GROUP COHOMOLOGY AND THE CYCLIC COHOMOLOGY OF THE CROSS-PRODUCT

by

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Introduction

In [6] A.Connes introduced the "cyclic cohomology groups" of an algebra A over $\mathbb{C}$. Connes original motivation was the existence of a pairing with K-theory [5],[6] . It is clear now that cyclic homology has important applications also to other branches of mathematics such as ring theory and topology.

The purpose of this paper is to study the cyclic cohomology of the (algebraic) cross-product $A \rtimes G$ of a unital associative algebra A (over a field of characteristic zero) with a discrete group G. Our interest in this problem is due to the fact that it may give hints for the computation of the K-theory groups of cross-product C^*-algebras.

Good results are obtained when the group G is torsion free and the class ξ_h of the extension $0 \to \mathbb{Z}h \to G_h \to N_h \to 0$ in $H^2(N_h, \mathbb{Z}) \otimes k$ is nilpotent. (Here we have denoted by G_h, for $h \in G$ the centralizer of h = the largest subgroup of G containing h in its center). The cyclic homology groups of $A \rtimes G$, denoted by $HC_*(A \rtimes G)$, decompose naturaly as a direct sum of two subgroups, called "the homogeneous" and "the inhomogeneous" parts of $HC_*(A \rtimes G)$. The homogeneous part can be obtained from a spectral sequence $E^2_{p,q} = H_p(G, HC_q(A))$ convergent to $HC_{p+q}(A \rtimes G)$. The inhomogeneous part vanishes after inverting S. These results were obtained in the topological situation, for $G = \mathbb{Z}$, by Nest [15] .

For general G but A = k and trivial action, thus for group rings, the computation of the cyclic homology groups of $A \rtimes G = k[G]$ is due to Burghelea [3].

The free groups satisfy our consitions. (See [8] for other classes of groups satisfying this condition.) In this case our results are compatible with the results of Pimsner and Voiculescu [16], [17] and suggests that for a large class of groups the γ-part of $K_*^{top}(\overline{A \rtimes G})$ is obtained from a convergent spectral sequence with $E^2_{p,q} = H_p(G, K_q^{top}(A))$. Here K_*^{top} denote the topological K-theory functors, see [1] and $\overline{A \rtimes G}$ is the completion of $A \rtimes G$ with respect to a suitable C^*-norm. Kasparov succeeded to prove this for groups having a "special manifold" as classifying space [11]. We expect that our spectral sequence will give more insight in Kasparov's spectral sequence. We also mention that results of Pimsner [18] also suggest connections between the homology of G and the K-theory of the cross-product. If G act on a tree X [20] with a tree as fundamental domain, then the periodic cyclic cohomology of $A \rtimes G$ satisfies a six term exact sequence analogous to Pimsner's exact sequence, see theorem 2.7.

1.

In this section we shall recall some definitions and results to be used in the sequel. We also fix our notations.

1.1 Recall that in [7] Connes has defined the notion of cyclic object in a category $\mathcal{M}$. A cyclic object is a simplicial object $(X_n)_{n \geq 0}$ in $\mathcal{M}$ with an extra structure given by an action of $\mathbb{Z}_{n+1}$ on the n-th component. If we denote by t_{n+1} a distinguished generator of $\mathbb{Z}_{n+1}$ then the following identities must hold for t, the face and the degeneracy operators $d_i : X_n \longrightarrow X_{n-1}$, $s_i : X_n \longrightarrow X_{n+1}$ $0 \leq i \leq n$; the simplicial identities [14]:

(S1) $\quad d_i d_j = d_{j-1} d_i \qquad i < j$

(S2) $\quad s_i s_j = s_{j+1} s_i \qquad i \leq j$

(S3) $$d_i s_j = \begin{cases} s_{j-1} d_i & i < j \\ 1 & i = j, i = j+1 \\ s_i d_{i-1} & i > j+1 \end{cases}$$

and the cyclic identities [7]:

(C1) $$d_i t_{n+1} = \begin{cases} t_n d_{i-1} & 1 \leq i \leq n \\ d_n & i = 0 \end{cases}$$

(C2) $$s_i t_{n+1} = \begin{cases} t_{n+2}\, s_{i-1} & 1 \leq i \leq n \\ t_{n+2}^2\, s_n & i = 0 \end{cases}$$

(C3) $\quad t_n^n = 1$

One can immediately see that a cyclic object is a contravariant functor $\Lambda \to \mathcal{M}$. The explicit definition of Λ is given in [7]. This agrees with Connes' definition since Λ is isomorphic to Λ^{opp} due to [7], lemma 1.

1.2. The main example is the cyclic object $A^{\natural}$ associated to a unital associative algebra A over a commutative ring k [7]. It is defined by $A_n = A^{\otimes n+1}$ ($\otimes = \otimes_k$) and

$$d_i(a_0,\dots,a_n) = \begin{cases} (a_0,\dots,a_i a_{i+1},\dots,a_n) & 0 \leqslant i \leqslant n-1 \\ (a_n a_0, a_1,\dots,a_{n-1}) & i = n \end{cases}$$

$$s_i(a_0,\dots,a_n) = (a_0,\dots,a_i,1,a_{i+1},\dots,a_n) \qquad 0 \leqslant i \leqslant n$$

$$t_{n+1}(a_0,\dots,a_n) = (a_n,a_0,\dots,a_{n-1})$$

(we have denoted $a_0 \otimes a_1 \otimes \dots \otimes a_n$ by $(a_0,a_1,\dots,a_n)$).

The identities S1-C3 are easely verified.

1.3. If $X = (X_n)_{n \geqslant 0}$ is a cyclic object in an abelian category its Hochschild and cyclic homology are defined as follows. Let $\partial, \partial' : X_n \to X_{n-1}$ be given by $\partial = \sum_{i=0}^{n} (-1)^i d_i$ and $\partial' = \sum_{i=0}^{n-1} (-1)^i d_i$. Then $HH_*(X)$, the Hochschild homology of X, is the homology of the complex (X_n, ∂). Let $\varepsilon = 1-(-1)^n t_{n+1}$, $N = \sum_{i=0}^{n} (-1)^{ni} t_{n+1}^i$.

