LEXICOGRAPHICAL ORDER, LEXICOGRAPHICAL INDEX AND LINEAR OPERATORS

by

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ABSTRACT

We study the lexicographical index of the elements of R^n, which we introduce here, and the linear operators on R^n which preserve the lexicographical order, or the lexicographical index, as well as those which do not increase or do not decrease the lexicographical index

§0. INTRODUCTION

In some previous papers, we have given applications of the lexicographical order in R^n to separation of convex sets by linear operators [9], surrogate duality in vector optimization [10] and the study of hemispaces, i.e., convex sets with convex complements [11] (see also [6]-[8] and [13], for other applications of the lexicographical order). In several proofs of these papers, there appears, implicitly, the "lexicographical index" of the elements of R^n, which we shall introduce below.

In the present paper we shall study the lexicographical index (§1), the lexicographical order preserving linear operators (§2), and the linear operators which preserve (§3), respectively, which do not increase or do not decrease (§4) the lexicographical index. Finally, in an Appendix (§5), we shall give some applications of the lexicographical order to the separation of p convex subsets of R^n.

Let us recall now some notions, notations and results which will be used in the sequel.

The elements of R^n (where $R = (-\infty, +\infty)$) will be considered column vectors, and the superscript T will mean transpose. We recall that $x = (\xi_1, \dots, \xi_n)^T \in R^n$ is said to be "lexicographically less than" $y = (\eta_1, \dots, \eta_n)^T \in R^n$ (in symbols, $x <_L y$) if $x \neq y$ and if for $k = \min \{i \in \{1, \dots, n\} | \xi_i \neq \eta_i\}$ we have $\xi_k < \eta_k$. We write $x \leq_L y$ if $x <_L y$ or $x = y$. The notations $y >_L x$ and $y \geq_L x$, respectively, will be also used. We shall denote by $\{e_j\}_1^n$, or, in some cases, by $\{e_j^n\}_1^n$, the unit vector basis of R^n. When $\{e'_j\}_1^n$ is an arbitrary basis of R^n, we shall also consider the lexicographical order on R^n. "in the basis $\{e'_j\}_1^n$", defined similarly to the above, with $x = (\xi_1, \dots, \xi_n)^T = \sum_{j=1}^n \xi_j e_j$ and $y = (\eta_1, \dots, \eta_n)^T = \sum_{j=1}^n \eta_j e_j$ replaced by $x = \sum_{j=1}^n \xi_j e'_j$ and $y = \sum_{j=1}^n \eta_j e'_j$ respectively.

We recall that, for any function $h : R^n \to \overline{R} = [-\infty, +\infty]$, and any $\lambda \in R$, the *λ-quasi-conjugate of h*, in the sense of Greenberg and Pierskalla [2], is the function $h_\lambda^\gamma : (R^n)^* \to \overline{R}$ defined by

$$h_\lambda^\gamma(\Psi) = - \inf_{\substack{y \in R^n \\ \Psi(y) \geq \lambda}} h(y) \qquad (\Psi \in (R^n)^*), \tag{0.1}$$

where $(R^n)^*$ is the conjugate space of R^n; in the sequel we shall sometimes identify $(R^n)^*$ with R^n, in the usual way (with the aid of the scalar product). Let us also recall that the *quasi-subdifferential of*

$h : R^n \to \overline{R}$ at $y_0 \in R^n$, in the sense of [2] and [14], is the subset $\partial^\gamma h(y_0)$ of $(R^n)^*$ defined by

$$\partial^\gamma h(y_0) = \{\Psi \in (R^n)^* \mid h(y_0) = -h^\gamma_{\Psi(y_0)}(\Psi)\}. \tag{0.2}$$

For a linear subspace S of R^n, we shall denote by $S^\perp$ its orthogonal complement in R^n.

We shall denote by $\mathcal{L}(R^n)$, $\mathcal{U}(R^n)$ and $\mathcal{O}(R^n)$ the families of all linear operators, all isomorphisms, and all linear isometries $v : R^n \to R^n$ respectively, and by $\mathcal{L}(R^n, R^k)$ the family of all linear operators $u : R^n \to R^k$. We shall identify each $u \in \mathcal{L}(R^n, R^k)$ with its $k \times n$ matrix with respect to the unit vector bases of R^n and R^k, that is, we shall write

$$u = (m_{ij}) = (c_1, \dots, c_n), \tag{0.3}$$

where $c_j = (m_{1j} \dots m_{nj})^T = u(e_j^n)$ $(j = 1, \dots, n)$ are the columns of (m_{ij}); hence, for example, instead of $u \in \mathcal{U}(R^n)$ we shall sometimes say, equivalently, that $u \in \mathcal{L}(R^n)$ and u is non-singular. We shall consider on $\mathcal{L}(R^n, R^k)$ the lexicographical order $u \geq_L 0$ in the sense of [7], defined columnwise (i.e., $u \geq_L 0$ if and only if all columns of u are $\geq_L 0$).

§1. LEXICOGRAPHICAL INDEX

Definition 1.1. a) Let $\{e'_j\}_1^n$ be a basis of R^n. For any $y = \sum_{j=1}^n \eta_j e'_j \in R^n$, we define *the lexicographical index* $\alpha'(y)$ *of* y *with respect to* $\{e'_j\}_1^n$, by

$$\alpha'(y) = \begin{cases} \min \quad \{j \in \{1, \dots, n\} | \eta_j \neq 0\} & \textit{if} \quad y \neq 0 \\ +\infty \quad \textit{if} \quad y = 0. \end{cases} \tag{1.1}$$

b) In the particular case when $\{e'_j\}_1^n = \{e_j\}_1^n$, the unit vector basis of R^n, we shall omit the words "with respect to $\{e_j\}_1^n$", and we shall denote the *lexicographical index of* $y = (\eta_1, \dots, \eta_n)^T = \sum_{j=1}^n \eta_j e_j \in R^n$ by $\alpha(y)$.

Remark 1.1. For $y \neq 0$, a generalization of the lexicographical index occurs implicitly, in [3], theorem 2.1; see also [1], § 3.3.

Given a basis $\{e'_j\}_1^n$ of R^n, we have the following characterization of the function $\alpha' : R^n \to \{1, \dots, n\} \cup \{+\infty\}$ defined by (1.1):

Theorem 1.1. *For a function* $\beta : R^n \to \{1, \dots, n\} \cup \{+\infty\}$ *and a basis* $\{e'_j\}_1^n$ *of* R^n, *the following statements are equivalent:*

1°. $\beta = \alpha'$, *the lexicographical index with respect to* $\{e'_j\}_1^n$.

2°. *We have*

$$\beta(0) = +\infty, \tag{1.2}$$

$$\beta(R^n) = \{1, \dots, n\} \cup \{+\infty\}, \tag{1.3}$$

$$\beta(\lambda y) = \beta(y) \qquad (y \in R^n,\ \lambda \neq 0), \tag{1.4}$$

$$y_1,\ y_2 \in R^n,\ y_1 \leq_{L'} y_2 \ \Rightarrow\ \beta(y_1) \geq \beta(y_2). \tag{1.5}$$

Proof. The implication $1^o \Rightarrow 2^o$ is obvious, by the definiton of α'.

$2^o \Rightarrow 1^o$. If 2^o holds, let $0 \neq y = \sum_{j=1}^{n} \eta_j e'_j \in R^n$. Then, since $\eta_1 = \ldots = \eta_{\alpha'(y)-1} = 0$, we have

$$-\big(|\eta_{\alpha'(y)}| + 1\big)e'_{\alpha'(y)} <_{L'} y <_{L'} \big(|\eta_{\alpha'(y)}| + 1\big)e'_{\alpha'(y)}, \tag{1.6}$$

whence, by (1.4) and (1.5),

$$\beta(e'_{\alpha'(y)}) = \beta\big(-(|\eta_{\alpha'(y)}| + 1)e'_{\alpha'(y)}\big) \geq \beta(y) \geq \beta\big((|\eta_{\alpha'(y)}| + 1)e'_{\alpha'(y)}\big) = \beta(e'_{\alpha'(y)}).$$

This proves that

$$\beta(y) = \beta(e'_{\alpha'(y)}) \qquad (y \in R^n \setminus \{0\}), \tag{1.7}$$

whence, by (1.3),

$$\{1, \dots, n\} \cup \{+\infty\} = \beta(R^n) = \beta\big(\{e'_1, \dots, e'_n, 0\}\big). \tag{1.8}$$

On the other hand, by $e'_n <_{L'} \ldots <_{L'} e'_1$, (1.5) and (1.2), we have

$$\beta(e'_1) \leq \ldots \leq \beta(e'_n) \leq +\infty = \beta(0). \tag{1.9}$$

From (1.8) and (1.9) it follows that

$$\beta(e'_j) = j \qquad (j = 1, \dots, n), \tag{1.10}$$

whence, by (1.7), we obtain

$$\beta(y) = \beta(e'_{\alpha'(y)}) = \alpha'(y) \qquad (y \in R^n \setminus \{0\}), \tag{1.11}$$

which, together with (1.2) and $\alpha'(0) = +\infty$, proves that $\beta = \alpha'$. ∎

Let us mention separately the particular case $\{e'_j\}_1^n = \{e_j\}_1^n$, $L' = L$, of theorem 1.1:

