SMASH PRODUCT AND APPLICATIONS TO FINITENESS CONDITIONS

by

C.NASTASESCU*)

November 1988

*) University of Bucharest, Faculty of Mathematics, Str.Academiei 14
70109 Bucharest, Romania.

Smash Product and Applications to Finiteness Conditions

by Constantin Năstăsescu
Universitatea din Bucureşti
Facultatea de Matematică
Str.Academiei nr.14
7o1o9 Bucharest, Romania

Introduction

The smash product, associated to a rings graded by a finitt groups, is a very porwerful tool for studying.
These rings, This was shown for the first time by M.Cohen and S.Montgomery in [3]. Lately, D.Quinn [1o] proved that the construction of the smash product may be performed even for rings graded by an infinite group. In this generalized form, the smash product was used in [2] and [1o].
The aim of this paper is to proved new properties of graded rings, using the smash product.
The first part of the paper is denoted to the study of the categor $\tilde{R} \# G^*$-mod, where $\tilde{R} \# G^*$ is the smash product (in the sense of D. Quinn) associated to the graded ring $R = \bigoplus_{\sigma \in G} R_\sigma$ (the group G is generally infinite).
In the second part, a series of finiteness properties of graded rings are given;(Theorems 2.1 and 2.2).The results generalize the ones obtained in the second part of [9] . In the last part of the paper, usinf Theorems 2.1 and 2.2 we show that if $R = \bigoplus_{i \in \mathbb{Z}} R_i$ is a graded ring of type Z and $M = \bigoplus_{i \in \mathbb{Z}} M_i$ is a graded R-module, then $\tau_\alpha(M)$ ($\alpha \geq 0$ is an arbitrary ordinal) is a graded submodule f M, i.e.the radical associated to the torsion theories defining the Gabriel dimension is a graded submodule (this is a partial answer to the general problem of deciding whethen an element of the latti of submodules of a graded module, described by some ungraded properties is a graded submodule).

0. Notation and Preliminaries

All rings considered in this paper will be unitary. If R is a ring, by an R-module we will mean a left R-module, and we will denote the category of R-modules by R-mod. Let G be a multiplicative group with identity element "1". A G-graded ring R is a ring with identity 1, together with a direct sum decomposition (as additive subgroups) $R = \bigoplus_{\sigma \in G} R_\sigma$ such that

(1) $R_\sigma R_\tau \subseteq R_{\sigma\tau}$ for all $\sigma, \tau \in G$

It is wellknoun [8] that R_1 is a subring of R, and that $1 \in R_1$. By a (left) G-graded module we understand a left R-module M, plus an internal direct sum decomposition $M = \bigoplus_{\sigma \in G} M_\sigma$, where M_σ's are subgroups of the additive group of M, such that $R_\sigma M_\tau \subseteq M_{\sigma\tau}$ for all $\sigma, \tau \in G$. We denote by R-gr the category of G-graded modules. In this category, if $M = \bigoplus_{\sigma \in G} M_\sigma$ and $N = \bigoplus_{\sigma \in G} N_\sigma$ are two objects then $\mathrm{Hom}_{R\text{-}gr}(M,N)$ denotes the set of morphisms in the category R-gr from M to N, i.e.

$\mathrm{Hom}_{R\text{-}gr}(M,N) = \{ f: M \longrightarrow N \mid f$ is R-linear and $f(M_\sigma) \subseteq N_\sigma \ \forall \sigma \in G \}$.

It is wellknown [8] that R-gr is a Grothendieck category. In particular, R-gr has enough injective objects. If M R-gr, we denote by $E^g(M)$ the injective envelope of M in R-gr, and by E(M) the injective envelope of M in R-mod.

If $M = \bigoplus_{\lambda \in G} M_\lambda$ is a graded R-module, and $\sigma \in G$, then $M(\sigma)$ is the graded module obtained form M by putting $M(\sigma)_\lambda = M_{\lambda\sigma}$; the graded module $M(\sigma)$ is called the σ -<u>suspension</u> of M.

Let R be a G-graded ring. Now we present the construction of the Smash Product associated to the ring R. We follow the construction given by D.Quinn in the paper [10] .

We denote by $M_G(R)$ the set of row and column finite matrices over R, with the rons and columns indexed by the elements of G. $M_G^*(R)$ is the ideal of $M_G(R)$ consisting of those matrices with only finitely many nonzero entries. Note that if G is finite, then $M_G^*(R) = M_G(R)$.

If $\alpha \in M_G(R)$, then we write $\alpha(x,y)$ for the entry in the (x,y) position of α . For α, $\beta \in M_G(R)$, the matrix product is given by:

$$(\alpha\beta)(x,y) = \sum_{z \in G} \alpha(x,z)\ \beta(z,y)$$

If $x,y \in G$, Then we let $e_{x,y}$ denote the matrix with 1 in the (x,y) position, and zero elsewhere.

Let $p_x = e_{x,x}$. Define $\eta : R \longrightarrow M_G(R)$ by $\eta(r) = \tilde{r}$, where $\tilde{r} = \sum_{x,y \in G} r_{xy^{-1}} e_{x,y}$, $r = \sum_{g \in G} r_g$, $r \in R$, $r_g \in R_g$ for any $g \in G$.

It is easy to see that η is a ring monomorphism.

Let $\tilde{R} = \mathrm{Im}(\eta)$, and $\tilde{R} \# G^*$ the subring of $M_G(R)$ generated by $\tilde{R}$ and the set of orthogonal idempotents $\{p_x \mid x \in G\}$. $\tilde{R} \# G^*$

called the <u>smash product</u> of R by G.

The group G embeds in $M_G(R)$ as permutation matrices:

each $g \in G$ is sent to $\overline{g} = \sum_{x \in G} e_{x,xy}$. Thus $\overline{G} = \{\overline{g} \mid g \in G\}$

is a subgroup of the group of units of $M_G(R)$, isomorphic to G.