Define then as in [7] and [13] the double complex

$\underline{\mathcal{C}} : C_{ij}(X) = X_j,\ i,j \geqslant 0$ by

$$\begin{tikzcd}
 & {} \arrow[d, "\partial"] & {} \arrow[d, "-\partial'"] & {} \arrow[d, "\partial"] & \\
{} & C_{2p,j}(X) \arrow[l, "N"'] \arrow[d, "\partial"] & C_{2p+1,j}(X) \arrow[l, "\varepsilon"'] \arrow[d, "-\partial'"] & C_{2p+2,j}(X) \arrow[l, "N"'] \arrow[d, "\partial"] & {} \arrow[l, "\varepsilon"'] \\
{} & C_{2p,j-1}(X) \arrow[l, "N"'] \arrow[d, "\partial"] & C_{2p+1,j-1}(X) \arrow[l, "\varepsilon"'] \arrow[d, "-\partial'"] & C_{2p+2,j}(X) \arrow[l, "N"'] \arrow[d, "\partial"] & {} \arrow[l, "\varepsilon"'] \\
 & {} & {} & {} &
\end{tikzcd}$$

The cyclic homology of X, denoted $HC_*(X)$, is the homology of the total complex Tot $\underline{\mathcal{C}}$ [7], [13].

Suppose that k is a commutative ring and $\underline{\mathcal{M}}$ is the abelian category of k-module. If M is an object in $\underline{\mathcal{M}}$ we shall denote by $M^* = \mathrm{Hom}_k(M,k)$.

The Hochschild cohomology and cyclic cohomology of a cyclic object X in $\mathcal{M}$ are defined by $HH^*(X)$ = the Hochschild cohomology of X = the cohomology of (X_n^*, ∂^*) ; $HC^*(X)$ = the cyclic cohomology of X = the cohomology of Tot $\mathcal{C}^*$ [6], [7], [13]. We shall denote $HH_n(A^\natural) = HH_n(A), \dots, HC^n(A^\natural) = HC^n(A)$ if A is as in 1.2.

1.4. Convention. From now on k will denote a commutative field of characteristic 0 and all cyclic objects will be k-vector spaces.

1.5. For a small category Σ we shall denote by $k[\Sigma]$ the free k-module generated by Hom(Σ) with the obvious k-algebra structure (without unit in general) $(af)(bg) = (ab)f\circ g$ if $f\circ g$ makes sense, 0 otherwise, for any $a,b \in k$, $f,g \in$ Hom(Σ). We denote, as usual, by Δ the simplicial category [14] and recall that $\Delta \subset \Lambda$ [7].

Recall also that there exist natural isomorphisms

$HH_n(X) \simeq Tor_n^{k[\Delta]}(X, k^{\natural *})$, $HC_n(X) \simeq Tor_n^{k[\Lambda]}(X, k^{\natural *})$

$HH^n(X) \simeq Ext^n_{k[\Delta]}(X, k^\natural)$, $HC^n(X) \simeq Ext^n_{k[\Lambda]}(X, k^\natural)$ [7].

The morphism $k[\Delta] \to k[\Lambda]$ gives, using the above isomorphisms, natural transformations denoted by $I: HH_n(X) \to HC_n(X)$,

$I: HC^n(X) \to HH^n(X)$. They coincide with the transformations obtained identifying the first coloumn of $\mathcal{C}$ with (X_n, ∂).

$HC^*(k) = Ext^*_{k[\Lambda]}(k^\natural, k^\natural)$ is a ring isomorphic to $k[\sigma]$, the polinomial ring in a generator of degree 2 [7].

We shall denote by $S: HC_n(X) \to HC_{n-2}(X)$ $(HC^n(X) \to HC^{n+2}(X))$ the product by σ using the well known pairing $Tor_* \otimes Ext^* \to Tor_*$ and $Ext^* \otimes Ext^* \to Ext^*$ [14]. (The second pairing is the Yoneda product.)

1.6. S may be obtained from the periodicity of the bicomplex $\mathcal{C}$ and it fits into a Gysin type exact sequence due to Connes [6]:

$$\to HH_n(X) \xrightarrow{I} HC_n(X) \xrightarrow{S} HC_{n-2}(X) \xrightarrow{B} HH_{n-1}(X) \to$$

$$\leftarrow HH^n(X) \xleftarrow{I} HC^n(X) \xleftarrow{S} HC^{n-2}(X) \xleftarrow{B} HH^{n-1}(X)$$

(see also [3], [7], [13]).

1.7. We shall need also the following lemma of homological algebra, which is the analogue of the Cartan-Leray spectral sequence relating the homology of X/G to the homology of X for a proper free G space.

If X,Y are filtered modules $X = \bigcup_{n\geq 0} X_n$, $Y = \bigcup_{n\geq 0} Y_n$, $X_k \subset X_{k+1}$, $Y_k \subset Y_{k+1}$ and $f:X \to Y$ is a morphism of filtered modules ($f(X_n) \subset Y_n$), then we shall denote by $\Gamma f = \bigoplus_{n\geq 0} f_n$, $f_n : X_{n+1}/X_n \to Y_{n+1}/Y_n$. We shall call Γf the graded morphism associated to f.

<u>Lemma</u> Let $M = (M_n, d)_{n \geq 0}$ be a complex and G a group operating on the right on M such that each M_n is a flat G-module. If M/G denotes the complex $(M_n \otimes_G \mathbb{Z}, d \otimes 1)$ then there exists a homology spectral sequence with $E^2_{p,q} = H_p(G, H_q(M))$ convergent to $H_{p+q}(M/G)$.

If $N = (N_n, d)_{n \geq 0}$ is an other such complex and $f:M \to N$ is a morphism of complexes, commuting with the action of G, then f defines a morphism of spectral sequences such that $E^{\infty}_{p,q}(f)$ is the graded operator associated to $H_{p+q}(f): H_{p+q}(M/G) \to H_{p+q}(N/G)$.

A similar result holds for cohomology.

A proof is given in [2].