Corollary 1.1. *For a function $\beta : R^n \to \{1, \dots, n\} \cup \{+\infty\}$, the following statements are equivalent:*

1^o. *$\beta = \alpha$, the lexicographical index with respect to the unit vector basis $\{e_j\}_1^n$ of R^n.*

2^o. *We have (1.2)-(1.4) and*

$$y_1,\ y_2 \in R^n, \quad y_1 \leq_L y_2 \ \Rightarrow \beta(y_1) \geq \beta(y_2). \quad ∎ \tag{1.12}$$

Now we shall give conditions on $\beta : R^n \to \{1, \dots, n\} \cup \{+\infty\}$, in order to have $\beta = \alpha'$ for *some* basis $\{e'_j\}_1^n$ of R^n. To this end, let us first prove

Lemma 1.1. *Let α' be the lexicographical index with respect to a basis $\{e'_j\}_1^n$ of R^n, and let $y_1 = \sum_{j=1}^n \eta_j^1 e'_j$, $y_2 = \sum_{j=1}^n \eta_j^2 e'_j \in R^n$ be such that*

$$\alpha'(y_1) \neq \alpha'(y_2). \tag{1.13}$$

Then

$$\alpha'(y_1 + y_2) = \min \{\alpha'(y_1), \alpha'(y_2)\}. \tag{1.14}$$

Proof. By (1.13) we may assume, without loss of generality, that

$$\alpha'(y_1) < \alpha'(y_2), \tag{1.15}$$

whence, by the definition of α', we have $y_1 \neq 0$ and

$$\eta_j^k = 0 \quad (j = 1, \dots, \alpha'(y_1) - 1; \quad k = 1, 2), \tag{1.16}$$

$$\eta^1_{\alpha'(y_1)} \neq 0, \quad \eta^2_{\alpha'(y_1)} = 0. \tag{1.17}$$

From (1.16), (1,17) and (1.15), we obtain

$$\alpha'(y_1 + y_2) = \min_{\eta_j^1 + \eta_j^2 \neq 0} j = \alpha'(y_1) = \min \{\alpha'(y_1), \ \alpha'(y_2)\}. \qquad \blacksquare$$

Lemma 1.2. *Let* $\beta : R^n \longrightarrow \{1, \dots, n\} \cup \{+\infty\}$ *be a function such that*

$$y_1, y_2 \in R^n, \ \beta(y_1) \neq \beta(y_2) \ \Rightarrow \ \beta(y_1 + y_2) = \min \{\beta(y_1), \beta(y_2)\}. \tag{1.18}$$

Then, for any $y_1, \dots, y_q \in R^n$ satisfying

$$\beta(y_i) \neq \beta(y_k) \qquad (i \neq k), \tag{1.19}$$

we have

$$\beta\Big(\sum_{k=1}^q y_k\Big) = \min_{1 \le k \le q} \beta(y_k). \tag{1.20}$$

Proof. For $q = 2$ the statement is true, by (1.18). Assume now that it holds for some $q \geq 2$ and let $y_1, \dots, y_{q+1} \in R^n$ satisfy (1.19). Then, by our induction assumption, we have (1.20), whence, by (1.18),

$$\beta\Big(\sum_{k=1}^{q+1} y_k\Big) = \beta\Big(\sum_{k=1}^{q} y_k + y_{q+1}\Big) = \min \Big\{\beta\Big(\sum_{k=1}^{q} y_k\Big), \ \beta(y_{q+1})\Big\} =$$
$$= \min \Big\{\min_{1 \le i \le q} \beta(y_i), \ \beta(y_{q+1})\Big\} = \min_{1 \le i \le q+1} \beta(y_i). \qquad \blacksquare$$

Theorem 1.2. *For a function $\beta : R^n \to \{1,\dots,n\} \cup \{+\infty\}$, the following statements are equivalent:*

1°. *There exists a basis $\{e'_j\}_1^n$ of R^n, such that $\beta = \alpha'$, the lexicographical index with respect to $\{e'_j\}_1^n$.*

2°. *We have* (1.2)-(1.4) *and* (1.18).

Proof. The implication $1^o \Rightarrow 2^o$ is immediate from the definition of α' and from lemma 1.1.

$2^o \Rightarrow 1^o$. If 2^o holds, then, by (1.3), for each $j \in \{1,\dots,n\}$ there exists $e'_j \in R^n$ such that we have (1.10). Then, by (1.4) and (1.10), we have

$$\beta(\lambda_j e'_j) = \beta(e'_j) = j \qquad (\lambda_j \neq 0), \tag{1.21}$$

whence, by (1.18) and lemma 1.2 we obtain, assuming that some λ_j is non-zero,

$$\beta\Big(\sum_{j=1}^{n} \lambda_j e'_j\Big) = \beta\Big(\sum_{\lambda_j \neq 0} \lambda_j e'_j\Big) = \min_{\lambda_j \neq 0} \beta\Big(\lambda_j e'_j\Big) = \min_{\lambda_j \neq 0} j \leq n. \tag{1.22}$$

Hence, by (1.2), we must have $\sum_{j=1}^{n} \lambda_j e'_j \neq 0$, which proves that $\{e'_j\}_1^n$ is a basis of R^n. Finally, by (1.22), (1.2) and the definition of α', we have $\beta = \alpha'$. ∎

Remark 1.2. a) Obviously, (1.10) and (1.2) imply (1.3). Moreover, the above argument shows that *if $\beta : R^n \to \{1,\dots,n\} \cup \{+\infty\}$ and $\{e'_j\}_1^n \subset R^n$ satisfy* (1.2)-(1.4), (1.18) *and* (1.10), *then $\{e'_j\}_1^n$ is a basis of R^n and $\beta = \alpha'$*, the lexicographical index with respect to $\{e'_j\}_1^n$.

b) By the above, conditions (1.2)-(1.4) and (1.18) (or, conditions (1.2)-(1.5) for a basis $\{e'_j\}_1^n$ of R^n) imply (1.22) (respectively, (1.11)), whence

$$\beta(R^n \backslash \{0\}) = \{1,\dots,n\}; \tag{1.23}$$

on the other hand, it is obvious that (1.2) and (1.23) imply (1.3).

Theorem 1.3. *Let $\beta : R^n \to \{1,\dots,n\} \cup \{+\infty\}$ be a function satisfying* (1.4) *and* (1.18). *Then*

$$\beta(y_1 + y_2) \geq \min \{\beta(y_1),\ \beta(y_2)\} \qquad (y_1, y_2 \in R^n). \tag{1.24}$$

Hence, β is quasi-concave and upper semi-continuous.

Proof. By (1.18), for the proof of (1.24) we only have to consider the case when $\beta(y_1) = \beta(y_2)$. Assume, a contrario, that $\beta(y_1+y_2) < \beta(y_1) = \beta(y_2)$. Then, by (1.4), we have $\beta(y_1+y_2) < \beta(y_2) = \beta(-y_2)$, whence, by (1.18), we obtain

$$\beta(y_1) = \beta\Big((y_1 + y_2) + (-y_2)\Big) = \min \Big\{\beta(y_1 + y_2),\ \beta(-y_2)\Big\} = \beta(y_1 + y_2),$$

in contradiction with our assumption. This proves (1.24).

Furthermore, by (1.24) and (1.4), we have

$$\beta\Big(\lambda y_1 + (1-\lambda)y_2\Big) \geq \min\Big\{\beta(\lambda y_1),\ \beta\big((1-\lambda)y_2\big)\Big\} = \\ = \min\Big\{\beta(y_1),\ \beta(y_2)\Big\} \qquad (y_1, y_2 \in R^n, 0 < \lambda < 1), \tag{1.25}$$

so β quasi-concave.

Finally, by (1.24) and (1.4), for any $c \in R$ we have

$$y_1, y_2 \in R^n,\ \beta(y_1), \beta(y_2) \geq c \ \Rightarrow\ \beta(y_1 + y_2) \geq c, \tag{1.26}$$

$$y \in R^n,\ \beta(y) \geq c,\ \lambda \neq 0 \ \Rightarrow\ \beta(\lambda y) \geq c, \tag{1.27}$$

$$y \in R^n \ \Rightarrow\ \beta(0) = \beta\big(y + (-y)\big) \geq \min\{\beta(y),\ \beta(-y)\} = \min\{\beta(y),\ \beta(y)\} = \beta(y), \tag{1.28}$$

whence (1.27) remains valid also for $\lambda = 0$. Thus, in this case, each upper level set

$$\{y \in R^n \mid \beta(y) \geq c\} \tag{1.29}$$

is a linear subspace of R^n, whence closed, which proves that β is upper semi-continuous. ∎

Corollary 1.2. *The lexicographical index* $\beta = \alpha' : R^n \to \{1, \ldots, n\} \cup \{+\infty\}$ *with respect to a basis* $\{e'_j\}_1^n$ *of* R^n *satifies* (1.24) *and is quasi-concave and upper semi-continuous.*

Proof. By the definition of α' and by lemma 1.1, $\beta = \alpha'$ satisfies (1.4) and (1.18), so theorem 1.3 applies. ∎

By corollary 1.2, for any basis $\{e'_j\}_1^n$ of R^n, the function $h = -\alpha'$ is *quasi-convex and lower semi-continuous.* For notational simplicity, let us consider only the case $\{e'_j\}_1^n = \{e_j\}_1^n$, i.e., let

$$h = -\alpha. \tag{1.30}$$

The following two propositions compute the Greenberg-Pierskalla quasi-conjugate (0.1) and the quasi-subdifferential (0.2) of h, respectively.