We denote $R\{G\} = (R = G)\, G = {}_{g \quad G} (R = G)\, g$.

The following properties of the ring $\widetilde{R} \# G^*$, given by D.Quinn in [10] , will be frecquently used in the sequels:

(a) Let R be a G-graded ring, with G infinite. Then

$$\widetilde{R} \# G^* = \widetilde{R} \oplus \left(\bigoplus_{x \in G} \widetilde{R}\, p_x \right)$$

is a free R-module with basis $\{I\} \cup \{p_x,\ x \in G\}$, where I is the identity matrix of $M_G(R)$.

If the group G is finite, then we have

$$\widetilde{R} \# G^* = \bigoplus_{x \in G} \widetilde{R}\, p_x$$

and in this case $\widetilde{R} \# G^*$ is exactly the smash product defined by M. Cohen and S.Montgomery in [3] .

(b) If $r, s \in R$, then

$$(\widetilde{r}\, p_x)(\widetilde{s}\, p_y) = \widetilde{r} \quad \widehat{s}_{xy^{-1}}\, p_y$$

In particular, if $r_\sigma \in R_\sigma$, then we have

$$p_x\, \widetilde{r}_\sigma = \widetilde{r}_\sigma\, p_{\sigma^{-1}x}$$

(c) If $g \in G$ and $\alpha \in \widetilde{R} \# G^*$, then

$$(\overline{g}^{-1} \alpha\, \overline{g})(x,y) = \alpha(xg^{-1}, yg^{-1})$$

In particular, $\overline{g}^{-1}\, p_x \overline{g} = p_{xg}$

(d) Iń G is an infinite group, then

$$R\{G\} = (\widetilde{R} \# G^*)\overline{G}$$

is the skew group ring of the group $\overline{G}$ over $\widetilde{R} \# G^*$, and

$$R\{G\} = (\bigoplus_{g \in G} \widetilde{R}\,\overline{g}) \oplus M_G^*(R)$$

If the group G is finite, then $R\{G\} = M_G(R)$.

1. The Category $\widetilde{R} \# G^*$ - mod

Let R be a G-graded ring. If $M \in \widetilde{R} \# G^*$-mod, then $M_0 = \bigoplus_{x \in G} p_x M$ is an $\widetilde{R} \# G^*$- submodule of M.

Indeed, it is sufficient to prove that $\widetilde{r}_\sigma M_0 \subseteq M_0$, where $r_\sigma \in R_\sigma$. But $\widetilde{r}_\sigma M \subseteq \sum_{x \in G} \widetilde{r}_\sigma p_x M = \sum_{x \in G} p_{\sigma x} \widetilde{r}_\sigma M \subseteq \sum_{x \in G} p_{\sigma x} M =$

$= M_0$, since $\{p_x | x \in G\}$ is a family of orthogonal idempotents.

Let us now denote by $\mathcal{C}^*$ the subclass of $\widetilde{R} \# G^*$ -mod defined by the property:

$$\mathcal{C}^* = \{ M \in \widetilde{R} \# G^* \text{-mod} \mid M = \sum_{x \in G} p_x M$$

<u>Proposition 1.1</u>. $\mathcal{C}^*$ is a localizing subcategory of $\widetilde{R} \# G^*$-mod (i.e. $\mathcal{C}^*$ is closed under subobjects, quotient objects, extensions and arbitrary direct sum).

<u>Proof</u>: It is obvious that $\mathcal{C}^*$ is closed under quotient objects and arbitrary direct sum.

Let $M \in \mathcal{C}^*$, and $N \leq M$ an $\widetilde{R} \# G^*$ -submodule.

If $n \in N$, then $n \in M = \bigoplus_{x \in G} p_x M$, and therefore $n = \sum_{x \in G} p_x m^x$.

Thus $p_x n = p_x m^x$, and thus $n = \sum_{x \in G} p_x n$, so $n \in \bigoplus_{x \in G} p_x N$,

hence $N = \bigoplus_{x \in G} p_x N$, i.e $N \in \mathcal{C}^*$.

Consider now the exact sequence of $\tilde{R} \# G^*$ -modules

$$0 \longrightarrow M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0$$

where $M', M'' \in \mathcal{C}^*$.

Let m N; then $V(m) \in M'' = \bigoplus_{x \in G} p_x M''$. Thus $v(m) = \sum_{x \in G} p_x v(m^x) =$

$= \sum_{x \in G} v(p_x m^x)$, and therefore $m - \sum_{x \in G} p_x m^x \in Ker(v) = Im(u)$.

Thus there exists $m' \in M'$ such that $m - \sum_{x \in G} p_x m^x = u(m')$

Since $M' \in \mathcal{C}^*$, then $m' = \sum_{x \in G} p_x m'^x$, and thus

$$m - \sum_{x \in G} p_x m^x = u(\sum_{x \in G} p_x m'^x) = \sum_{x \in G} p_x \ u(m'^x). \text{ Hence}$$

$m = \sum_{x \in G} p_x(m^x + u(m'^x))$, i.e. $M \in \mathcal{C}^*$. Therefore $\mathcal{C}^*$ is a localizing subcategory.

Let now $M \in R$-gr. Then M has a natural structure of $\tilde{R} \# G^*$-module if we put for all $\tilde{r} \in \tilde{R}$ and $x \in G$ $\tilde{r}m = rm$, and $p_x m = m_x$, where $m \in M$ and $m = \sum_{x \in G} m_x$ ($m_x \in M_x$ are the homogeneous components of m).