1.8. For later use denote by $\beta_n(G)$ the free $\mathbb{Z}$-module generated by symbols $[g_0, \dots, g_n]$ with $g_i \in G$.
Let $d_i[g_0, \dots, g_n] = [g_0, \dots, \hat{g}_i, \dots, g_n]$, $s_i[g_0, \dots, g_n] = [g_0, \dots, g_i, g_i, \dots, \dots, g_n]$. $\beta_n(G)$ is a free left G-module if we let

$g[g_0,\ldots,g_n] = [gg_0,\ldots,gg_n]$. $(\beta_n^{\updownarrow}(G),\partial)$ is the standard resolution of the trivial G-module $\mathbb{Z}$ [14]. We have adopted the convention that $\hat{g}_i$ means that g_i is omitted.) $\beta_n^{\updownarrow}(G)$ has an obvious action of $\mathbb{Z}_{n+1}$, $t_{n+1}[g_0,\ldots,g_n] = [g_n,g_0,\ldots,g_{n-1}]$ making $\beta_*(G)$ a right $k[\Lambda]$ module.

2.

In this section we compute the homogeneous part of the cyclic homology of the cross-product and give a simple proof for the nilpotency of S on the inhomogeneous parts, under the hypothesis that the normaliser has finite homological dimension over k.

2.1. Let A be a unital associative algebra over a field k of characteristic 0. Suppose that we are given a discrete group G acting on A by unit preserving automorphisms $\alpha : G \longrightarrow \mathrm{Aut}(A)$.
We shall denote by $B = A \rtimes G$ = the algebraic cross-product of A by G. It consits of finite sums $\sum a_g u_g$ and $(au_g)(bu_h) = a\alpha_g(b)u_{gh}$ for any $a,b \in A, g,h \in G$.

2.2. If $h \in G$, denote by $G_h = \{g \in G, gh = hg\}$ = the centraliser of h in G and $G_h/\mathbb{Z}h = N_h$ = the normaliser of h. Let $\langle G \rangle$ denote the set of conjugacy classes of G.

2.3. If $x \in G$, define $L(A,G,x)$ by $L(A,G,x)_n$ = the k-submodule of B_n generated by those $(a_0, a_1, \ldots, a_n, g_0, g_1, \ldots, g_n)$ such that $g_0 g_1 \ldots g_n \in x$. Here $(a_0, a_1, \ldots, a_n, g_0, g_1, \ldots, g_n)$ stands for

$$a_0 u_{g_0} \otimes a_1 u_{g_1} \otimes \ldots \otimes a_n u_{g_n}.$$

<u>Lemma</u> $L(A,G,x)$ is a $k[\Lambda]$ submodule of $B^{\natural}$ and $B^{\natural} \simeq \bigoplus_{x \in \langle G \rangle} L(A,G,x)$.

<u>Proof</u>. Obvious.

We obtain the following theorem:

2.4. Theorem $HC_*(B) \simeq \bigoplus_{x \in \langle G\rangle} HC_*(L(A,G,x))$,

$$HC^*(B) \simeq \prod_{x \in \langle G\rangle} HC^*(L(A,G,x)).$$

Proof. Obvious.

The part corresponding to $x = \{e\}$ will be called ,"the homogeneous part", the other part will be called "the inhomogeneous part".

2.5. Fix $x \in \langle G\rangle$ and $h \in x$. We define now a "simply connected covering" of $L(A,G,x)$, corresponding to the covering $EG \longrightarrow BG$. For $A = k$ and $x = \{e\}$ it is due to Karoubi [10]. For $A = k$ and general x it is an "algebraic" alternative to the topological reasoning used by Burghelea [3].

Let $\tilde{L}(A,G,h)$ be defined by $\tilde{L}(A,G,h)_n = B^{\otimes n+1}$.

The covering $p:\tilde{L}(A,G,h) \longrightarrow L(A,G,x)$ is defined on generators by

$$p(a_0,\ldots,g_n) = (\alpha_{g_n}^{-1}(a_0),\alpha_{g_0}^{-1}(a_1),\ldots,\alpha_{g_{n-1}}^{-1}(a_n),g_n^{-1}hg_0,g_0^{-1}g_1,\ldots,g_{n-1}^{-1}g_n)$$

Define on $\tilde{L}(A,G,h)$ the operators d_0, s_0, T_{n+1} by

$$d_0(a_0,\ldots,g_n) = (a_0\alpha_h(a_1),a_2,\ldots,a_n,g_1,\ldots,g_n)$$

$$s_0(a_0,\ldots,g_n) = (a_0,1,a_1,\ldots,a_n,g_0,g_0,g_1,\ldots,g_n)$$

$$T_{n+1}(a_0,\ldots,g_n) = (a_n,\alpha_h^{-1}(a_0),a_1,\ldots,a_{n-1},h^{-1}g_n,g_0,\ldots,g_{n-1})$$

Define also a right action of G_h by

$$g(a_0,\ldots,g_n) = (\alpha_g^{-1}(a_0),\ldots,\alpha_g^{-1}(a_n),g^{-1}g_0,\ldots,g^{-1}g_n)$$

$L(A,G,h)$ becomes a free G_h module.

Let $d_i = T_n^i\, d_0\, T_{n+1}^{-i}$, $s_i = T_{n+2}^i\, s_0\, T_{n+1}^{-i}$, $1 \leq i \leq n$.

The explicit formulae are

$$d_i(a_0,\dots,g_n) = (a_0,\dots,a_i a_{i+1},\dots,g_n,g_0,\dots,\hat{g}_i,\dots,g_n) \quad 1 \leq i \leq n-1$$

(i.e. g_i is omitted, see also 1.8.)

$$d_n(a_0,\dots,g_n) = (a_n a_0,\dots,a_{n-1},g_0,\dots,g_{n-1})$$

$$s_i(a_0,\dots,g_n) = (a_0,\dots,a_i,1,a_{i+1},\dots,a_n,g_0,\dots,g_i,g_i,\dots,g_n)$$

One easily verifies

Lemma a) The operators d_i, s_i, T_n satisfy (S1) to (C2),

b) $T_n^n = h$,

c) $gd_i = d_i g, gs_i = s_i g, gT_n = T_n g$,

d) $pd_i = d_i p, ps_i = s_i p, pT_n = t_n p$,

e) p factors to give an isomorphism $\tilde{L}(A,G,h)/G_h \to L(A,G,x)$.