Proposition 1.1. *We have, for each* $\Psi \in (R^n)^*$,

$$h_\lambda^\gamma(\Psi) = \begin{cases} \delta(\Psi) & if \quad \lambda > 0 \\ +\infty & if \quad \lambda \leq 0, \end{cases} \tag{1.31}$$

where

$$\delta(\Psi) = \begin{cases} \max \ \{j \in \{1, \ldots, n\} \mid \zeta_j \neq 0\} \ if \ \Psi = (\zeta_1, \ldots, \zeta_n) \in (R^n)^* \backslash \{0\} \\ -\infty \ if \ \Psi = 0. \end{cases} \tag{1.32}$$

Proof. Let $\lambda > 0$, $\Psi = (\zeta_1, \ldots, \zeta_n) \in (R^n)^* \setminus \{0\}$, $y = (\eta_1, \ldots, \eta_n)^T \in R^n$. Then

$$\Psi(y) = \sum_{j=1}^{n} \zeta_j \eta_j = \sum_{j=\alpha(y)}^{\delta(\Psi)} \zeta_j \eta_j . \tag{1.33}$$

Hence, if $\alpha(y) > \delta(\Psi)$, we have $\Psi(y) = 0 < \lambda$. On the other hand, for $y_0 = (\eta_1^0, \ldots, \eta_n^0)^T \in R^n$ defined by

$$\eta_j^0 = \begin{cases} 0 & if \quad j \neq \delta(\Psi) \\ \frac{1}{\zeta_{\delta(\Psi)}}\lambda & if \quad j = \delta(\Psi), \end{cases} \tag{1.34}$$

we have $\alpha(y_0) = \delta(\Psi)$ and, by (1.33), $\Psi(y_0) = \zeta_{\delta(\Psi)}\eta^0_{\delta(\Psi)} = \lambda$. Hence

$$h_\lambda^\gamma(\Psi) = -\inf_{\substack{y \in R^n \\ \Psi(y) \geq \lambda}} h(y) = \sup_{\substack{y \in R^n \\ \Psi(y) \geq \lambda}} \alpha(y) = \delta(\Psi).$$

Furthermore, let $\lambda > 0$ and $\Psi = 0$. Then $\{y \in R^n \mid \Psi(y) \geq \lambda\} = \emptyset$, whence $h_\lambda^\gamma(0) = -\inf \emptyset = -\infty = \delta(0)$.

Finally, let $\lambda \leq 0$. Then $\Psi(0) = 0 \geq \lambda$, whence

$$h_\lambda^\gamma(\Psi) = \sup_{\substack{y \in R^n \\ \Psi(y) \geq \lambda}} \alpha(y) \geq \alpha(0) = +\infty. \qquad \blacksquare$$

Remark 1.3. The "reverse lexicographical index" $\delta(\Psi)$ can be also expressed with the aid of the lexicographical index, namely,

$$\delta(\Psi) = n + 1 - \alpha(p(\Psi)) \qquad (\Psi \in (R^n)^*), \tag{1.35}$$

where we use the notation

$$p(\Psi) = (\zeta_n, \ldots, \zeta_1)^T \qquad (\Psi = (\zeta_1, \ldots, \zeta_n) \in (R^n)^*). \tag{1.36}$$

Proposition 1.2. *We have*

$$\partial^\gamma h(0) = R^n, \tag{1.37}$$

and, for any $y_0 = (\eta_1^0, \ldots, \eta_n^0)^T \in R^n \setminus \{0\}$,

$$\partial^\gamma h(y_0) = \Big\{\Psi = (\zeta_1, \ldots, \zeta_n) \in (R^n)^* \mid \operatorname{sign} \zeta_{\alpha(y_0)} = \operatorname{sign} \eta^0_{\alpha(y_0)} (\neq 0),\ \zeta_{\alpha(y_0)+1} = \ldots = \zeta_n = 0\Big\}. \tag{1.38}$$

Proof. Since $h(0) = -\alpha(0) = -\infty = \min h(R^n)$, we have (1.37) (see [14], theorem 1).

Let $y_0 = (\eta_1^0, \ldots, \eta_n^0)^T \neq 0$. Then, by (1.31), (1.23), (1.33) and (1.32),

$$\begin{aligned}
\partial^\gamma h(y_0) &= \Big\{\Psi \in (R^n)^* \mid h(y_0) = -h^\gamma_{\Psi(y_0)}(\Psi)\Big\} = \\
&= \Big\{\Psi \in (R^n)^* \mid \Psi(y_0) > 0,\ -\alpha(y_0) = -\delta(\Psi)\Big\} = \\
&= \Big\{\Psi = (\zeta_1, \ldots, \zeta_n) \in (R^n)^* \mid \sum_{j=\alpha(y_0)}^{\delta(\Psi)} \zeta_j \eta_j^0 > 0,\ \zeta_{\alpha(y_0)+1} = \ldots = \zeta_n = 0\Big\} = \\
&= \Big\{\Psi = (\zeta_1, \ldots, \zeta_n) \in (R^n)^* \mid \operatorname{sign} \zeta_{\alpha(y_0)} = \operatorname{sign} \eta^0_{\alpha(y_0)},\ \zeta_{\alpha(y_0)+1} = \ldots = \zeta_n = 0\Big\}.
\end{aligned} \qquad \blacksquare$$

§2. LEXICOGRAPHICALLY ISOTONE LINEAR OPERATORS

Definition 2.1. a) Let $\leq_{L'}$ and $\leq_{L''}$ be the lexicographical orders on R^n and R^k, with respect to bases $\{e'_j\}_1^n$ of R^n and $\{e''_i\}_1^k$ of R^k, respectively. We shall say that an operator $u \in \mathcal{L}(R^n, R^k)$ is $(L', L'')-$ *isotone*, if

$$u(y) \geq_{L''} 0 \qquad (y \in R^n,\ y \geq_{L'} 0). \tag{2.1}$$

b) In the particular case when $\{e'_j\}_1^n = \{e_j\}_1^n$, $\{e''_i\}_1^k = \{e_i^k\}_1^k$, the unit vector bases of R^n and R^k respectively, whence $\leq_{L'}$ and $\leq_{L''}$ are the usual lexicographical orders $\leq_L$ on R^n and R^k respectively, an (L, L)-isotone operator $u \in \mathcal{L}(R^n, R^k)$, i.e., which satifies

$$u(y) \geq_L 0 \qquad (y \in R^n,\ y \geq_L 0), \tag{2.2}$$

will be called a *lexicographically isotone* (or a *lexicographical order preserving*) *linear operator.*

The following lemma reduces the study of (L', L'')-isotone linear operators to that of lexicographically isotone linear operators.

Lemma 2.1. *Let* $\{e'_j\}_1^n$, $\{e''_i\}_1^k$, $\{e_j\}_1^n$, $\{e_i^k\}_1^k$, *be as in definition 2.1. An operator* $u \in \mathcal{L}(R^n, R^k)$ *is* (L', L'')*-isotone if and only if* $v_2^{-1}uv_1 \in \mathcal{L}(R^n)$ *is lexicographically isotone, where* $v_1 \in \mathcal{U}(R^n)$ *and* $v_2 \in \mathcal{U}(R^k)$ *are defined by*

$$v_1(e_j) = e'_j \qquad (j = 1, \ldots, n), \tag{2.3}$$

$$v_2(e_i^k) = e''_i \qquad (i = 1, \ldots, k). \tag{2.4}$$

Proof. By (2.4),

$$x = \sum_{i=1}^k \xi''_i e''_i \in R^k \iff v_2^{-1}(x) = \sum_{i=1}^k \xi''_i e_i^k = (\xi''_1, \ldots, \xi''_k)^T, \tag{2.5}$$

and hence, for any $\overline{y} \in R^n$, we have the equivalence

$$u(\overline{y}) \geq_{L''} 0 \iff v_2^{-1}(u(\overline{y})) \geq_L 0. \tag{2.6}$$

Similarly, by (2.3), for any $\overline{y} \in R^n$ we have the equivalence

$$\overline{y} \geq_{L'} 0 \iff v_1^{-1}(\overline{y}) \geq_L 0. \tag{2.7}$$

Therefore, (2.1) holds if and only if

$$v_2^{-1}(u(\overline{y})) \geq_L 0 \qquad (\overline{y} \in R^n,\ v_1^{-1}(\overline{y}) \geq_L 0). \tag{2.8}$$

Hence, writing $\overline{y} = v_1(y)$, $y = v_1^{-1}(\overline{y})$ in (2.8), we obtain that (2.1) holds if and only if

$$v_2^{-1}uv_1(y) \geq_L 0 \qquad (y \in R^n,\ y \geq_L 0). \quad \blacksquare \tag{2.9}$$

Theorem 2.1. *For* $u \in \mathcal{L}(R^n, R^k)$, *the following statements are equivalent:*

1^o. *u is lexicographically isotone.*

2^o. $u \geq_L 0$ *and* $\alpha(u(y))$ *depends only on* $\alpha(y)$.