If $r, s \in R$ and $x, y \in G$, then we have

$(\tilde{r}p_x)((\tilde{s}\ p_y)m) = (\tilde{r}\ p_x)(\tilde{s}\ m_y) = r(s\ m_y)_x$, where $(s\ m_y)_x$ is the homogeneous component of degree x of the element sm_y.

Since $s\ m_y = (\sum_{x \in G} s_x) m_y = \sum_{x \in G} s_x m_y$, then $(s\ m_y)_x =$

$= s_{xy^{-1}} m_y$. On the other hand,

$((\tilde{r}\ p_x)(\tilde{s}\ p_y))m = (\tilde{r}\ \tilde{s}_{xy^{-1}} p_y)m = (r\ s_{xy^{-1}})m_y = r(s_{xy^{-1}} m_y)$.

Hence $((\tilde{r}\ p_x)(\tilde{s}\ p_y))m = (\tilde{r}\ p_x((\tilde{s}\ p_y)m)$.

Hence M has a natural structure of an $\widetilde{R} \# G^*$-module.

We will denote the module M, considered with this structure, by M^*.

Now if $M, N \in R\text{-gr}$, and $f \in \mathrm{Hom}_{R\text{-gr}}(M,N)$, then $f: M^* \longrightarrow N^*$ is also an $\widetilde{R} \# G^*$-morphism.

Therefore we obtain the exact functor

$$F: R\text{-gr} \longrightarrow \widetilde{R} \# G^*\text{-mod}$$

$F(M) = M^*$ for all $M \in R\text{-gr}$, and $F(f) = f$ for all $f \in \mathrm{Hom}_{R\text{-gr}}(M,N)$, $M, N \in R\text{-gr}$.

We remark that if $M \in R\text{-gr}$, then $M^* \in \mathcal{C}^*$ and therefore F can be considered as a functor $F: R\text{-gr} \longrightarrow \mathcal{C}^*$.

We can define another functor $F': R\text{-gr} \longrightarrow \widetilde{R} \# G^*\text{-mod}$ if we put for $M = \bigoplus_{x \in G} M_x$, $M \in R\text{-gr}$, $F'(M) = \prod_{x \in G} M_x$, where $F'(M)$ has the following structure of an $\widetilde{R} \# G^*$-module: if $\widetilde{r_\sigma} \in \widetilde{R}$, $r_\sigma \in R_\sigma$ $x \in G$, and $\overline{m} = (m_x)_{x \in G} \in \prod_{x \in G} M_x$, then $\widetilde{r_\sigma} \cdot \overline{m} = \overline{n}$, where

$\overline{n} = (n_y)_{y \in G}$, $n_y = r_\sigma \; m_{\sigma^{-1}y}$ and $p_x \overline{m} = \overline{m}'$, where $\overline{m}' = (m'_y)_{y \in G}$ $m'_y = o$ for $y \neq xm$ and $m'_x = m_x$. It is easy to see that $\prod_{x \in G} M_x$ is an $\widetilde{R} \# G^*$-module. It is obvious that F' is an exact functor.

We remark that F is a subfunctor of F'. Indeed if $M \in R\text{-gr}$, we define the map

$$\alpha_M : F(M) \longrightarrow F'(M), \quad \alpha_M(m) = (m_x)_{x \in G} \quad \text{where}$$

$m = \sum_{x \in G} m_x$, $m_x \in M_x$, i.e. $\{m_x, \; x \in G\}$ are the homogeneous composants of m. It is obviously that α_M is injective and it is also $\widetilde{R} \# G^*$-linear. In case G is a finite group, then $F = F'$.

Let now $M \in \widetilde{R} \# G^{*}$-mod. We have seen that $M_{o} = \sum_{x \in G} p_{x} M$ is an $\widetilde{R} \# G^{*}$-submodule of M. M_{o} has a natural structure of a graded R-module of type G, if we put $(M_{o})_{x} = p_{x} M$ and we consider M_{o} as an R-module via the morphism $\eta : R \longrightarrow \widetilde{R} \# G^{*}$. It is easy to see that the map $M \longrightarrow M_{o}$ defines an exact functor

$$H : \widetilde{R} \# G^{*} \text{-mod} \longrightarrow R\text{-gr}$$

Proposition 1.2. With the above notation we have:

a) The functor F is a left adjoint of the functor H;

b) The functor F' is a right adjoint of the functor H.

Proof:

a) We define the functdoial morphisms:

$$\mathrm{Hom}_{\widetilde{R} \# G^{*}\text{-mod}}(F(-),(-)) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathrm{Hom}_{R\text{-gr}}(-, H(-))$$

as follows:

if $M \in R\text{-gr}$, $N \in \widetilde{R} \# G^{*}$-mod, then

$$\alpha(M,N) : \mathrm{Hom}_{\widetilde{R} \# G^{*}\text{-mod}}(F(M),N) \longrightarrow \mathrm{Hom}_{R\text{-gr}}(M,H(N))$$

is defined by $\alpha(M,N)(u)(x) = u(x)$, where $u: F(M) \longrightarrow N$ and $x \in M$. Since $F(M) = M^{*} = \bigoplus_{x \in G} p_{x} M$, then $u(M) = u(\bigoplus_{x \in G} p_{x} M) =$

$= \sum_{x \in G} p_{x} u(M) \subseteq H(N)$.

We define $\beta(M,N): \mathrm{Hom}_{R\text{-gr}}(M?H(N)) \longrightarrow \mathrm{Hom}_{\widetilde{R}=G^{*}\text{-mod}}(F(M),N)$

as follows: if $v \in \mathrm{Hom}_{R\text{-gr}}(M,H(N))$, then $\beta(M,N)(v) = i \circ v$, where $i: H(N) \hookrightarrow N$ is the inclusion.

It is obvious that $\alpha \circ \beta = 1$ and $\beta \circ \alpha = 1$. Consequently, F is a left adjoint of H.

b) See [1] ... .