2.6. The previous discussion shows that $\tilde{L}(A,G,e)$ is a right $k[\Lambda]$ module. (e is the unit of G). Moreover $L(A,G,e)_n = A_n \otimes \beta_n(G)$ and the face, degeneracy and cyclic actions are those comming from factors. This gives the following

<u>Theorem</u> There exist homology (cohomology) spectral sequences with $E^2(E_2)$-terms given by $EH^2_{p,q} = H_p(G,HH_q(A))$, $EC^2_{p,q} = H_p(G,HC_q(A))$, $EH_2^{p,q} = H^p(G,HH^q(A))$, $EC_2^{p,q} = H^p(G,HC^q(A))$, which converge to $HH_{p+q}(L(A,G,\{e\}))$,

$HC_{p+q}(L(A,G,\{e\}))$, $HH^{p+q}(L(A,G,\{e\})$, $HC^{p+q}(L(A,G,\{e\}))$ respectively

The morphisms I and S define morphisms $E^r_{p,q}(I):EH^r_{p,q} \longrightarrow EC^r_{p,q}$ and $E^r_{p,q}(S):EC^r_{p,q} \longrightarrow EC^r_{p,q-2}$ $\quad r = 0,1,\dots,\infty$ such that $E^{\infty}_{p,q}(I)$ and $E^{\infty}_{p,q}(S)$ are the graded operators defined by I and S of the Connes' exact sequence of $L(A,G,\{e\})$. A similar statement holds for the cohomology.

Proof. Everything will follow from lemma 1.7. applied to $(L(A,G,e),\partial)$ or to $\mathscr{C}(L(A,G,e))$ provided that we prove that $\varphi : A^{\natural} \to \tilde{L}(A,G,e)$ defined by $A_n \ni (a_0,\dots,a_n) \to (a_0,\dots,a_n,e,\dots,e) \in \tilde{L}(A,G,e)_n$ gives G-isomorphisms $HH_*(A) \to HH_*(\tilde{L}(A,G,e)),\dots,HC^*(A) \to HC^*(\tilde{L}(A,G,e))$.

Consider the following homotopy commutative diagram:

$$\begin{tikzcd} A^{\natural}_n \arrow[rr, "\varphi"] \arrow[dr, "\psi"'] & & L(A,G,e)_n \arrow[dl, "f"] \\ & \bigoplus\limits_{i+j=n} A_i \otimes \beta_j(G) & \end{tikzcd}$$

Here f is the Alexander-Whitney morphism [14] and
$\psi(a_0,\dots,a_n) = (a_0,\dots,a_n) \otimes [e] \in A^{\natural}_n \otimes \beta_0(G)$

The augmentation $\varepsilon : \beta_0(G) \to \mathbb{Z}$, $\varepsilon[g] = 1$ gives an isomorphism $H_0(\beta_*(G)) \simeq \mathbb{Z}$, the other homology groups vanish. It is easy to see that ψ induces in homology the product with the generator of $H_0(\beta_*(G))$ and, since k is a field, gives an isomorphism in homology [14] . Since f gives a quasiisomorphism [14] (i.e. an isomorphism in homology), it follows that φ is also a quasiisomorphism. This quasiisomorphism commutes with the action of G since the class of $(a_0,\dots,a_n,g_0,\dots,g_n)$ in $HH_*(\tilde{L}(A,G,e))$ depends only on $(a_0,\dots,a_n)$.

This is enough to conclude that $HC_*(\varphi)$ is an isomorphism since, due to the spectral sequence of Loday and Quillen relating HH_* to HC_* [13] , any morphism of cyclic modules which gives isomorphism for HH_* gives isomorphism also for HC_*.

Unfortunat~~e~~ely this gives no precise result about $HC_*(\varphi)g - gHC_*(\varphi)$, it gives only the vanishing of its graded operator However this argument suffices to show that $j_g : k_n^\natural \ni 1 \longrightarrow (g,g,\dots,g) \in \beta_n(G)$ give isomorphisms independent of $g \in G$. Observe that we have to prove that $\varphi_g : A_n^\natural \ni (a_0,\dots,a_n) \longrightarrow (a_0,\dots,a_n,g,\dots,g) \in \tilde{L}_n(A,G,e)$ satisfies $HC_*(\varphi_g) = HC_*(\varphi)$. We shall use here a theorem of C.Kassel [12], Theorem 2.4.Denote by $\square$ the cotensor product over the coalgebra $HC_*(k) = k[u]$ with commultiplication $u \to u \otimes 1 + 1 \otimes u$. Kassel's theorem shows that if X,Y,Z are cyclic modules and $X_n = Y_n \otimes Z_n$ with induced diagonal $k[\Lambda]$ structure then there exists a natural isomorphism

$$0 \to \mathrm{Cotor}_1^{k[u]}(HC_*(Y), HC_{*-1}(Z)) \to HC_*(X) \to HC_*(Y) \,\square\, HC_*(Z) \to 0$$

(Cotor_1 is the first derived functor of $\square$) see also [4].

We use this exact sequence, for $A^\natural, A^\natural, k^\natural$: $A_n^\natural = A_n^\natural \otimes k_n^\natural$ and $\tilde{L}(A,G,e), A^\natural, \beta_*(G)$: $\tilde{L}(A,G,e)_n = A_n^\natural \otimes \beta_n(G)$. In fact we mainly use its naturality for $(id, j_g) : (A^\natural, k^\natural) \longrightarrow (A^\natural, \beta_*(G))$. We obtain, since $HC_*(j_g)$ does not depend on g and the Cotor_1-group vanishes, that also $HC_*(\varphi_g) = id \,\square\, HC_*(j_g)$ does not depend on g.

For cohomology we use the universal coeficient theorem.

2.7. Let us turn now to the case $h \neq e$.

Denote by M_h the complex $(M_h)_n = A^{\otimes n+1}$ with differential $d = d_0' - d_1 + \dots + (-1)^n d_n$, $d_0'(a_0,\dots,a_n) = (a_0 \alpha_h(a_1), a_2, \dots, a_n)$.

One can prove as in theorem 2.6. that $HH_*(\tilde{L}(A,G,h)) \simeq H_*(M_h)$. If h acts trivial one can proceed further as in theorem 2.6. It seems, in general, that there is no connection between $HC_*(\tilde{L}(A,G,h))$ and $HC_*(A)$. However one has the following lemma.

Lemma. The inclusion $L(A,G_h,\{h\}) \to L(A,G,x)$ induces isomorphisms for both Hochschild and cyclic homology.