3^o. $u \geq_L 0$ *and there exists a unique mapping* $\varphi_u : \{1, \ldots, n, +\infty\} \to \{1, \ldots, k, +\infty\}$, *such that*

$$\alpha(u(y)) = \varphi_u(\alpha(y)) \qquad (y \in R^n). \tag{2.10}$$

4^o. $u \geq_L 0$ *and*

$$\alpha(u(e_{j-1})) + 1 \leq \alpha(u(e_j)) \qquad (j = 2, \ldots, n). \tag{2.11}$$

Proof. $1^o \Rightarrow 2^o$. If 1^o holds, then, since $e_j \geq_L 0$, we have $c_j = u(e_j) \geq_L 0$ $(j = 1, \ldots, n)$, that is, $u \geq_L 0$. Assume now, a contrario, that there exist $y_1, y_2 \in R^n$ with $\alpha(y_1) = \alpha(y_2)$ and $\alpha(u(y_1)) \neq \alpha(u(y_2))$; say, $y_1 <_L y_2$. Then, by 1^o, we have $u(y_1) \leq_L u(y_2)$, whence $-u(y_2) \leq_L -u(y_1)$. Hence, by (1.12) and (1.4) for $\beta = \alpha$, we obtain

$$\alpha(u(y_1)) > \alpha(u(y_2)) = \alpha(-u(y_2)) \geq \alpha(-u(y_1)) = \alpha(u(y_1)),$$

which is impossible. This proves that

$$y_1, y_2 \in R^n, \quad \alpha(y_1) = \alpha(y_2) \ \Rightarrow \ \alpha(u(y_1)) = \alpha(u(y_2)). \tag{2.12}$$

$2^o \iff 3^o$. If 2^o holds, then, by $\alpha(e_j) = j$ $(j = 1, \ldots, n)$ and $\alpha(0) = +\infty$, we have 3^o, with

$$\varphi_u(j) = \varphi_u(\alpha(e_j)) = \alpha(u(e_j)) \qquad (j = 1, \ldots, n), \tag{2.13}$$

$$\varphi_u(+\infty) = \varphi_u(\alpha(0)) = \alpha(u(0)) = \alpha(0) = +\infty. \tag{2.14}$$

On the other hand, the converse implication $3^o \Rightarrow 2^o$ is obvious.

$2^o \Rightarrow 4^o$. Assume that 2^o holds, but not 4^o, so there exists $j \in \{2, \ldots, n\}$ such that

$$\alpha(u(e_{j-1})) + 1 > \alpha(u(e_j)). \tag{2.15}$$

If $\alpha(u(e_{j-1})) = +\infty$, then $\alpha(u(e_j)) < +\infty$ and thus $u(e_{j-1}) = 0 \neq u(e_j)$, whence

$$\alpha(u(e_{j-1} + e_j)) = \alpha(u(e_j)) \neq \alpha(u(e_{j-1}), \tag{2.16}$$

which, together with

$$\alpha(e_{j-1} + e_j) = j - 1 = \alpha(e_{j-1}), \tag{2.17}$$

contradicts 2^o.

On the other hand, if $\alpha(u(e_{j-1})) < +\infty$, then (2.15) is equivalent to

$$\alpha(u(e_{j-1})) \geq \alpha(u(e_j)). \tag{2.18}$$

If $+\infty > \alpha(u(e_{j-1})) > \alpha(u(e_j))$, we can write

$$u(e_{j-1}) = \delta_i e_i^k + \ldots, \quad \delta_i \neq 0; \quad u(e_j) = \gamma_\ell e_\ell^k + \ldots, \quad \gamma_\ell \neq 0, \tag{2.19}$$

where $i > \ell$, whence

$$\alpha(u(e_{j-1} + e_j)) = \alpha(\gamma_\ell e_\ell^k + \ldots + \gamma_{i-1} e_{i-1}^k + (\delta_i + \gamma_i) e_i^k + \ldots) = \ell < i = \alpha(u(e_{j-1})),$$

which, together with (2.17), contradicts 2^o.

Assume, finally, that $+\infty > \alpha(u(e_{j-1})) = \alpha(u(e_j))$. Then we can write

$$u(e_{j-1}) = \delta_i e_i^k + \ldots, \quad \delta_i \neq 0; \quad u(e_j) = \gamma_i e_i^k + \ldots, \quad \gamma_i \neq 0, \tag{2.20}$$

whence

$$\alpha(u(-\gamma_i e_{j-1} + \delta_i e_j)) = \alpha((-\gamma_i \delta_{i+1} + \delta_i \gamma_{i+1}) e_{i+1}^k + \ldots) \geq i + 1 > i = \alpha(u(e_{j-1})),$$

which, together with $\alpha(-\gamma_i e_{j-1} + \delta_i e_j) = j - 1 = \alpha(e_{j-1})$, contradicts 2^o.

$4^o \Rightarrow 1^o$. Assume 4^o and let $y = (\eta_1, \ldots, \eta_n)^T >_L 0$, $j_0 = \alpha(y)$. Then, by (1.1) (for $\alpha' = \alpha$), we have $\eta_1 = \ldots = \eta_{j_0 - 1} = 0$ (and $\eta_{j_0} > 0$), whence, by (0.3),

$$u(y) = \Big(\sum_{j=j_0}^{n} m_{1j} \eta_j \ , \ldots, \ \sum_{j=j_0}^{n} m_{kj} \eta_j\Big)^T. \tag{2.21}$$

If $c_{j_0} = 0$, then $\alpha(c_{j_0}) = +\infty$, whence, applying (2.11) successively to $j = j_0 + 1, \ldots, n$, we obtain, $c_{j_0+1} = \ldots = c_n = 0$. Hence, by (2.21) and (0.3), we obtain $u(y) = 0$, so (2.2) holds for any $y >_L 0$ with $c_{\alpha(y)} = 0$.

Assume now that $c_{j_0} = (m_{1j_0}, \ldots, m_{k,j_0})^T \neq 0$. Then, by (1.1) (for $\alpha' = \alpha$), and $u \geq_L 0$, we have $m_{1j_0} = \ldots = m_{\alpha(c_{j_0})-1, j_0} = 0$ and $m_{\alpha(c_{j_0}), j_0} > 0$. If $c_{j_0+1} = 0$, then, as above, we obtain $c_{j_0+1} = \ldots = c_n = 0$, whence

$$m_{1j} = \ldots = m_{\alpha(c_{j_0})-1, j} = 0 \qquad (j = j_0, \ldots, n). \tag{2.22}$$

On the other hand, if $c_{j_0+1} \neq 0$, then, by (2.11) and (1.1) (for $\alpha' = \alpha$), we have $\alpha(c_{j_0}) + 1 \leq \alpha(c_{j_0+1}) < +\infty$, whence, by (1.1) (for $\alpha' = \alpha$), we obtain

$$m_{1,j_0+1} = \ldots = m_{\alpha(c_{j_0})-1, j_0+1} = m_{\alpha(c_{j_0}), j_0+1} = \ldots = m_{\alpha(c_{j_0}+1)-1, j_0+1} = 0. \tag{2.23}$$

Passing to c_{j_0+2} and continuing in this way, we obtain that, whenever $c_{j_0} \neq 0$, we have (2.22) and

$$m_{\alpha(c_{j_0}), j} = 0 \qquad (j = j_0 + 1, \ldots, n). \tag{2.24}$$

But, from (2.21), (2.22), (2.24), $m_{\alpha(c_{j_0}),j_0} > 0$ and $\eta_{j_0} > 0$, it follows that

$$u(y) = (0, \ldots, 0, \ m_{\alpha(c_{j_0}),j_0}\eta_{j_0}, \ldots)^T >_L 0,$$

and thus (2.2) holds for any $y >_L 0$ with $c_{\alpha(y)} \neq 0$, too. ∎

Remark 2.1. a) As shown by the above proof, the equivalence $2^o \iff 3^o$ and the implication $2^o \implies 4^o$ remain valid for any $u \in \mathcal{L}(R^n, R^k)$ (not necessarily satisfying $u \geq_L 0$).

b) By the above, *if there exists a mapping* $\varphi_u : \{1, \ldots, n, +\infty\} \to \{1, \ldots, k, +\infty\}$ *satisfying* (2.10), *then it is unique and it satisfies*

$$\varphi_u(j-1) + 1 \leq \varphi_u(j) \qquad (j = 2, \ldots, n), \tag{2.25}$$

or, equivalently, φ_u *is strictly increasing on* $\varphi_u^{-1}(\{1, \ldots, k\})$. For a proof of the latter equivalence, note that if we have (2.25) and $j_1, \ j_2 \in \varphi_u^{-1}(\{1, \ldots, k\})$, $j_1 < j_2$, then $\varphi_u(j_1) < \varphi_u(j_2) < +\infty$. Conversely, if φ_u is strictly increasing on $\varphi_u^{-1}(\{1, \ldots, k\})$ and there exists $j \in \{2, \ldots, n\}$ such that $\varphi_u(j-1)+1 > \varphi_u(j)$, then $\varphi_u(j) < +\infty$, and hence $\varphi_u(j-1) = +\infty$ (since otherwise $j-1, \ j \in \varphi_u^{-1}(\{1, \ldots, k\})$, whence $\varphi_u(j-1) < \varphi_u(j) < \varphi_u(j-1)+1$, which is impossible). Thus, by (2.10), $\alpha(u(e_j - 1)) = \varphi_u(\alpha(e_{j-1})) = \varphi_u(j-1) = +\infty$, whence $u(e_{j-1}) = 0$; similarly, by $\varphi_u(j) < +\infty$, we have $\alpha(u(e_j)) < +\infty$. Hence, by (2.10), we obtain

$$\varphi_u(\alpha(e_{j-1} + e_j)) = \alpha(u(e_{j-1} + e_j)) = \alpha(u(e_j)) < +\infty = \alpha(u(e_{j-1})) = \varphi_u(\alpha(e_{j-1})),$$

which contradicts (2.17).