If $M = \bigoplus_{\sigma \in G} M_{\sigma}$ is an object of R-gr, we define the <u>support</u> of M by

$$\mathrm{Supp}(M) = \left\{ \sigma \in G \;\middle|\; M_{\sigma} \neq 0 \right\}$$

We remark that is $|\mathrm{Supp}(M)| < \infty$, then $F(M) = F'(M)$.

<u>Corollary 1.1.</u> The following assertions hold:

1) If $M \in$ R-gr is projective, then $M^{*} = F(M)$ is a projective $\widetilde{R} \# G^{*}$ -module.

2) If $M \in$ R-gr, with $|\mathrm{Supp}(M)| < \infty$ is gr-injective (i.e. it is an injective object of R-gr) then $M^{*} = F(M)$ is an injective $\widetilde{R} \# G^{*}$-module.

<u>Proof</u>: 1) Since F is a left adjoint of H, and H is an exact functor, it is wellknown from the theory of adjoint functors [11] that $M^{*} = F(M)$ is projective in $\widetilde{R} \# G^{*}$-mod.

2) Since F' is a right adjoint of H, and H is an exact functor, we deduce again (see [11]) that $F'(M)$ is injective in $\widetilde{R} \# G^{*}$-mod. Since $|\mathrm{Supp}(M)| < \infty$, then $M^{*} = F(M) = F'(M)$, and the assertion follows.

Now since $\mathcal{C}^{*}$ is a localizing subcategory of $\widetilde{R} \# G^{*}$-mod, then, following Gabriel [4] (see also [11]) we can form the quotient category $\widetilde{R} \# G^{*}$ -mod $/\mathcal{C}^{*}$.

<u>Corollary 1.2.</u> With the above notation, the following assertions hold:

1) The categories R-gr and $\mathcal{C}^{*}$ are isomorphic via the functors F and H.

2) If the group G is infinite, then the quotient category $\widetilde{R} \# G^{*}$ -mod$/\mathcal{C}^{*}$ is isomorphic to the category R-mod.

<u>Proof:</u> 1) From the construction of the functors F and H, we see that $H \circ F = 1_{R-gr}$. Now if $M \in \mathcal{C}^*$, then $(F \circ H)(M) = M$, and therefore $F \circ H' = 1\!\!1_{\mathcal{C}^*}$, where H' denotes the restriction of the functor H to $\mathcal{C}^*$.

2) Since $\mathcal{C}^*$ is a localizing subcategory, for any $M \in \tilde{R} \# G^*$-mod we let $t_*(M)$ denote the largest $\tilde{R} \# G^*$-submodule of M such that $t_*(M) \in \mathcal{C}^*$. We remark that $t_*(M) = \sum_{x \in G} p_x M$. If $t_*(M) = o$, we say that M is $\mathcal{C}^*$ torsion free.

Thus, if M is $\mathcal{C}^*$-torsion free, we have that $p_x M = o$ for any $x \in G$. Thus, $IM = o$, where I is the two-sided ideal $\sum_{x \in G} \tilde{R} p_x$.

Therefore, if M is $\mathcal{C}^*$-torsion free, M is a $(\tilde{R} \# G^* / I)$-module.

Since $\tilde{R} \# G^* / I \simeq R$, then M is an $\tilde{R}$-module.

Let now

$$\tilde{R} \# G^* \text{-mod} \underset{V}{\overset{U}{\rightleftarrows}} \tilde{R} \# G^* \text{-mod} / \mathcal{C}^*$$

be the canonical functors (see [4] ch III). It is wellknown [4] that U is an exact functor, and V is a right adjoint of U.

If $\phi : U \circ V \longrightarrow 1\!\!1_{\tilde{R} \# G^* \text{-mod} / \mathcal{C}^*}$ and $\psi : 1\!\!1_{\tilde{R} \# G^* \text{-mod}} \longrightarrow V \circ U$ are the natural transformations of functors, then ϕ is an isomorphism. Furthermore, if $M \in \tilde{R} \# G^*$-mod, then we have the exact sequence

$$0 \longrightarrow \text{Ker}\,\psi(M) \longrightarrow M \xrightarrow{\psi(M)} (V \circ U)(M) \longrightarrow \text{Coker}\,\psi(M) \longrightarrow 0$$

where $\text{Ker}\,\psi(M)$ and $\text{Coker}\,\psi(M)$ belong to $\mathcal{C}^*$. In particular, if M is $\mathcal{C}^*$-torsionfree, we have the exact sequence

$$0 \longrightarrow M \xrightarrow{\psi(M)} (V \circ U)(M) \longrightarrow \text{Coker}\,\psi(M) \longrightarrow 0$$

Since $(V \circ U)$ is also $\mathcal{C}^*$-torsion free, we have that $I(V \circ U)(M)) = 0$ and therefore $I\,\text{Coker}\,\psi(M) = 0$. Since $\text{Coker}\,\psi(M) \in \mathcal{C}^*$, then

tehn I Coker ψ (M) = Coker ψ (M), so Coker ψ (M) = 0.
Thus if M is $\mathcal{C}^*$ -torsion free, ψ (M) is an isomorphism.
It follows that the category $\tilde{R} \# G^*$-mod /$\mathcal{C}^*$ is isomorphic to the category $\tilde{R}$-mod. Since $\tilde{R}$-mod is isomorphic to R-mod, our result follows.

Remarks. 1)From Corollary 1.2, it follows that if the group G is infinite, then $\tilde{R} \# G^*$-mod is an "extension of the category R-gr by R-mod", i.e. we have the exact sequence of categories.

$$0 \longrightarrow R\text{-gr} \longrightarrow \tilde{R} \# G^* \text{-mod} \longrightarrow R\text{-mod} \longrightarrow 0$$

2) $I = \sum_{x \in G} Rp_x$ is a two-sided idempotent ideal of the ring $\tilde{R} \# G^*$. Moreover, $\tilde{R} \# G^*/I \simeq \tilde{R} \simeq R$ (the group G is infinite).