Proof. We observe first that there exists an inclusion $\tilde{L}(A,G_h,h) \to \tilde{L}(A,G,h)$ which induces isomorphism on Hochschild homology since $HH_*(\tilde{L}(A,G_h,h)) \simeq H_*(M_h) \simeq HH_*(\tilde{L}(A,G,h))$. Then, using lemma 1.7. we obtain that $HH_*(L(A,G_h,\{h\})) \simeq HH_*(L(A,G,x))$. The other isomorphisms follow from the spectral sequence connecting Hochschild to cyclic homology [13] and from the universal coeficient theorem.

Suppose that G acts on a tree X without inversion such that it has a tree as fundamental domain [20]. Let $Y = G\backslash X$. Denote by G_P and G_y the stabilisers of the vertex $P \in X^0$ and of the edge $y \in X^1$. Identify Y with a subtree of X [20]. Then there exists a six-term exact sequence connecting the periodic cyclic cohomology of B, denoted $PHC^*(B)$ $\underset{\to}{=} \varinjlim(HC^*(A),S)$ with the periodic cyclic cohomology of $A \rtimes G_P$ and $A \rtimes G_y$ with $P \in Y^0, y \in Y^1$. Our notations are taken from [20].

Theorem a) There exists an exact sequence

$$\begin{array}{ccccc}
\bigoplus_{y\in Y^{1+}} PHC^{even}(A \rtimes G_y) & \xleftarrow{\delta} & \bigoplus_{P\in Y^0} PHC^{even}(A \rtimes G_P) & \longleftarrow & PHC^{even}(A \rtimes G) \\
\downarrow & & & & \uparrow \\
PHC^{odd}(A \rtimes G) & \longrightarrow & \bigoplus_{P\in Y^0} PHC^{odd}(A \rtimes G_P) & \xrightarrow{\delta} & \bigoplus_{y\in Y^{+1}} PHC^{odd}(A \rtimes G_y)
\end{array}$$

b) There exists a subgroup $Q_n \subseteq HC_n\ (A \rtimes G)$ such that $Q_n + \ker S = HC_n\ (A \rtimes G)$, $SQ_n \subseteq Q_{n-2}$ and an exact sequence

$$\to Q_{n+1} \to \bigoplus_{y \in Y^{1+}} HC_n\ (A \rtimes G_y) \to \bigoplus_{P \in Y^0} HC_n\ (A \rtimes G_P) \to Q_n \to$$

A dual statement holds for $HC^n\ (A \rtimes G)$. (Here Y^{1+} is the set of positive oriented arrows of Y).

The morphisms are induced by the inclusions $A \rtimes G_P \to A \rtimes G$, $A \rtimes G_y \to A \rtimes G_P$ if $P = t(y)$ and by the opposite if $P = 0(y)$ as in [20]

Proof. Let $h \in G$. Suppose that G_h contains no conjugate of an element in $G_P, P \in Y^0$. Then G_h acts free on X and hence is a free group [20]. Since h is central $\neq e$ in G_h it follows that $G_h \simeq \mathbb{Z}$. It follows that $L(A,G,\langle h\rangle) \subseteq \ker S$ (lemma 2.8.). $\langle h\rangle$ stands for the conjugacy class of h in G.

If $gg_Pg^{-1} \in G_h$ and $g_P \in G_P$ it follows that g_P commutes with $g^{-1}hg$ and hence $g^{-1}hg \in G_P$ ([20], 4.5. theorem 9). This shows that we may suppose $h \in G_P$. Let Y_h be the subtree consisting of those $P \in Y^0$ and $y \in Y^1$ such that $h \in G_P$ and $h \in G_y$. Then $N_{P,h} = (G_P)_h / \mathbb{Z}h$ and $N_{y,h} = (G_y)_h / \mathbb{Z}h$ define a graph of groups and $N_h = G_h / \mathbb{Z}h$ is easely seen to be isomorphic to the fundamental group of this graph [20].

Define $Q_n = \bigoplus_{x \in \cup G_P} L(A,G,x)$

Claim If $M = (M_n, d)$ is as in lemma 1.7. then there exists an exact sequence

$$\to H_{n+1}\ (M/G) \to \bigoplus_{y \in Y^{1+}} H_n(M/G_y) \to \bigoplus_{P \in Y^0} H_n(M/G_P) \to H_n(M/G) \to$$

Proof of the claim: The tree X has an obvious simplicial structure: it is a CW complex of dimension 1. Its homology is computed as the homology of the complex $0 \to C^1(X) \xrightarrow{d} C^0(X) \to 0$. $C^1(X)$ is the free abelian group on X^{1+} and $C^0(X)$ is the free abelian group on X^0. The tree is acyclic and hence we obtain an exact sequence

$$0 \to C^1(X) \xrightarrow{d} C^0(X) \to \mathbb{Z} \to 0$$

Since each M_n is projective as G-module we obtain an exact sequence of complexes:

$$0 \to M \otimes_G C^1(X) \to M \otimes_G C^0(X) \to M \otimes_G \mathbb{Z} \to 0$$

The lemma is nothing but the long homology exact sequence of this short exact sequence of complexes.

<u>End of the proof of the theorem:</u> Let $h \in G_P$. The inclusion $L(A,G_P,h) \subseteq L(A,G,h)$ gives isomorphism for Hochschild homology as follows from the discussion in the beginning of this section. It follows that the inclusion of cyclic objects $L(A,G_P,h)/\mathcal{G}_h \to L(A,G,h)/\mathcal{G}_h$ also gives isomorphism for Hochschild homology and hence also for cyclic homology (use the spectral sequence relating HC_* to HH_*). Using the same argument it follows that $L(A,G_P,\langle h\rangle) \to (L(A,G_P,h)/\mathbb{Z}h)/N_{P,h} \to (L(A,G\ ,h)/\mathbb{Z}h)/N_{P,h}$ gives isomorphism for cyclic homology. Then use the claim for N_h acting on the Connes complex of $L(A,G,h)/\ _h$. The corresponding long exact sequences for all $x \in \langle G\rangle$ containing at least one $h \in G_P$ for some $P \in X^0$ build up the exact sequence of the theorem, part b, in view of the preceding isomorphisms.

Part a) follows from part b) using the exactness of lim.

2.8. As anticipated in the proof of theorem 2.7. it may happen that S is nilpotent on certain components $HC^*(L(A,G,x)), x \in \langle G\rangle$.

It turns out that this is related to the cohomology of N_h. The next section contains a more detailed analysis of this phenomenon. Here we content ourselves to give a short proof to a particular case of theorem 3.