Definition 2.2. a) For $\leq_{L'}$ and $\leq_{L''}$ as in definition 2.1 a), we shall say that an operator $u \in \mathcal{L}(R^n, R^k)$ is *strictly* (L', L'')*-isotone*, if

$$u(y) >_{L''} 0 \qquad (y \in R^n, \ y >_{L'} 0). \tag{2.26}$$

b) In particular, a strictly (L, L)-isotone operator $u \in \mathcal{L}(R^n, R^k)$, i.e., which satisfies

$$u(y) >_L 0 \qquad (y \in R^n, y >_L 0), \tag{2.27}$$

wil be called *lexicographically strictly isotone* (or, a *linear operator preserving the strict lexicographical order*).

Remark 2.2. $u \in \mathcal{L}(R^n, R^k)$ *is strictly* (L', L'')*-isotone if and only if it is* (L', L'')*-isotone and one-to-one*; hence, in this case $k \geq n$. Since lemma 2.1 remains valid for $u \in \mathcal{L}(R^n, R^k)$ strictly (L', L'')-isotone and $v_2^{-1}uv_1 \in \mathcal{L}(R^n)$ lexicographically strictly isotone, it will be enough to consider lexicographically strictly isotone linear operators $u \in \mathcal{L}(R^n, R^k)$.

Theorem 2.2. *For* $u \in \mathcal{L}(R^n, R^k)$, *the following statements are equivalent:*

1°. *u is lexicographically strictly isotone.*

2°. $u \geq_L 0$ *and there exists a unique strictly increasing mapping* $\varphi_u : \{1, \ldots, n, +\infty\} \to \{1, \ldots, k, +\infty\}$ *satisfying* (2.10).

3°. *We have* $u \geq_L 0$ *and*

$$\alpha(u(e_{j-1})) < \alpha(u(e_j)) \leq k \qquad (j = 2, \ldots, n). \tag{2.28}$$

When $k = n$, *these statements are equivalent to*

4°. $u \geq_L 0$ *and* u *is non-singular and lower triangular.*

Proof. 1° ⇒ 2°. If (2.27) holds, then, by theorem 2.1 and remark 2.1 b) we have $u \geq_L 0$ and there exists a unique mapping $\varphi_u : \{1, \ldots, n, +\infty\} \to \{1, \ldots, k, +\infty\}$, satisfying (2.10) and strictly increasing on $\varphi_u^{-1}(\{1, \ldots, k\})$. But, we have now $\varphi_u^{-1}(\{1, \ldots, k\}) = \{1, \ldots, n\}$; indeed, if $j \in \{1, \ldots, n\}$, $\varphi_u(j) = +\infty$, then, by (2.10), we obtain $\alpha(u(e_j)) = \varphi_u(\alpha(e_j)) = \varphi_u(j) = +\infty$, whence $u(e_j) = 0$, in contradiction with 1°.

2° ⇒ 3°. If 2° holds, then, by (2.10), we have

$$\alpha(u(e_{j-1})) = \varphi_u(j-1) < \varphi_u(j) = \alpha(u(e_j)) \qquad (j = 2, \ldots, n).$$

Furthermore, if $\alpha(u(e_j)) = +\infty$ for some $j \leq n$, then φ_u is not strictly increasing on $\{1, \ldots, n, +\infty\}$, since $\varphi_u(j) = +\infty = \alpha(0) = \alpha(u(0)) = \varphi_u(\alpha(0)) = \varphi_u(+\infty)$. Thus, $\alpha(u(e_j)) \leq k$ $(j = 2, \ldots, n)$.

3° ⇒ 1°. If 3° holds, then, by the above proof of theorem 2.1, implication 4° ⇒ 1° (case $c_{j_0} \neq 0, \ldots, c_{j_n} \neq 0$), we have (2.27).

1° ∩ 3° ⇒ 4°, when $k = n$. By 1°, u is non-singular. Also, by $1 \leq \alpha(u(e_1)) < \ldots < \alpha(u(e_n)) \leq k = n$, we have

$$\alpha(u(e_j)) = j \qquad (j = 1, \ldots, n), \tag{2.29}$$

so u is lower triangular.

4° ⇒ 3°, when $k = n$. If 4° holds and $k = n$, then we have (2.29), whence (2.28). ∎

Remark 2.3. a) For $k = n$, one can also give the following alternative proof of the implication 4° ⇒ 1°: If 4° holds and $k = n$, then we have (2.29), whence (2.11), and thus, by theorem 2.1, there holds (2.2). Hence, since u is non-singular, we obtain (2.27).

b) From theorem 2.2 one obtains again [9], lemma 1.2, according to which, every unitary lower triangular $u \in \mathcal{L}(R^n)$ is lexicographically (strictly) isotone.

Proposition 2.1. *For each* $u \in \mathcal{L}(R^n, R^k)$ *there exist a basis* $\{e'_j\}_1^n$ *of* R^n *and a basis* $\{e''_i\}_1^k$ *of* R^k, *such that* u *is* (L', L'')*-isotone.*

Proof. Given $u \in \mathcal{L}(R^n, R^k)$, let $\{e'_j\}_1^\ell$ and $\{e'_j\}_{\ell+1}^n$ be bases of $(\text{Ker u})^\perp$ $(\subseteq R^n)$ and Ker $u(= \{y \in R^n \mid u(y) = 0\})$ respectively, and let

$$e''_i = u(e'_i) \qquad (i = 1, \ldots, \ell). \tag{2.30}$$

Then, since $u|_{(\mathrm{Ker}\ u)^\perp}$ is an isomorphism, $\{e_i''\}_1^\ell$ is a basis of $u((\mathrm{Ker}\ u)^\perp) = u(R^n)$, so it can be extended to a basis $\{e_i''\}_1^k$ of R^k. Then,

$$u(y) = \sum_{j=1}^{\ell} \alpha_j u(e_j') = \sum_{i=1}^{\ell} \alpha_i e_i'' \qquad (y = \sum_{j=1}^{n} \alpha_j e_j' \in R^n), \tag{2.31}$$

and hence we have (2.1). ∎

However, if we require that $k = n$ and $\{e_j'\}_1^n = \{e_i''\}_1^n$, then the situation is different. Let us give

Definition 2.3. a) Let $\{e_j'\}_1^n$ be a basis of R^n. We shall say that an operator $u \in \mathcal{L}(R^n)$ is *lexicographically isotone* (or, that u *preserves the lexicographical order*) *in the basis* $\{e_j'\}_1^n$, if u is (L', L')-isotone (in the sense of definition 2.1 a)).

b) We shall say that an operator $u \in \mathcal{L}(R^n)$ is *lexicographically isotone* (or, that u *preserves the lexicographical order*) *in some basis*, if there exists a basis $\{e_j'\}_1^n$ of R^n, in which u is lexicographically isotone.

Theorem 2.3. *For* $u \in \mathcal{L}(R^n)$, *the following statements are equivalent:*

1^o. u *lexicographically isotone in some basis.*

2^o. *The eigenvalues of* u *are real and non-negative.*

For $u \in \mathcal{L}(R^n)$, *these statements are equivalent to*

3^o. *The eigenvalues of* u *are real and positive.*

For $u \in \mathcal{O}(R^n)$, *these statements are equivalent to*

4^o. u *is the identity operator.*

Proof. $1^o \Rightarrow 2^o$. If 1^o holds, then, by lemma 2.1 (with $n = k$, $\{e_j'\}_1^n = \{e_i''\}_1^n$, $v_1 = v_2$), $v^{-1}uv \in \mathcal{L}(R^n)$ is lexicographically isotone, where $v \in \mathcal{U}(R^n)$ is defined by

$$v(e_j) = e_j' \qquad (j = 1, \ldots, n). \tag{2.32}$$

Hence, by theorem 2.1, $v^{-1}uv$ is lower triangular, and its diagonal elements are real and non-negative. But, the eigenvalues of $v^{-1}uv$ are its diagonal elements (since $v^{-1}uv$ is lower triangular) and they coincide with the eigenvalues of u (since $u(y) = \lambda y$ if and only if $v^{-1}uv(v^{-1}(y)) = \lambda v^{-1}(y)$).

