq 4$.

Let us put

Let us put

$$\mu = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$$
$$\nu = \mu + \lambda_1 + \lambda_3$$

Then $0 \leq \mu \leq \nu \leq 2\mu$ and simple computations show that $w = (-\lambda_1 - \lambda_2 + \lambda_3 + \lambda_4, \mu, \nu)$. Since $-1 \circ w = (\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4, \mu, \nu)$ and $w = -1 \circ w$, it follows that $-\lambda_1 - \lambda_2 + \lambda_3 + \lambda_4 = 0$, i.e., $w = (0, \mu, \nu)$, $0 \leq \mu \leq \nu \leq 2\mu$, which proves the inclusion $\subset$ in (6.1). To show the other inclusion in (6.1), let $(0, \mu, \nu) \in R^3$ such that $0 \leq \mu \leq \nu \leq 2\mu$. If $(0, \mu, \nu) \in X$, then $-1 \circ (0, \mu, \nu) = (0, \mu, \nu)$, i.e., $(0, \mu, \nu) \in W_X$. So it remains to show that $(0, \mu, \nu) \in X$ if $0 \leq \mu \leq \nu \leq 2\mu$.

<u>Case</u> 1. $(3\mu/2) - \nu \leq 0$. Let $\lambda_1 = \nu - 3\mu/2$, $\lambda_2 = 2\mu - \nu$, $\lambda_3 = \mu/2$ and $\lambda_4 = 0$. We have $(0, \mu, \nu) = \sum_{i=1}^{4} \lambda_i x_i \in X$ since $\lambda_i \geq 0$, $1 \leq i \leq 4$.

<u>Case</u> 2. $(3\mu/2) - \nu > 0$. Let $\lambda_1 = 0$, $\lambda_2 = \mu/2$, $\lambda_3 = \nu - \mu$ and $\lambda_4 = (3\mu/2) - \nu$. We have $(0, \mu, \nu) = \sum_{i=1}^{4} \lambda_i x_i \in X$ since $\lambda_i \geq 0$, $1 \leq i \leq 4$.

Consequently, we have the equality in (6.1).

For $x = (\alpha, \beta, \gamma) \in X$ define $|||x||| = |\alpha| + \beta + \gamma$. Then X is a normed almost linear space. We have $E_X = R^3$ equipped with the norm given by (2.1) where $\omega_X(x) = x$, $x \in X$. Consequently ω_X is one-to-one and $\omega_X(X)(=X)$ is closed in E_X.

Since $V_X = \{0\}$ and for $1 \leq i \leq 4$ no x_i is a positive linear combination of the elements $\{x_j : j \neq i, 1 \leq j \leq 4\}$ it follows that X has no basis.

For $1 \leq i \leq 3$, let f_i be the functionals defined by

$$f_1((\alpha, \beta, \gamma)) = -\beta + \gamma \qquad ((\alpha, \beta, \gamma) \in X)$$

$$f_2((\alpha,\beta,\gamma)) = 2\beta - \gamma \qquad ((\alpha,\beta,\gamma)\in X)$$
$$f_3((\alpha,\beta,\gamma)) = \alpha \qquad ((\alpha,\beta,\gamma)\in X)$$

Then f_i are additive and positively homogeneous and for $w\in W_X$, $w = (0,\mu,\nu)$, $0\leq\mu\leq\nu\leq 2\mu$ we have $f_1(w) = -\mu+\nu\geq 0$, $f_2(w) = 2\mu-\nu\geq 0$ and $f_3(w) = 0$. Consequently $f_i\in X^{\#}$, $1\leq i\leq 3$, and clearly $f_i\in X^*$. We have $f_1,f_2\in W_{X^*}$ and $f_3\in V_{X^*}$. We claim that $\{f_1,f_2,f_3\}$ is a basis of X^*. Let $f\in X^*$ and let us put

$$\lambda_1 = f(x_1+x_3)/2$$
$$\lambda_2 = f(x_2+x_4)/2$$
$$\lambda_3 = (f(x_3)-f(x_1))/2$$

Using (6.1) we get that $x_1+x_3\in W_X$ and $x_2+x_4\in W_X$, hence $\lambda_1,\lambda_2\in R_+$. We first show that

(6.2) $$f = \sum_{i=1}^{3}\lambda_i\circ f_i$$

For this it is enough to show that

(6.3) $$f(x_j) = (\sum_{i=1}^{3}\lambda_i\circ f_i)(x_j)(=\sum_{i=1}^{3}\lambda_i f_i(x_j)) \qquad 1\leq j\leq 4$$

Since $x_1+x_4 = x_2+x_3$ we get

$$\lambda_3 = (f(x_3)-f(x_1))/2 = (f(x_4)-f(x_2))/2$$

Simple computations show that we have $f_1(x_1)=f_1(x_3)=f_2(x_2)=f_2(x_4)= =f_3(x_3)=f_3(x_4)=1$, $f_1(x_2)=f_1(x_4)=f_2(x_1)=f_2(x_3)=0$ and $f_3(x_1)=f_3(x_2)=-1$; whence (6.3) follows taking for $j=1,3$, $\lambda_3 = (f(x_3)-f(x_1))/2$ and for $j=2,4$, $\lambda_3 = (f(x_4)-f(x_2))/2$. Consequently we have (6.2).