<u>Proposition</u> Suppose that $h \in x$ is torsion free and that N_h has finite homological dimension over k. Then the S operator of the Connes exact sequence of $L(A,G,x)$ is nilpotent (i.e there exists $m \in \mathbb{N}$ such that $S^m = 0$).

<u>Lemma</u> If $G = \mathbb{Z}$, $h = 1$ then $S = 0$.

<u>Proof of the lemma</u> Consider the bicomplex

$$\begin{tikzcd}
{} & \tilde{L}(A,\mathbb{Z},1)_n \arrow[l, "-\partial'"'] \arrow[d, "1-(-1)^n T_{n+1}"] & L(A,\mathbb{Z},1)_{n+1} \arrow[l, "-\partial'"'] \arrow[d, "1-(-1)^{n+1} T_{n+2}"] & {} \arrow[l, "-\partial'"'] \\
{} & \tilde{L}(A,\mathbb{Z},1)_n \arrow[l, "\partial"'] & \tilde{L}(A,\mathbb{Z},1)_{n+1} \arrow[l, "\partial"'] & {} \arrow[l, "\partial"']
\end{tikzcd}$$

The homology of the total complex is $HC_*(L(A,\mathbb{Z},\{1\}))$. It follows that there exists a commutative diagram

$$\begin{tikzcd}
& HH_n(\tilde{L}(A,\mathbb{Z},1)) \arrow[dl] \arrow[dr, "q"] & \\
HH_n(L(A,\mathbb{Z},\{1\})) \arrow[rr, "I"] & & HC_n(L(A,\mathbb{Z},\{1\}))
\end{tikzcd}$$

q being obtained by identifying the Hochschild complex of $\tilde{L}(A,\mathbb{Z},1)$ with the bottom line of the previous bicomplex. It follows that q is an isomorphism since the top line is acyclic. This shows that I is onto and hence S vanishes.

<u>Proof of the proposition.</u> As in lemma 2.7.1 $L(A,\mathbb{Z},1) \to \tilde{L}(A,G,h)/\mathbb{Z}h$ induces isomorphisms in both Hochschild and cyclic homology. This shows that $S = 0$ on $HC_*(L(A,G,h)/\mathbb{Z}h)$. Note that $\tilde{L}(A,G,h)/\mathbb{Z}h$ is indeed a cyclic object. Since $L(A,G,x) \simeq (\tilde{L}(A,G,h)/\mathbb{Z}h)/N_h$ it follows from lemma 1.7. that $HC_*(L(A,G,x))$ has a filtration such that the graded operator of S vanishes. The length of this filtration is at most the homological dimension of N_h over k and hence S is nilpotent.

3

In this section we make a more detaliate discussion of the action of S on $HC_*(L(A,G\ ,x))$ for $h \in x$ torsion free. It follows that $HC_*(L(A,G,x))$ is a module over $H^*(N_h,k)$ and we identify the action of S as the multiplication by the class of $0 \to \mathbb{Z}h \to G_h \to N_h \to 0$ in $H^2(N_h,k)$.

3.1. Define the category $\Lambda \rtimes G_h$ to be the opposite of the category having $(\beta_n(G))_{n\in\mathbb{N}}$ as objects and as morphisms all compositions of d_i, s_i, T_i and g $(g \in G_h)$ of 2.5 for $A = k$. Note that $\Lambda \rtimes G_h$ contains the category $\Delta \times G_h$. If we denote by $(\Lambda \rtimes G_h)_n = \beta_n(G)^{opp}$ = the n-th object of $\Lambda \rtimes G_h$ then the group of automorphisms of $(\Lambda \rtimes G_h)_n$ is generated by T_{n+1} and G_h and is determined by the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, "n+1"] & G_h \arrow[r] \arrow[d] & N_h \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathrm{Aut}(\Lambda \rtimes G_h)_n \arrow[r, "\pi"] & N_h \arrow[r] & 0
\end{tikzcd}$$

It follows form the definition of $\Lambda \rtimes G_h$ and [7], lemma 2, that every morphism φ in $\Lambda \rtimes G_h$ can be uniquely written as $\varphi_0\varphi_1$ with $\varphi_0 \in \Delta$ and $\varphi_1 \in \mathrm{Aut}(\Lambda \rtimes G_h)_n$ for some n.

It is obvious form the definition that $\tilde{L}(A,G,h)$ has a right $k[\Lambda \rtimes G_h]$-structure.

3.2. Observe that there exists a functor $\Lambda \rtimes G_h \to \Lambda$ defined by $(\Lambda \rtimes G_h)_n \to \Lambda_n$, $d_i \to d_i$, $s_i \to s_i$, $T_n \to t_n$ and $G_h \to 1$. This gives a morphism of k-algebras $k[\Lambda \rtimes G_h] \to k[\Lambda]$.

Let M be a $k[\Lambda \rtimes G_h]$ module, using the above morphism we define a transformation of functors $\mathrm{Tor}_*^{k[\Lambda \rtimes G_h]}(M, k^{\natural *}) \longrightarrow \mathrm{Tor}_*^{k[\Lambda \rtimes G_h]}(M/G_h, k^{\natural *}) \longrightarrow \mathrm{Tor}_*^{k[\Lambda]}(M/G_h, k^{\natural *})$. We denote by f_0 this composition. Note that M/G_h is a $k[\Lambda]$ module.
$f_1 : \mathrm{Ext}^*_{k[\Lambda]}(M_{G_h}, k^{\natural}) \longrightarrow \mathrm{Ext}^*_{k[\Lambda \rtimes G_h]}(M, k^{\natural})$ is defined similar~~l~~ly.

<u>Lemma</u> Suppose that M is a right $k[\Lambda \rtimes G_h]$ module which is projective as G_h module, then f_0 and f_1 are isomorphisms.

<u>Proof.</u> The general strategy for proving this is that used for prooving the uniqueness of the derived functors. We show that :

a) f_0 and f_1 are isomorphisms for Tor_0 and Ext^0;

b) the functors satisfy long (co)-homology exact sequences;

c) f_0 and f_1 commute with the connecting morphisms of these exact sequences;

d) the category of right $k[\Lambda \rtimes G_h]$ modules, projective as G_h modules, contains the kernel of each epimorphism;

e) f_0 and f_1 are isomorphisms for free $k[\Lambda \rtimes G_h]$ modules.