Since I is an idempotent ideal, the class

$$\mathcal{C}^{**} = \{ M \in \tilde{R} \# G^* \text{-mod} \mid IM = 0 \}$$

is a localizing subcategory of $\tilde{R} \# G^*$-mod. We remark that in fact, $\mathcal{C}^{**}$ is equivalent to the categroy $\tilde{R}$-mod.

Since R and $\tilde{R}$ are isomorphic rings, then $\mathcal{C}^{**}$ is equivalent to the category R-mod.

On the other hand, it is shown in [1] that the quotient categroy $\tilde{R} \# G^*$ -mod/$\mathcal{C}^{**}$ is equivalent to the category R-gr, via the functors F' and H.

Thus, we have an exact sequence of categories

$$0 \longrightarrow R\text{-mod} \longrightarrow \tilde{R} \# G^*\text{-mod} \longrightarrow R\text{-gr} \longrightarrow 0$$

3) If the group is finite then $\mathcal{C}^* = \tilde{R} \# G^*$-mod and therefore the assertion 1) of corollary 1.2. give that the categories R-gr and $\tilde{R} \# G^*$-mod are isomorphic, which is a wellknown result [3] .

2. Finiteness Conditions for Modules

Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a G-graded ring and $M \in$ R-gr. Let N be a (not necessarily graded) R-submodule. We denote by $(N)_g$ (resp. $(N)^g$) the greatest graded submodule of M contained in N (resp. the smallest graded submodule of M containing N). Thus $(N)_g = \sum_P P$, where P runs through all graded submodules of M such that $P \leq N$, and $(N)^g = \bigcap_Q Q$ where Q runs through all graded submodules of M, such the $N \leq Q$.

As we have seen in section 1, each $M \in$ R-gr can be considered as an $\tilde{R} \# G^*$-module. We have denoted this module by M^*. In this case, N will be a subset of M weach that N is an $\tilde{R}$-submodule of M by restriction of scalars ($\tilde{R}$ is a subring of $\tilde{R} \# G^*$).

Lemma 2.1. With the above notation, we have

1) $(N)_g = \bigcap_{x \in G} (N : p_x)$

2) $(N)^g = (\tilde{R} \# G^*) N = \bigoplus_{x \in G} p_x N$

Proof: 1) is obvious

2) Since $\tilde{r}_\sigma \; p_x = p_{\sigma x} \; \tilde{r}_\sigma$, we have that $\bigoplus_{x \in G} p_x N$ is an $\tilde{R} \# G^*$-submodule of N.

If $n \in N$, then $n \in M^*$, and therefore $n = \sum_{x \in G} n_x$, where $n_x \in M_x$ for any $x \in G$. Thus $p_x n = n_x$, and therefore $n = \sum_{x \in G} p_x n$, so

$N \subseteq \bigoplus_{x \in G} p_x N$. Hence, $(N)^g \subseteq \bigoplus_{x \in G} p_x N$.

Conversely, if $P \leq M$ is a graded submodule such that $N \leq P$, then P is an $\widetilde{R} \# G^*$-submodule of M^*, and thus $\bigoplus_{x \in G} p_x N \leq P$.

Therefore, $\bigoplus_{x \in G} p_x N \subseteq (N)^g$, or $(N)^g = \bigoplus_{x \in G} p_x N$.

If $M \in R$-mod, we denote by $Col_G(M)$ the set of column matrices over M with elements indexed by G, and with finitely many non-zero entries. Since the elements of $M_G(R)$ are both row and column finite, $Col_G(M)$ is a left $M_G(R)$-module and hence a left $R\{G\}$-module. Since $\widetilde{R} \# G^*$ is a subring of $R\{G\}$, then $Col_G(M)$ is a left $\widetilde{R} \# G^*$ -module.

If M R-gr, then by Lemma 1.3 of [2] we have the canonical isomorphism of $R\{G\}$-modules

$$Col_G(M) \simeq R\{G\} \otimes_{\widetilde{R} \# G^*} M^*$$

(so $Col_G(M)$ is the $R\{G\}$-module induced by M^*)

Assume now that M is a graded R-module, and $N \leq M$ is an arbitrary R-submodule of M.

We define the maps:

$$\alpha : Col_G(M) \longrightarrow M^*$$

by $\alpha(\widetilde{n}) = \sum_{x \in G} p_x n^x$, where $\widetilde{n} \in Col_G(N)$, and n^x is the element of the column $\widetilde{n}$ in the position x, and

$$\beta : M^* \longrightarrow Col_G(M/N)$$

by $\beta(m) = \begin{pmatrix} \vdots \\ \widehat{p_x m} \\ \vdots \end{pmatrix} - x$, i.e $\beta(m)$ is the column having on the position x the class of the element $p_x m$ modulo the submodu[le] M.

<u>Lemma 2.2.</u> 1) α is an $\widetilde{R} \# G^*$ -morphism, and Im $\alpha = (N)^g$

2) β is an $\widetilde{R} \# G^*$ -morphism, and Ker $\beta = (N)_g$.

<u>Proof</u>: 1) We must prove that $\alpha(\widetilde{r}_\sigma \widetilde{n}) = \widetilde{r}_\sigma \alpha(\widetilde{n})$ for any $r_\sigma \in R_\sigma$, $\sigma \in G$, and $\alpha(p_x \widetilde{n}) = p_x \alpha(\widetilde{n})$. Indeed, if

$$\widetilde{n} = \begin{pmatrix} \vdots \\ n^y \\ \vdots \end{pmatrix} - y \text{ , we have } p_x \widetilde{n} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ n^x \\ 0 \\ \vdots \\ 0 \end{pmatrix} - x, \text{ and}$$

therefore $\alpha(p_x \widetilde{n}) = p_x n^x = p_x(\sum_{y \in G} p_y \widetilde{n}) = p_x \alpha(\widetilde{n})$.