$2^o \Rightarrow 1^o$. Assume that $u \in \mathcal{L}(R^n)$ satisfies 2^o, and let $\{e_j'\}_1^n$ be the "canonical" (Jordan) basis of R^n, for which the 1's in the matrix of u (in $\{e_j'\}_1^n$) are below the diagonal and the e_j''s such that $u(e_j') = 0$ are the last ones; such a basis of R^n exists, by 2^o (see e.g. [5], pp. 397-399). Define $v = v_1 \in \mathcal{L}(R^n)$ by (2.3) and let A and B be the matrices of u and v, respectively (in the unit vector basis $\{e_j\}_1^n$). Then, since B is the matrix of the exchange of basis, $B^{-1}AB$ is the matrix of u in the canonical basis $\{e_j'\}_1^n$, and thus $B^{-1}AB$ satisfies 4^o of theorem 2.1 (since it is lower triangular, so its diagonal elements are the eigenvalues of u, which, by our assumption, are real and non-negative). Hence, since $B^{-1}AB$ is the matrix of $v^{-1}uv$ in the unit vector basis, from theorem 2.1 it follows that $v^{-1}uv$ is lexicographically isotone, and therefore, by lemma 2.1, u is lexicographically isotone in the basis $\{e'_j\}_1^n$.

Finally, for $u \in \mathcal{U}(R^n)$ and $u \in \mathcal{O}(R^n)$, respectively, the equivalences $2^o \iff 3^o$ and $2^o \iff 4^o$ are obvious. ∎

§ 3. LEXICOGRAPHICAL INDEX PRESERVING LINEAR OPERATORS

Definition 3.1. a) Let $\leq_{L'}$ and $\leq_{L''}$ be the lexicographical orders of R^n and R^k, with respect to bases $\{e'_j\}_1^n$ of R^n and $\{e''_i\}_1^k$ of R^k, respectively. We shall say that an operator $u \in \mathcal{L}(R^n,\ R^k)$ is *$(L',\ L'')$-index preserving,* if

$$\alpha''\big(u(y)\big) = \alpha'(y) \qquad (y \in R^n). \tag{3.1}$$

b) In the particular case when $\{e'_j\}_1^n = \{e_j\}_1^n$, $\{e''_i\}_1^k = \{e_i^k\}_1^k$ (the unit vector bases), an (L, L)-index preserving operator $u \in \mathcal{L}(R^n,\ R^k)$, i.e., which satisfies

$$\alpha\big(u(y)\big) = \alpha(y) \qquad (y \in R^n), \tag{3.2}$$

will be called a *lexicographical index preserving linear operator.*

Remark 3.1. *If $u \in \mathcal{L}(R^n,\ R^k)$ is (L', L'')-index preserving, then u is one-to-one,* and hence, in this case, $k \geq n$. Indeed, if $y \in R^n$, $u(y) = 0$, then $\alpha'(y) = \alpha''\big(u(y)\big) = \alpha''(0) = +\infty$, whence $y = 0$.

The following lemma reduces the study of $(L',\ L'')$-index preserving linear operators to that of linear operators preserving the lexicographical index.

Lemma 3.1. *An operator $u \in \mathcal{L}(R^n,\ R^k)$ is $(L',\ L'')$-index preserving if and only if $v_2^{-1}uv_1 \in \mathcal{L}(R^n)$ is lexicographical index preserving, where $v_1 \in \mathcal{U}(R^n)$ and $v_2 \in \mathcal{U}(R^k)$ are defined by* (2.3) *and* (2.4)*, respectively.*

Proof. By (2.5),

$$\alpha''\big(u(\overline{y})\big) = \alpha\Big(v_2^{-1}\big(u(\overline{y})\big)\Big) \qquad (\overline{y} \in R^n), \tag{3.3}$$

and, similarly, by (2.3),

$$\alpha'(\overline{y}) = \alpha\big(v_1^{-1}(\overline{y})\big) \qquad (\overline{y} \in R^n). \tag{3.4}$$

Therefore, (3.1) holds if and only if

$$\alpha\Big(v_2^{-1}\big(u(\overline{y})\big)\Big) = \alpha\Big(v_1^{-1}(\overline{y})\Big) \qquad (\overline{y} \in R^n). \tag{3.5}$$

Hence, writing $\overline{y} = v_1(y)$, $y = v_1^{-1}(\overline{y})$ in (3.5), we obtain that (3.1) holds if and only if

$$\alpha\big(v_2^{-1} u v_1(y)\big) = \alpha(y) \qquad (y \in R^n). \quad \blacksquare \tag{3.6}$$

Theorem 3.1. *Let* $\{e'_j\}_1^n \subset R^n$ *be any set of* n *elements of* R^n, *such that*

$$\alpha(e'_j) = j \qquad (j = 1, \dots, n). \tag{3.7}$$

For an operator $u \in \mathcal{L}(R^n, R^k)$, *the following statements are equivalent:*

1°. u *is lexicographical index preserving.*

2°. *We have*

$$\alpha\big(u(e'_j)\big) = j \qquad (j = 1, \dots, n). \tag{3.8}$$

3°. *We have*

$$\alpha\big(u(e_j)\big) = j \qquad (j = 1, \dots, n). \tag{3.9}$$

When $k = n$, *these statements are equivalent to*

4°. u *is non-singular and lower triangular.*

Proof. 1° $\Rightarrow$ 2°. If 1° holds, then, by (3.2) and (3.7), we have $\alpha\big(u(e'_j)\big) = \alpha(e'_j) = j$ $(j = 1, \dots, n)$.

2° $\Rightarrow$ 3°. By (3.7), there exist $\eta_i^j \in R$ such that

$$e'_j = \sum_{i=j}^{n} \eta_i^j e_i\,, \qquad \eta_j^j \neq 0 \qquad (j = 1, \dots, n), \tag{3.10}$$

and hence there exist $\gamma_i^j \in R$ such that

$$e_j = \sum_{i=j}^{n} \gamma_i^j e'_i\,, \qquad \gamma_j^j \neq 0 \qquad (j = 1, \dots, n) \tag{3.11}$$

(moreover, this also shows that $\{e'_j\}_1^n$ is a basis of R^n). Hence,

$$u(e_j) = \sum_{i=j}^{n} \gamma_i^j u(e'_i) = \sum_{i|\gamma_i^j \neq 0} \gamma_i^j u(e'_i) \qquad (j = 1, \dots, n). \tag{3.12}$$

But, by (1.4) (for $\beta = \alpha$) and (3.8),

$$\alpha\big(\gamma_i^j u(e'_i)\big) = \alpha\big(u(e'_i)\big) = i \qquad (1 \leq j \leq i \leq n,\ \gamma_i^j \neq 0). \tag{3.13}$$

Thus, by (3.12), (3.13) and lemma 1.2, we obtain

$$\alpha\big(u(e_j)\big) = \min_{\substack{j \leq i \leq n \\ \gamma_i^j \neq 0}} \alpha\big(\gamma_i^j u(e'_i)\big) = \min_{\substack{j \leq i \leq n \\ \gamma_i^j \neq 0}} i = j \qquad (j = 1, \dots, n).$$

3° $\Rightarrow$ 1°. Assume that 3° holds, and let $y = (\eta_1, \dots, \eta_n)^T \neq 0$. Then $y = \sum_{j=\alpha(y)}^{n} \eta_j e_j$, whence

$$u(y) = \sum_{j=\alpha(y)}^{n} \eta_j u(e_j). \tag{3.14}$$

But, by (3.9), there exist $\delta_i^j \in R$ such that

$$u(e_j) = \sum_{i=j}^{n} \delta_i^j e_i \ , \ \delta_j^j \neq 0 \qquad (j = 1, \ldots, n), \tag{3.15}$$

whence, by (3.14),

$$u(y) = \sum_{j=\alpha(y)}^{n} \eta_j \sum_{i=j}^{n} \delta_i^j e_i. \tag{3.16}$$

Thus, by (3.16), $\eta_{\alpha(y)} \neq 0$ and $\delta_{\alpha(y)}^{\alpha(y)} \neq 0$, we obtain $\alpha(u(y)) = \alpha(y)$.

Finally, when $k = n$, the equivalence $3^o \iff 4^o$ is obvious, since $u(e_j)$ $(j = 1, \ldots, n)$ are the columns of the matrix of u. ∎

Remark 3.2. a) By theorem 3.1, *if for an operator* $u \in \mathcal{L}(R^n)$ *there exists a basis* $\{e_j'\}_1^n$ *of* R^n *such that*

$$\alpha(e_j') \neq \alpha(e_i') \qquad (j, i \in \{1, \ldots, n\}; j \neq i), \tag{3.17}$$

$$\alpha(u(e_j')) = \alpha(e_j') \qquad (j = 1, \ldots, n), \tag{3.18}$$

then u *is lexicographical index preserving*; indeed, by (3.17), there exists a permutation π of $\{1, \ldots, n\}$, such that $\alpha(e'_{\pi(j)}) = j$ $(j = 1, \ldots, n)$, so one can apply theorem 3.1 to $\{e'_{\pi(j)}\}_1^n$. *Condition* (3.17) *cannot be omitted here*, as shown by the following example: Let $y_1 = (1,1)^T$, $y_2 = (1,2)^T \in R^2$, and define $u \in \mathcal{L}(R^2)$ by $u(e_1) = e_2$, $u(e_2) = e_1$. Then $\alpha(u(y_j)) = \alpha(y_j) = 1$ $(j = 1, 2)$, but $y_2 - y_1 = (0,1)^T$, $u(y_2 - y_1) = (1,0)^T$, whence $\alpha(u(y_2 - y_1)) = 1 \neq 2 = \alpha(y_2 - y_1)$.

b) From theorems 2.2 and 3.1 it follows that, *if* $k > n$, *there exist operators* $u \in \mathcal{L}(R^n, R^k)$ *which are lexicographically strictly isotone but not lexicographical index preserving*. However, for $k = n$ the situation is different; indeed, again by theorems 2.2 and 3.1, *every lexicographically strictly isotone operator* $u \in \mathcal{L}(R^n)$ *is lexicographical index preserving*. Moreover, the above results also show that *a lexicographical index preserving* $u \in \mathcal{L}(R^n)$ *is lexicographically isotone if and only if* $u \geq_L 0$.