With this in mind everything is simple. a) is an obvious computation. b) and c) follow from definition, the proprieties of Tor, Ext and of the functor of changing the ring [14]. d) is a consequence of the projectivity. $k[\Lambda \rtimes G_h]/G_h \simeq k[\Lambda]$ as $k[\Lambda]$ right modules.
It follows that for M a free $k[\Lambda \rtimes G_h]$ module, M/G_h is a free $k[\Lambda]$ module and hence both groups are 0 for $* > 0$. This proves e) and concludes the proof.

3.3. Let us contemplate the following commutative diagrams (denote $k[\Lambda \rtimes G_h]$ by R, $k[\Lambda]$ by R_0)

$$\begin{tikzcd}
\mathrm{Tor}^{R}_{m}(M,k^{\natural *}) \otimes \mathrm{Ext}^{p}_{R}(k^{\natural *},k^{\natural *}) \arrow[r] & \mathrm{Tor}^{R}_{m-p}(M,k^{\natural *}) \\
\mathrm{Tor}^{R_0}_{m}(M/G_h,k^{\natural *}) \otimes \mathrm{Ext}^{p}_{R_0}(k^{\natural *},k^{\natural *}) \arrow[r] \arrow[u, "f_0^{-1}\otimes\iota"] & \mathrm{Tor}^{R_0}_{m-p}(M/G_h,k^{\natural *}) \arrow[u, "f_0^{-1}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Ext}^{m}_{R}(M,k^{\natural}) \otimes \mathrm{Ext}^{p}_{R}(k^{\natural},k^{\natural}) \arrow[r] & \mathrm{Ext}^{m+p}_{R}(M,k^{\natural}) \\
\mathrm{Ext}^{m}_{R_0}(M/G_h,k^{\natural}) \otimes \mathrm{Ext}^{p}_{R_0}(k^{\natural},k^{\natural}) \arrow[r] \arrow[u, "f_1\otimes\iota"] & \mathrm{Ext}^{m+p}_{R}(M/G_h,k^{\natural}) \arrow[u, "f_1"']
\end{tikzcd}$$

Here M is a right R module projective as G_h-module and $\iota : \mathrm{Ext}_{R_0} \longrightarrow \mathrm{Ext}_R$ is defined using $R \longrightarrow R_0$.

It follows that $\mathrm{Tor}^R_*(-,k^{\natural *})$ has a right $\mathrm{Ext}^*_R(k^{\natural *},k^{\natural *})$-module structure and that the action of S is obtained via the multiplication with the image of $\sigma \in k[\sigma] \simeq$ $\mathrm{Ext}_{R_0}(k^{\natural *},k^{\natural *}) \longrightarrow \mathrm{Ext}_{R_0}(k^{\natural *},k^{\natural *}) \longrightarrow \mathrm{Ext}_R(k^{\natural *},k^{\natural *})$ (we used the fact that $R_0 \simeq R_0^{opp}$ since Λ is isomorphic to Λ^{opp} [7], lemma 1).
The same is true of Ext.

3.4. We are left with the determination of $\mathrm{Ext}_R(k^{\natural},k^{\natural})$ and of the morphism $\mathrm{Ext}_{R_0}(k^{\natural},k^{\natural}) \longrightarrow \mathrm{Ext}_R(k^{\natural},k^{\natural})$

It is well known that
$\mathrm{Ext}^*_R(k^{\natural},k^{\natural}) \simeq H^*(B(\Lambda\rtimes G_h),k) \simeq H^*(B(\Lambda\rtimes G_h)^{opp}, k) \simeq \mathrm{Ext}_R(k^{\natural *}, k^{\natural *})$
[19]. Here we have denoted by $B(\Lambda\rtimes G_h)$ the classifying space of the small category $\Lambda\rtimes G_h$ [9], see also [19].

Consider the following commutative diagram of categories and functors

$$\begin{tikzcd} & \Delta\times G_h \arrow[dl] \arrow[d] \arrow[r] & \Delta \arrow[d] \\ N_h & \Lambda\rtimes G_h \arrow[l, "\rho"'] \arrow[r] & \Lambda \end{tikzcd}$$

$B\Delta$ is contractible, $B\Lambda = P_\infty(\mathbb{C}) = BS^1$ due to [7], theorem 10. $B(\Delta\times G_h) = B\Delta \times BG_h = BG_h\cdot\Delta\times G_h \to N_h$ is defined using the morphism $G_h \to N_h$. ρ is given by $\rho((\Lambda\rtimes G_h)_n) = *$, the unique object of N_h, if $\varphi\in \mathrm{Hom}((\Lambda\rtimes G_h)_n, (\Lambda\rtimes G_h)_m)$ write $\varphi=\varphi_0\varphi_1$ with $\varphi_1 \in \mathrm{Aut}(\Lambda\rtimes G_h)_n$, $\varphi_0\in\Delta$ and let $\rho(\varphi) = \pi(\varphi_1)$.

3.5. <u>Lemma</u> ρ gives a homotopy equivalence $B(\Lambda\rtimes G_h) \to BN_h$.

<u>Proof.</u> Recall [19] the definition of the category $*\backslash\rho$. Its objects are pairs (n,t) , $n\in\mathbb{N}$, $t\in N_h$. A morphism $(n,t)\to(m,s)$ is a morphism φ: $(\Lambda\rtimes G_h)_n \to (\Lambda\rtimes G_h)_m$ such that $\rho(\varphi)t = s$. According to [19] , theorem A , it is enough to show that $*\backslash\rho$ is contractible (i.e. $B(*\backslash\rho)$ is contractible).

Let $j : \Lambda\rtimes\mathbb{Z} \to *\backslash\rho$ be defined by $j(\Lambda\rtimes\mathbb{Z})_n = (n,e)$, $j(\varphi) = \varphi$ (1 is sent to h). A left adjoint of j is defined by $j^*(n,t) = (\Lambda\rtimes\mathbb{Z})_n$; let τ: $\mathrm{Ob}(*\backslash\rho) \to \mathrm{Aut}(\Lambda\rtimes G_h)$ satisfying $\tau(n,t)\in \mathrm{Aut}(\Lambda\rtimes G_h)_n$, $\pi(\tau(n,e)) = e$. Then $j^*(\varphi) = \tau(m,s)^{-1}\varphi\tau(n,t)$ for any φ: $(n,t)\to(m,s)$. It follows that j is a homotopy equivalence [19].