Suppose now that $f = \sum_{i=1}^{3} \lambda_i \circ f_i = \sum_{i=1}^{3} \mu_i \circ f_i$, where $\lambda_i, \mu_i \in R_+$ for i=1,2. Then $(\sum_{i=1}^{3} \lambda_i \circ f_i)(x_j) = (\sum_{i=1}^{3} \mu_i \circ f_i)(x_j)$, $1 \leq j \leq 4$, whence it follows that $\lambda_i = \mu_i$, i=1,2,3. This completes the proof that $\{f_1, f_2, f_3\}$ is a basis of X^*.

In conclusion we have $\dim X^* = 3$, ω_X is one-to-one and $\omega_X(X)$ is closed in E_X but X has no basis.

6.8. EXAMPLE. Let $X = \{(\alpha_i) \in \ell^1;\ \alpha_i \geq 0,\ i=1,2,\ldots,\ \text{card}\{i;\ \alpha_i > 0\} < \infty\}$. Define the addition as in ℓ^1 and for $x = (\alpha_i) \in X$ and $\lambda \in R$, define $\lambda \circ x = (|\lambda| \alpha_i) \in X$. Then X is an almost linear space such that $X = W_X$. For $x = (\alpha_i) \in X$, define $|||x||| = \sum_{i=1}^{\infty} \alpha_i$. Then X is a normed almost linear space. We have $E_X = \{(\alpha_i) \in \ell^1;\ \text{card}\{i;\ \alpha_i \neq 0\} < \infty\}$, $\omega_X(x) = x$, $x \in X$ and we show that for $z = (\alpha_i) \in E_X$ we have $\|z\| = \sum_{i=1}^{\infty} |\alpha_i|$, where $\|z\|$ is given by (2.1). By Remark 2.8 we have $\sum_{i=1}^{\infty} |\alpha_i| \leq$ $\leq \|z\|$. For the other inequality, define $(\beta_i) \in X$ and $(\gamma_i) \in X$ in the following way; for $i \in N$ where $\alpha_i \geq 0$ take $\beta_i = \alpha_i$ and $\gamma_i = 0$ and for $i \in N$ where $\alpha_i < 0$ take $\beta_i = 0$ and $\gamma_i = -\alpha_i$. Then $(\alpha_i) = (\beta_i) - (\gamma_i)$ and by (2.1), $\|z\| \leq |||(\beta_i)||| + |||(\gamma_i)||| =$ $= \sum_{i=1}^{\infty} |\alpha_i|$, which proves that $\|z\| = \sum_{i=1}^{\infty} |\alpha_i|$. For $n \in N$, let $e_n = (\delta_{nj})_{j=1}^{\infty}$, where $\delta_{nj} = 1$ for n=j and $\delta_{nj} = 0$ for $n \neq j$. Then $e_n \in X$ and $\{e_n;\ n \in N\}$ is a basis of X ($\dim X = \infty$).

By Theorem 2.7 (iv) we have $X^* = \{(\alpha_i) \in \ell^\infty;\ \alpha_i \geq 0\}$ and since $X = W_X$ we get $X^* = W_{X^*}$. We claim that X^* has no basis. Indeed, suppose that X^* has a basis $\mathcal{B}$. Clearly, for each $n \in N$, $e_n \in X^*$, and we show that there exists $\lambda_n > 0$ such that $\lambda_n \circ e_n \in \mathcal{B}$. We prove this only for n=1, the proof for n > 1 being similar. Since $e_1 \in X^*$, there exist unique $b_i \in \mathcal{B}$, $1 \leq i \leq k$ and $\lambda_i > 0$, $1 \leq i \leq k$ such that $e_1 = \sum_{i=1}^{k} \lambda_i \circ b_i$. Suppose $b_i = (\alpha_{ij})_{j=1}^{\infty}$, where $\alpha_{ij} \geq 0$, j=1,2,.... and $1 \leq i \leq k$. Then we have

$$\sum_{i=1}^{k} \lambda_i \alpha_{i1} = 1$$
$$\sum_{i=1}^{k} \lambda_i \alpha_{ij} = 0 \qquad j=2,3,\ldots.$$

Since $\lambda_i > 0$, $1 \leq i \leq k$ and $\alpha_{ij} \geq 0$, $1 \leq i \leq k$, $j \in N$, it follows that $\alpha_{ij} = 0$, $1 \leq i \leq k$, $j = 2,3,\ldots.$, whence $k=1$ and $b_1 = \lambda_1 \circ e_1$. By Remark 2.1 (i) we can suppose $e_n \in \mathcal{B}$, $n \in N$.

Let $y = (1,1,\ldots,1,\ldots) \in X^*$. Then there exist unique $b_i \in \mathcal{B}$, $1 \leq i \leq k$, $\lambda_i > 0$, $1 \leq i \leq k$ such that $y = \sum_{i=1}^{k} \lambda_i \circ b_i$. Let $n_o \in N$ such that $e_{n_o} \notin \{b_1,\ldots,b_k\}$ and let $y_o = (\beta_i) \in X^*$ be defined by $\beta_i = 1$, $i \neq n_o$ and $\beta_{n_o} = 0$. Then $y = e_{n_o} + y_o$ and since $y_o = \sum_{i=1}^{m} \mu_i \circ b_i'$, $b_i' \in \mathcal{B}$, $\mu_i > 0$ and $e_{n_o} \in \mathcal{B} \setminus \{b_1,\ldots,b_k\}$ it follows that y has no unique representation, a contradiction which proves that X^* has no basis.

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