As an application of theorems 3.1 and 2.2, let us give the following result on classification of the elements of $(R^n, \leq_L)$:

Theorem 3.2. *For* $y, y' \in R^n$, *the following statements are equivalent:*

1^o. *There exists a lexicographically strictly isotone linear isomorphism* $u \in \mathcal{U}(R^n)$ *such that* $u(y) = y'$.

2^o. $\alpha(y) = \alpha(y')$ *and* y, y' *"have the same lexicographical sign"* (e.e., *either* $y >_L 0$, $y' >_L 0$, *or* $y = y' = 0$, *or* $y <_L 0$, $y' <_L 0$).

Proof. $1^o \Rightarrow 2^o$. If 1^o holds, then, by theorem 2.2, $u \geq_L 0$ and u is lower triangular. Hence, by theorem 3.1, u is lexicographical index preserving, and thus $\alpha(y) = \alpha(u(y)) = \alpha(y')$. Also, since u is lexicographically strictly isotone, y and $y' = u(y)$ have the same lexicographical sign.

$2^o \Rightarrow 1^o$. If $y = y' = 0$, one can take $u = I$, the identity operator. Assume now that 2^o holds and $y, y' \neq 0$. Then we can write

$$y = \sum_{j=\ell}^{n} \eta_j e_j \ , \ y' = \sum_{j=\ell}^{n} \eta'_j e_j \ , \tag{3.19}$$

where $\ell = \alpha(y)$, $\eta_\ell \eta'_\ell > 0$. Define $u \in \mathcal{U}(R^n)$ by

$$u(e_j) = e_j \qquad (j = 1, \ldots, \ell - 1, \ell + 1, \ldots, n), \tag{3.20}$$

$$u(e_\ell) = \frac{1}{\eta_\ell} \Big(\eta'_\ell e_\ell + \sum_{j=\ell+1}^{n} (\eta'_j - \eta_j) e_j \Big). \tag{3.21}$$

Then, $\alpha\big(u(e_j)\big) = \alpha(e_j) = j$, $u(e_j) >_L 0$ $(j = 1, \ldots, \ell-1, \ell+1, \ldots, n)$ and, by (3.21) and $\eta_\ell \eta'_\ell > 0$, we have $\alpha\big(u(e_\ell)\big) = \ell$, $u(e_\ell) >_L 0$. Thus, we have (2.28) and $u \geq_L 0$, and hence, by theorem 2.2, u is lexicographically strictly isotone. Finally, by (3.19)-(3.21),

$$u(y) = \sum_{j=\ell}^{n} \eta_j u(e_j) = \eta'_\ell e_\ell + \sum_{j=\ell+1}^{n} (\eta'_j - \eta_j) e_j + \sum_{j=\ell+1}^{n} \eta_j e_j = \sum_{j=\ell}^{n} \eta'_j e_j = y'. \qquad \blacksquare$$

Proposition 3.1. *For $u \in \mathcal{L}(R^n, R^k)$, the following statements are equivalent:*

1^o. *There exist a basis $\{e'_j\}_1^n$ of R^n and a basis $\{e''_i\}_1^k$ of R^k, such that u is (L', L'')-index preserving.*

2^o. *u is one-to-one.*

Hence, if 1^o holds, then $k \geq n$.

Proof. The implications $1^o \Rightarrow 2^o \Rightarrow k \geq n$ are nothing else than remark 3.1.

Finally, the proof of the implication $2^o \Rightarrow 1^o$ is similar to that of proposition 2.1 (with $\ell = n$, since by 2^o we have now $\mathrm{Ker}\ u = \{0\}$, $(\mathrm{Ker}\ u)^\perp = R^n$). ■

Finally, let us consider the case when $k = n$ and $\{e'_j\}_1^n = \{e''_i\}_1^n$. Let us first give

Definition 3.2. a) Let $\{e'_j\}_1^n$ be a basis of R^n. We shall say that an operator $u \in \mathcal{L}(R^n)$ *preserves the lexicographical index in the basis* $\{e'_j\}_1^n$, if u is (L', L')-index preserving (in the sense of definition 3.1. a)).

b) We shall say that an operator $u \in \mathcal{L}(R^n)$ *preserves the lexicographical index in some basis*, if there exists a basis $\{e'_j\}_1^n$ of R^n, in which u preserves the lexicographical index.

Theorem 3.3. *For $u \in \mathcal{L}(R^n)$, the following statements are equivalent:*

1^o. *u preserves the lexicographical index in some basis.*

2^o. *The eigenvalues of u are real and $\neq 0$.*

Proof. The proof is similar to the above proof of theorem 2.3, using now lemma 3.1 and theorem 3.1, equivalence $1^o \Longleftrightarrow 4^o$ (instead of lemma 2.1 and theorem 2.1, respectively). ■

§ 4. LINEAR OPERATORS WHICH DO NOT INCREASE OR DO NOT DECREASE THE LEXICOGRAPHICAL INDEX

Definition 4.1. a) Let $\leq_{L'}$ and $\leq_{L''}$ be the lexicographical orders on R^n and R^k, with respect to bases $\{e'_j\}_1^n$ of R^n and $\{e''_i\}_1^k$ of R^k, respectively. We shall say that an operator $u \in \mathcal{L}(R^n, R^k)$ is

i) (L', L'')-*index non-increasing*, if

$$\alpha''(u(y)) \leq \alpha'(y) \qquad (y \in R^n); \tag{4.1}$$

ii) (L', L'')-*index non-decreasing*, if

$$\alpha''(u(y)) \geq \alpha'(y) \qquad (y \in R^n); \tag{4.2}$$

b) In the particular case when $\{e'_j\}_1^n = \{e_j\}_1^n$, $\{e''_i\}_1^k = \{e_i^k\}_1^k$ (the unit vector bases), an (L, L)-index non-increasing (non-decreasing) operator $u \in \mathcal{L}(R^n, R^k)$ will be called a *linear operator which does not increase* (respectively *does not decrease*) *the lexicographical index*.

Remark 4.1. *If* $u \in \mathcal{L}(R^n, R^k)$ *is* (L', L'')-*index non-increasing, then* u *is one-to-one, and hence, in this case,* $k \geq n$. Indeeed, if $y \in R^n$, $u(y) = 0$, then $\alpha'(y) \geq \alpha''(u(y)) = \alpha''(0) = +\infty$, whence $y = 0$.

The following lemma reduces the study of (L', L'')-index non-increasing (non-decreasing) linear operators to that of linear operators which do not increase (decrease) the lexicographical index.

Lemma 4.1. *An operator* $u \in \mathcal{L}(R^n, R^k)$ *is* (L', L'')-*index non-increasing (non-decreasing) if and only if* $v_2^{-1} u v_1 \in \mathcal{L}(R^n)$ *does not increase (decrease) the lexicographical index, where* $v_1 \in \mathcal{U}(R^n)$ *and* $v_2 \in \mathcal{U}(R^k)$ *are defined by* (2.3) *and* (2.4), *respectively.*

Proof. The proof is similar to that of lemma 3.1. ∎

Theorem 4.1. *For an operator* $u \in \mathcal{L}(R^n, R^k)$, *the following statements are equivalent:*

1^o. u *does not increase the lexicographical index.*

2^o. u *is lexicographical index preserving.*

Proof. The implication $2^o \Rightarrow 1^o$ is obvious.

$1^o \Rightarrow 2^o$. If 1^o holds, then

$$\alpha(u(e_j)) \leq \alpha(e_j) = j \qquad (j = 1, \ldots, n). \tag{4.3}$$

By theorem 3.1, it will be enough to show that in (4.3) we have the equality sign, for all j. Assume not, and let

$$\ell = \min \{j \leq n \mid \alpha(u(e_j)) < j\}. \tag{4.4}$$

Since $\alpha(u(e_1)) \geq 1$, we have $\ell \geq 2$. Let

$$p = \alpha(u(e_\ell)). \tag{4.5}$$

Then $p < \ell$, and there exist $\gamma_p, \ldots, \gamma_k \in R$ and $\delta_p, \ldots, \delta_k \in R$ such that

$$u(e_\ell) = \sum_{i=p}^{k} \gamma_i e_i^k, \ \gamma_p \neq 0; \ u(e_p) = \sum_{i=p}^{k} \delta_i e_i^k, \ \delta_p \neq 0. \tag{4.6}$$

Let

$$y = \gamma_p e_p - \delta_p e_\ell. \tag{4.7}$$

Then, since $p < \ell$, we have $\alpha(y) = p$. On the other hand,

$$u(y) = \gamma_p u(e_p) - \delta_p u(e_\ell) = \sum_{i=p}^{k} (\gamma_p \delta_i - \delta_p \gamma_i) e_i^k,$$

whence, since the term corresponding to $i = p$ is 0, we obtain

$$\alpha\big(u(y)\big) \geq p + 1 > p = \alpha(y),$$

in contradiction with 1^o. ∎

Theorem 4.2. *For an operator $u \in \mathcal{L}(R^n, R^k)$, the following statements are equivalent:*