Let now $r = r_\sigma \in R_\sigma$; then $\widetilde{r} = \widetilde{r}_\sigma = \sum_{y \in G} r_\sigma e_{\sigma y, y} =$

$= r_\sigma \sum_{y \in G} e_{\sigma y, y}$.

If we denote $\widetilde{r}_\sigma \widetilde{n} = \widetilde{n}'$, then $\widetilde{n}'$ is the column matrix with component n'^x on the x position, where

$$n'^x = \sum_{u \in G} \widetilde{r}_\sigma (x,u) n^u = r_\sigma n^{\sigma^{-1} x}$$

We have $\alpha(\widetilde{r}_\sigma \widetilde{n}) = \sum_{x \in G} p_x (r_\sigma n^{\sigma^{-1} x}) = \sum_{x \in G} p_x (\widetilde{r}_\sigma n^{\sigma^{-1} x}) =$

$= \sum_{x \in G} (p_x \widetilde{r}_\sigma) n^{\sigma^{-1} x} = \sum_{x \in G} \widetilde{r}_\sigma p_{\sigma^{-1} x} n^{\sigma^{-1} x} = \widetilde{r}_\sigma \sum_{x \in G} p_{\sigma^{-1} x} n^{\sigma^{-1} x} =$

$= \widetilde{r}_\sigma \alpha(\widetilde{n})$.

Therefore α is an $\widetilde{R} \# G^*$ -morphism. It is obvious that Im $\alpha = (N)^g$.

2) We must show that $\beta(p_x m) = p_x \beta(m)$ and $\beta(\widetilde{r}_\sigma m) = \widetilde{r}_\sigma \beta(m)$ for any $x \in G$ and $r_\sigma \in R_\sigma$.

Indeed, $\beta(p_x m) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ p_x m \\ 0 \\ \vdots \\ 0 \end{pmatrix} - x = p_x \beta(m)$

Now $\beta(\tilde{r}_\sigma\, m) = \left(p_x\, \tilde{r}_\sigma\, m \right) - x = \left(\tilde{r}_\sigma\, p_{\sigma^{-1}x}\, m \right) - x =$

$= \tilde{r}_\sigma \left(p_{\sigma^{-1}x}\, m \right) - x = \tilde{r}_\sigma\, \beta(m).$

Therefore β is an $\tilde{R} \# G^{*}$ -morphism. Now $\beta(m) = 0$ if an only if $p_x m \in N$ for all $x \in G$, and therefore $m \in \bigcap_{x \in G} (N : p_x) =$ $= (N)_g$, hence $\operatorname{Ker} \beta = (N)_g$.

If M is an R-module, we denote by $K.\dim_R(M)$ the Krull dimension of M over the ring R, and by $G.\dim_R(M)$ the Gabriel dimension of M over the ring R (see Gordon and Robson [5], [6]).

If $M \in R$-gr and M is a noetherian (resp.artinian, semisimple,etc object in the category R-gr, then we say that M is gr-noetherian (resp.gr-artinian, gr-semisimple etc). Analogously we say that M has gr-Krull dimension (resp.gr.Gabriel dimension) if M as Krull (resp.Gabriel) dimension in the category R-gr.

<u>Theorem 2.1.</u> Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a graded ring, G a finite group. Let $M \in R$-gr, and $N \leq M$ an R-submodule of M. Then the following assertions hold.

1) If N is noetherian (resp.artinian) then $(N)^g$ is noetherian (resp.artinian).

2) If N has a Krull dimension, then $(N)^g$ has a Krull dimension. Moreover, $K.\dim_R(N) = K.\dim_R(N)^g)$.

3) If N has a Gabriel dimension, then $(N)^g$ has a Gabriel dimension . Moreover, $G.\dim_R(N) = G.\dim_R((N^g)$.

4) If N is simple, then $(N)^g$ is gr-semisimple of finite length.

Proof: Since $M_G(\bar{R}) = R\{G\}$, then the $R\{G\}$-submodules of $Col_G(N)$ are of the form $Col_G(N')$, where N' is an R-submodule of N.

We prove the assertions 1), 2) and 3). If N is noetherian simple (resp.artinian, has Krull dimension, has Gabriel dimension), then $Col_G(N)$ is an $R\{G\}$-module which is noetherian (resp.artinian, has Krull dimension, has Gabriel dimension simple). Now since $R\{G\}$ is the group ring of $\tilde{R} \# G^*$ by the finite group $\bar{G} \simeq G$, then by Theorems I 8.1o, I.8.12 and I.8.14 of [8], we obtain that $Col_G(N)$ is noetherian (resp.artinian, has Krull dimension, has Gabriel dimension) as an $\tilde{R} \# G^*$-module.

Now by Lemma 2.2 assertion 1), and Corollary 1.2 assertion 1) we obtain that $(N)^g$ is gr-noetherian (resp.gr-artinian, has gr-Krull dimension, has gr-Gabriel dimension).
Moreover, we have in the case that N has Krull (resp.Gabriel) dimension, that $\text{gr-K.dim}\,(N)^g) = \text{K.dim}_{\tilde{R} \# G^*}((N)^g) \leq \text{K.dim}_{\tilde{R} \# G^*}(Col_G(N)) = \text{K.dim}_{R\{G\}}(Col_G(N)) = \text{K.dim}_R(N)$ (analogously, for Gabriel dimension, we have that $\text{gr-G.dim}((N)^g) \leq \text{G.dim}_R(N)$).