$B(\Lambda\rtimes\mathbb{Z})$ is connected since $\mathrm{Hom}((\Lambda\rtimes\mathbb{Z})_n, (\Lambda\rtimes\mathbb{Z})_m)$ is not empty for any $n,m \geq 0$. The homology of $B(\Lambda\rtimes\mathbb{Z})$ with coeficients in $\mathbb{Z}$ is isomorphic to $\mathrm{Tor}_*^{\mathbb{Z}[\Lambda\rtimes\mathbb{Z}]}(\mathbb{Z}^h, \mathbb{Z}^h)$ [19]. Let us compute these groups. Let $e_n \in \mathbb{Z}[\Lambda\rtimes\mathbb{Z}]$ be the identity of $(\Lambda\rtimes\mathbb{Z})_n$. $E_n = \mathbb{Z}[\Lambda\rtimes\mathbb{Z}]e_n$ is a projective module. Let $\partial(\partial')$: $E_n \to E_{n-1}$ be the right multiplication by $\partial(\partial')$, $1-(-1)^{n-1}T_n$: $E_{n-1}\to E_{n-1}$ be the right multiplication by $1-(-1)^{n-1}T_n$. The double complex $\mathcal{E}$:

$$\begin{tikzcd}
\arrow[from=1-2, l, "-\partial'"'] & E_{n-1} \arrow[d, "1-(-1)^{n-1}T_n"] & E_n \arrow[l, "-\partial'"'] \arrow[d, "1-(-1)^{n}T_n"] & \arrow[l, "-\partial'"'] \\
\arrow[from=2-2, l, "\partial"'] & E_{n-1} & E_n \arrow[l, "\partial"'] & \arrow[l, "\partial"']
\end{tikzcd}$$

gives a projective resolution of $\mathbb{Z}^{\natural}$ over $\mathbb{Z}[\Lambda \rtimes \mathbb{Z}]$ (compare with [7], lemma 6).

$\operatorname{Tor}_*^{\mathbb{Z}[\Lambda\rtimes\mathbb{Z}]}$ is the homology of $\mathbb{Z}^{\natural} \underset{\Lambda\rtimes\mathbb{Z}}{\otimes} \mathcal{E}$,that is,the homology of *the total complex associated with*

$$\begin{tikzcd}
\mathbb{Z} \arrow[d, "0"] & \mathbb{Z} \arrow[l, "-1"'] \arrow[d, "1"] & \mathbb{Z} \arrow[l, "0"'] \arrow[d, "0"] & \mathbb{Z} \arrow[l, "-1"'] \arrow[d, "1"] & \arrow[l, "0"'] \\
\mathbb{Z} & \mathbb{Z} \arrow[l, "0"'] & \mathbb{Z} \arrow[l, "1"'] & \mathbb{Z} \arrow[l, "0"'] & \arrow[l, "1"']
\end{tikzcd}$$

It follows that $H_n(B(\Lambda\rtimes\mathbb{Z})) = 0$ if $n \geq 1$ and hence all homotopy groups of $B(\Lambda\rtimes\mathbb{Z})$ vanish (use Hurewich's theorem [21]). It follows from Whitehead's theorem that $B(\Lambda\rtimes\mathbb{Z})$ is contractible [21].

3.6. <u>Theorem</u> $\operatorname{Ext}^*_{k[\Lambda\rtimes G_h]}(k^{\natural}, k^{\natural}) \simeq H^*(N_h,k)$.
The morphism $\operatorname{Ext}^*_{k[\Lambda]}(k^{\natural}, k^{\natural}) \longrightarrow \operatorname{Ext}^*_{k[\Lambda\rtimes G_h]}(k^{\natural}, k^{\natural})$ sends the generator σ to the class of the extension $0 \to \mathbb{Z}h \to G_h \to N_h \to 0$ in $H^2(N_h,k)$.

<u>Proof.</u> The first part follows from lemma 3.5. and the isomorphism $H^*(BN_h,k) \simeq H^*(N_h,k)$.

The commutative diagram of 3.4 gives, using lemma 3.5., a commutative diagram

$$\begin{tikzcd}
BG_h \arrow[r] \arrow[d] & B\Delta = ES^1 \arrow[d] \\
BN_h \arrow[r, "g"] & B\Lambda = BS^1
\end{tikzcd}$$

which ~~it~~ is a morphism of fibrations. σ is the obstruction to a lifting of $B\Delta \longrightarrow BS^1$. It follows that $q^*\sigma = \xi \in H^2(BN_h, \mathbb{Z}) = H^2(BN_h, \pi_1(S^1))$ is the obstruction to a lifting for $BG_h \longrightarrow BN_h$.

Let us compute this obstruction. BN_h is a CW-complex, its low dimensional cells are: one 0-dimensional cell, denoted $*$; 1-dimensional cells denoted $[n]$, $n \in N_h \setminus \{e\}$, $d_0[n] = d_1[n] = *$; 2-dimensional cells $[m,n]$, $m,n \in N_h \setminus \{e\}$, $d_0[m,n] = [n]$, $d_1[m,n] = [mn]$ if $mn \neq e$, $d_1[n,n^{-1}] = *$, $d_2[m,n] = [m]$. Let $\tau : N_h \longrightarrow G_h$ satisfy $\pi(\tau(n)) = n$, $\tau(e) = e$. A lifting s on the 1-dimensional squeleton is obtained from $s* = *$, $s[n] = [\tau(n)]$. The obstruction for lifting s on $[m,n]$ coincides with the obstruction of extending $s|\partial[m,n] = S^1 \to BG_h$ to all of $[m,n]$. It coincide with the class of $s|\partial[m,n]$ in $\pi_1(BG_h) = G_h$ i.e. with $\tau(m)\tau(n)\tau(mn)^{-1}$. It follows that the same cocycle represents both ξ and the extension $0 \to \mathbb{Z} \to G_h \to N_h \to 0$ in $H^2(N_h, \mathbb{Z})$.

3.7. We obtain

<u>Corollary</u> $HC_*(L(A,G,x))$ and $HC^*(L(A,G,x))$ are modules over $H^*(N_h,k)$. The action of S corresponds to the multiplication by $\xi \in H^2(N_h,k)$, the class of the extension

$$0 \to \mathbb{Z} \to G_h \to N_h \to 0.$$

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