1^o. *u does not decrease the lexicographical order.*

2^o. *We have*

$$\alpha\big(u(e_j)\big) \geq j \qquad (j = 1, \ldots, n). \tag{4.8}$$

When $k = n$, the above statements are equivalent to

3^o. *u is lower triangular.*

Proof. If 1^o holds, then

$$\alpha\big(u(e_j)\big) \geq \alpha(e_j) = j \qquad (j = 1, \ldots, n).$$

$2^o \Rightarrow 1^o$. For $y = 0$ we have $u(y) = 0$, whence $\alpha\big(u(y)\big) = +\infty = \alpha(y)$. Assume now 2^o and let $y = \sum_{j=\alpha(y)}^{n} \eta_j e_j \in R^n \setminus \{0\}$. Then

$$u(y) = \sum_{j=\alpha(y)}^{n} \eta_j u(e_j),$$

whence, by theorem 1.3, (1.4) and (1.2) with $\beta = \alpha$ and by 2^o, we obtain

$$\begin{aligned}\alpha\big(u(y)\big) &\geq \min\Big\{\alpha\big(\eta_{\alpha(y)} u(e_{\alpha(y)})\big), \ldots, \alpha\big(\eta_n u(e_n)\big)\Big\} \geq \\ &\geq \min\big\{\alpha(y), \ldots, n\big\} = \alpha(y).\end{aligned}$$

Finally, for $k = n$ the equivalence $2^o \iff 3^o$ is obvious. ∎

For the case when $k = n$ and $\{e'_j\}_1^n = \{e''_i\}_1^k$, one can introduce, similarly to definition 3.2, the concepts of *operators $u \in \mathcal{L}(R^n)$ which do not increase (decrease) the lexicographical index in the basis $\{e'_j\}_1^n$* (of R^n), respectively, *in some basis* (of R^n). Then, from lemma 4.1 and theorems 4.1 and 3.3, we obtain

Corollary 4.1. *For an operator $u \in \mathcal{L}(R^n)$, the following statements are equivalent:*

1^o. *u does not increase the lexicographical index in some basis.*

2^o. *The eigenvalues of u are real and $\neq 0$.*

Corollary 4.2. *For an operator $u \in \mathcal{L}(R^n)$, the following statements are equivalent:*

1^o. *u does not decrease the lexicographical index in some basis.*

2^o. *The eigenvalues of u are real.*

Proof. The proof is similar to the above proof of theorem 2.3, using now lemma 4.1 and theorem 4.2, equivalence $1^o \iff 3^o$ (instead of lemma 2.1 and theorem 2.1, respectively). ∎

§ 5. APPENDIX: LEXICOGRAPHICAL SEPARATION OF p SETS

Let us recall the following "lexicographical separation theorem"([9], theorem 2.1); for another lexicographical separation theorem, see also [4], § 2.4):

Theorem 5.1. *For any sets, $G_1, G_2 \subset R^n$, the following statements are equivalent:*

1^o. co $G_1 \cap$ co $G_2 = \emptyset$ *(where* co G_i *denotes the convex hull of G_i).*

2^o. *There exists $u \in \mathcal{L}(R^n)$ such that*

$$u(y_1) <_L \ u(y_2) \qquad\qquad (y_1 \in G_1, \ y_2 \in G_2). \tag{5.1}$$

We shall now give an extension of this theorem to p subsets of R^n.

Theorem 5.2. *For any sets $G_1, \ldots, G_p \subset R^n$, the following statements are equivalent:*

1^o. $\cap_{i=1}^p$ co $G_i = \emptyset$.

2^o. *There exist p linear operators $u_1, \ldots, u_p \in \mathcal{L}(R^n, R^{nq})$, where $q = \min\ \{p-1, n\}$, such that*

$$\sum_{i=1}^p u_i = 0, \tag{5.2}$$

$$\sum_{i=1}^p u_i(y_i) <_L 0 \qquad\qquad (y_i \in G_i, \ i = 1, \ldots, p). \tag{5.3}$$

Proof. $1^o \Rightarrow 2^o$. Assume 1^o. We shall first prove that there exist $u_1, \ldots, u_p \in \mathcal{L}(R^n, R^{n(p-1)})$, satisfying (5.2) and (5.3). Indeed, 1^o is equivalent to

$$\left(\prod_{i=1}^p \text{co } G_i\right) \cap D = \emptyset, \tag{5.4}$$

where Π denotes the cartesian product and where

$$D = \left\{(y_1, \ldots, y_p) \in (R^n)^p \mid y_1 = \ldots = y_p\right\}. \tag{5.5}$$

Hence, by Zorn's lemma, there exists a maximal convex set H in $(R^n)^p$, such that

$$\prod_{i=1}^{p} \text{co } G_i \subseteq H, \qquad H \cap D = \emptyset. \tag{5.6}$$

Then, by [11], theorem 3.2, H is a hemi-space of type $<_L$ (in the sense of [11], definition 2.1) and D is the linear manifold associated to H (in the sense of [11], definition 3.1). Hence, since codim $D = n(p-1)$, there exist $u \in \mathcal{L}(R^{np}, R^{n(p-1)})$ and $x \in R^{n(p-1)}$ such that

$$H = \Big\{(y_1, \dots, y_p) \in (R^n)^p \ \big| \ u(y_1, \dots, y_p) <_L x\Big\}, \tag{5.7}$$

$$D = \Big\{(y_1, \dots, y_p) \in (R^n)^p \ \big| \ u(y_1, \dots, y_p) = x\Big\}. \tag{5.8}$$

But, by (5.5), we have $0 \in D$, so D is a linear subspace of $(R^n)^p$, and hence we must have $x = 0$ in (5.8) and (5.7). Now, define $u_i \in \mathcal{L}(R^n, R^{n(p-1)})$ by

$$u_i(y) = u(\underbrace{0, \dots, 0}_{i-1}, y, 0, \dots, 0) \qquad (y \in R^n, \ i = 1, \dots, p). \tag{5.9}$$

Then, we have

$$u(y_1, \dots, y_p) = \sum_{i=1}^{p} u_i(y_i) \qquad (y_i \in R^n, \ i = 1, \dots, p). \tag{5.10}$$

Hence, by the first part of (5.6) and by (5.7) with $x = 0$, we obtain (5.3). Finally, by (5.10), (5.5) and (5.8) with $x = 0$, we get $\sum_{i=1}^{p} u_i(y) = u(y, \dots, y) = 0$, i.e., (5.2). This proves our assertion on the existence of $u_1, \dots, u_p \in \mathcal{L}(R^n, \ R^{n(p-1)})$.

Now, if $q = \min \{p-1, n\} = p-1$, then $nq = n(p-1)$, so we are done. On the other hand, assume now that $q = \min \{p-1, \ n\} = n$, or, equivalently, $n+1 \leq p$. Then, by 1° and Helly's theorem for a finite collection of sets (see e.g. [12], p. 196, theorem 21.6), there exist distinct $i_1, \dots, i_{n+1} \in \{1, \dots, p\}$ such that

$$\bigcap_{j=1}^{n+1} \text{co } G_{i_j} = \emptyset. \tag{5.11}$$

By (5.11) and the first part of the above proof, there exist $u_{i_1}, \dots, u_{i_{n+1}} \in \mathcal{L}(R^n, \ R^{n^2})$ satisfying

$$\sum_{j=1}^{n+1} u_{i_j} = 0, \tag{5.12}$$

$$\sum_{j=1}^{n+1} u_{i_j}(y_{i_j}) <_L 0 \qquad (y_{i_j} \in G_{i_j}, \ j = 1, \dots, n+1). \tag{5.13}$$

Hence, if we set

$$u_i = 0 \in \mathcal{L}(R^n, \ R^{n^2}) \qquad (i \in \{1, \dots, p\} \backslash \{i_1, \dots, i_{n+1}\}), \tag{5.14}$$

then $u_1,\dots,u_p \in \mathcal{L}(R^n,\ R^{n^2}) = \mathcal{L}(R^n,\ R^{nq})$ satisfy (5.2) and (5.3).

$2^o \Rightarrow 1^o$. Assume 2^o and non-1^o, say, $y \in \bigcap_{i=1}^p$ co G_i. We claim that for each $i \in \{1,\dots,p\}$ there exists $y_i \in G_i$ such that

$$u_i(y_i) \geq_L u_i(y). \tag{5.15}$$

Indeed, if for some i we had

$$u_i(y') <_L u_i(y) \qquad (y' \in G_i),$$

that is, $G_i \subseteq \{y' \in R^n \mid u_i(y') <_L u_i(y)\}$, then we would obtain $y \in$ co $G_i \subseteq \{y' \in R^n \mid u_i(y') <_L u_i(y)\}$, which is impossible. This proves the claim. Then, for $y_i \in G_i$ satisfying (5.15) we get, by (5.2) and (5.3),

$$0 = \sum_{i=1}^p u_i(y) \leq_L \sum_{i=1}^p u_i(y_i) <_L 0,$$

which is impossible. ∎

Remark 5.1. For a recent result on separation of p sets in R^n by hemi-spaces, see [11], theorem 5.1.

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