On the other hand, by Corollaries II.3.3 and II.5.21 of [8] we have that $(N)^g$ is noetherian (resp.artinian, has Krull dimension, has Gabriel dimension). Moreover, $\text{gr.K.dim}((N)^g) = \text{K.dim}((N)^g)$ (resp.$\text{gr.G.dim}((N)^g) = \text{G.dim}((N)^g)$. Hence $\text{K.dim}_R(N)^g) \leq \text{K.dim}_R(N)$.
Since $N \subseteq (N)^g$, then we also have $\text{K.dim}_R(N) \leq \text{K.dim}_R((N^g)$ and therefore $\text{K.dim}_R(N) = \text{K.dim}_R((N)^g)$. Analogously, for the case of Gabriel dimension, we obtain that $\text{G.dim}_R((N)^g) = \text{G.dim}_R N$.

4) If N is a simple R-module, then $Col_G(N)$ is a simple $R\{G\}$-module; and by Proposition I.7.9 of [8] we obtain

that $Col_G(N)$ is a semisimple $\tilde{R} \# G^*$-module of finite length By Lemma 2.2, $(N)^g$ is a semisimple $\tilde{R} \# G^*$-module of finite length Now by Corollary 1.2, $(N)^g$ is gr-semisimple.

Corollary 2.1. [9] . Assume that the group G is finite. Then the following assertions hold.

1) If M R-mod is noetherian (resp.artinian) then M is isomorphic to a submodule of a noetherian (resp.artinian) graded R-module

2) If M R-mod has Krull dimension (resp.Gabriel dimension)then M is isomorphic to a submodule of a graded R-module having the same Krull (resp.Gabriel) dimension.

3) If M is a simple R-module, then M is isomorphic to a submodule of a gr-simple module.

Proof: If M R-mod, we consider the coinduced R-module $M' = Hom_{R_1}({}_{R_1}R_{R_1}, M)$, which has the grading

$$M'_\sigma = \left\{ f \in Hom_{R_1}(R, M) \;\middle|\; f(R_\tau) = 0 \;\forall \tau \neq \sigma^{-1} \right\} \text{ (see [7])}$$

On the other hand, M is isomorphic to an R-submodule of M' via the canonical R-monomorphism

$$\alpha_M : M \longrightarrow Hom_{R_1}(R, M)$$

$\alpha_M(m)(a) = am$. We can apply now Theorem 2.1.

Using Corollary 1.2 and Lemma 2.2, assertion 2), we obtain the following:

Theorem 2.2. Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a G-graded ring, where G is a finite group. If M is a graded R-module and $N \leq M$ is a submodule, then the following assertions hold.

1) If $M/_N$ is noetherian (resp.artinian), then so is $M/_{(N)_g}$.

2) If $M/_N$ has Krull dimension, then so does $M/_{(N)_g}$, and

$$K.\dim_R(M/_N) = K.\dim_R(M/_{(N)_g})$$

3) If $M/_N$ has Gabriel dimension, then so does $M/(N)_g$, and

$$G.\dim_R(M/N) = G.\dim_R(M/(N)_g)$$

4) If M/N is a simple R-module, then $M/_{(N)_g}$ is gr-semisimple of finite length.

We give now sdme applications of theorems 2.1 and 2.2.

Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a graded ring of type G. If $H \triangleleft G$ is a normal subgroup of G, then R has, in natural may a graduation of type $G/_H$: if $C = \sigma H = H\sigma$ is an element of $G/_H$, then we put

$$R_C = \bigoplus_{h \in H} R_{\sigma h} = \bigoplus_{h \in H} H_{h\sigma}$$

Obviously, $R = \bigoplus_{C \in G/_H} R_C$, and $R_C R_{C'} \subseteq R_{CC'}$ for any $C, C' \in G/_H$ (see [8]). Analogously, if $M = \bigoplus_{\sigma \in G} M_\sigma \in R\text{-gr}$, we can consider on M a graduation of type $G/_H$ constructed as above.

Now if M R-modmx has Krull (resp.Gabriel) dimension then for any ordinal $\alpha \geqslant 0$ there exist a largest submodule $\mathcal{T}_\alpha(M)$ of M, having Krull (resp.Gabriel) dimension less then or equal to α (see [5] and [6]).

<u>Corollary 2.2</u>. Ler $R = \bigoplus_{\sigma \in G} R_\sigma$ be a G-graded ring, G a finite group. Let $M \in R\text{-gr}$, and assume that M has a Krull (resp.Gabriel) dimension. Then for any ordinal $\alpha \geqslant 0$. $\mathcal{T}_\alpha(M)$ is a graded submodule of M.

Proof. By Theorem 2.1 , $\tau_\alpha(M) = (\tau_\alpha(M))^g$, and therefore (M) is a graded submodule of M.

Corollary 2.3. Let $R = \bigoplus_{i \in \mathbb{Z}} R_i$ be a $\mathbb{Z}$-graded ring. Let $M \in R$-gr and assume that M has Krull (resp.Gabriel) dimension Then for any ordinal $\alpha \geq 0$. $\tau_\alpha(M)$ is a graded submodule of M.

Proof: If n is a natural number, $n > 1$, let $\mathbb{Z}_n = \mathbb{Z} / n\mathbb{Z}$ and consider R with a graduation of type $\mathbb{Z}_n$.

We denote by$(\tau_\alpha(M))^{g,n}$ the smallest graded submodule of M which contain $\tau_\alpha(M)$ when R and M being considered with the graduation of type $\mathbb{Z}_n$.

By Corollary 2.2, we have $\tau_\alpha(M) = (\tau_\alpha(M))^{g,n}$ for any $n \geq 1$

Let now $x \in \tau_\alpha(M)$, we can write

$$x = x_{-s} + x_{-(s-1)} + x_0 + \ldots + x_t, \quad \text{where } x_{-s}, \ldots, x_0, \ldots, x_t$$

are the homogeneous components of x in the initial graduation of M, $M = \bigoplus_{i \in \mathbb{Z}} M_i$. We remark that for any $n > s+t$, $x_{-s}, \ldots, x_0, \ldots$ $\ldots, x_t$, remain also the homogeneous components of x if M is considered as a $\mathbb{Z}_n$-graded module. Since in this case $\tau_\alpha(M) =$ $= (\tau_\alpha(M))^{g,n}$, then $x_{-s}, \ldots, x_t \in (\tau_\alpha(M))^{g,n}$ and therefore $x_{-s}, \ldots, x_t \in \tau_\alpha(M)$. Thus $\tau_\alpha(M)$ is a graded submodule of M.

For to give another application we recall the definition of the "Gabriel filtration" on the category R-mod (to see [6] , pag.3). Consider the localizing subcategories $\mathcal{A}_\alpha$ of R-mod and the canonicall functors T_α : R-mod $\longrightarrow$ R-mod $/\mathcal{A}_\alpha$ defined recursivel as follows: $\mathcal{A}_0 = \{0\}$, T_0 = identity functor on R-mod. If α is not a limit ordinal, $\mathcal{A}_\alpha$ is the smallest localizing subcategor containingn all R-modules M, such that $T_{\alpha-1}(M)$ has finite lengt

If α is a limit ordinal $\mathcal{A}_\alpha$ is the smallest localizing subcategory containing $\bigcup_{\beta<\alpha} \mathcal{A}_\beta$.

If a module $M \in \mathcal{A}_\alpha$ for some α, we recall that we say that M has Gabriel dimension and on this case Gabriel dimension of M, G-dim M, is the least such α.

We denote by F_α the Gabriel topology associated to the localizing subcategory $\mathcal{A}_\alpha$; i.e.

$$F_\alpha = \left\{ I \text{ left ideal of } R \;\middle|\; R/_I \in \mathcal{A}_\alpha \right\}$$

<u>Corollary 2.4</u>. Assume that $R = \bigoplus_{\sigma \in G} R_\sigma$ is a G-graded ring where G is a finite group.

If $I \in F_\alpha$ then $(I)_g \in F_\alpha$ i.e. F_α has a cofinal system of left graded ideals.

<u>Proof</u>. Since $I \in F_\alpha$ then $G.\dim R/_I \leq \alpha$. By Theorem 2.2 we have $G.\dim R/_{(I)_g} = G.\dim R/_I$ and therefore $(I)_g \in F_\alpha$.

<u>Remark</u>. 1) If the group G is infinite the corollary 2.4 is not true. Indeed we consider the graded ring of type $\mathbb{Z}$. Let $R = K[X, X^{-1}]$ be a Laurent polynomials ring where K is a field (deg X = 1) with the grading

$$R_n = \left\{ aX^n \;\middle|\; a \in X \quad \text{for any } n \in \mathbb{Z} \right\}$$

It is obviously that R has two graded ideals $\{0\}$ and R.

On the other hand R is noetherian with the Krull dimension $K.\dim R = 1$ (hence R has also Gabriel dimension and $G.\dim R = 2$).

We observe that in this case we have

$$F_1 = \left\{ I \text{ ideal of } R \;\middle|\; G.\dim R/_I \leq 1 \right\} =$$

$$= \left\{ I \text{ ideal of } R \;\middle|\; K.\dim R/_I = 0 \right\} = \left\{ I \text{ ideal of } R \quad R/_I \text{ has the finite length.} \right\}$$

If $I \in F_1$ then $(I)_g = \{0\}$ and it is obviously that $\{0\} \notin F_1$.

2) Assume that $R = \bigoplus_{i \in Z} R_i$ is a Z-graded ring and $M = \bigoplus_{i \in Z} M_i$ is a graded R-module of finite support i.e. $Supp(M) = \{ i \in \mathbb{Z} \mid M_i \neq 0 \}$ is a finite set.

Then the assertions from Theorems 2.1 and 2.2 remain holds for a submodule N of M.

Indeed there exist a natural nombre $n \geq 1$ such that a nonzero homogeneous composants of M remain homogeneous composants of M when consider M and R are considered as graded rings of type $Z_n = \mathbb{Z}/_{n\mathbb{Z}}$.

REFERENCES

1) T.Albu and C.Năstăsescu, Infinite group-graded rings. Rings of endomorphisms and localization (to appear in JPAA).

2) W.Chin and D.Quinn, Rings graded by polycyclic by-finite groups, preprint.

3) M.Cohen and S.Montgomery, Group graded rings, Smash product and group actions. Trans.Amer.Math.Soc.282(1984), 237-258.

4) P.Gabriel, Des categories abeliennes, Bull.Soc.Math.France 9o(1962), 323-448.

5) R.Gordon and J.C.Robson, Krull Dimension Amer.Math.Soc. Memoires 133 (1973).

6) R.Gordon şi J.C.Robson, The Gabriel Dimension of a module, Journal of Algebra 29(1974), 459-473.

7) C.Năstăsescu, Some Construction ones Graded Rings. Applications (to appear in Journal of Algebra).

8) C.Năstăsescu and F.Van Oystaeyen, Graded Ring Theory, North Holland, Math.Library, vol.28(1982).

9) C.Năstăsescu, M.Van den Bergh and F.Van Oystaeyen Separable Functors and Some Constructions over graded ring (to appear in J.of.Algebra).

1o) D.Quinn, Group-graded ring and duality. Trans.Amer.Math.Soc. (1985), 154-167.

11) Bo Stenstrom, Rings of Quotients, Berlin-Heidelberg-New York 1975.