AUTOMORPHISMS OF AF-ALGEBRAS

by

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INTRODUCTION

Given $T \in L(H)$, where H is a separable infinite dimensional Hilbert space, it is easy to see that we can find an increasing sequence of finite dimensional subspaces H_n with $H=\overline{\cup H_n}$ and $T(H_n) \subset H_{n+1}$ for every n.

Given an unital AF-algebra A, it is natural to consider automorphisms α for which there exists an increasing sequence of finite dimensional C^*-subalgebras A_n such that $A=\overline{\cup A_n}$ and $\alpha(A_n) \subset A_{n+1}$ for every n. Such an α will be called (s)-automorphism.

In his paper [6], Voiculescu shows that the almost inductive limit automorphisms of an AF-algebra (a notion analogous to that of quasitriangular operator) are approximable by inductive limit automorphisms.

In this note we prove that every automorphism of an unital AF-algebra can be approximated by (s)-automorphisms. Also, given A and $\alpha \in Aut(A)$, we define a new AF-algebra $A(\alpha)$ which depends only on A and α and reflects some properties of α. In general A and $A(\alpha)$ are not isomorphic; however, if $A \rtimes_\alpha \mathbb{Z}$ embedds into an AF-algebra, it seems that $A(\alpha) \simeq A$.

§1°. THE FORM OF AUTOMORPHISMS

Let A be an unital AF-algebra and denote by $\mathcal{F}(A)$ the set of all finite dimensional C^*-subalgebras of A. By a nest of finite dimensional C^*-subalgebras of A we shall mean an increasing sequence

$$\mathbb{C}\cdot 1 = A_0 \subset A_1 \subset \ldots \subset A_n \subset \ldots$$

with $A_n \in \mathcal{F}(A)$ for every n and $A=\overline{\bigcup_{n\geq 0} A_n}$.

If C_1, C_2 are C^*-subalgebras of an arbitrary C^*-algebra C and $\varepsilon > 0$ we shall write $C_1 \subset^{\varepsilon} C_2$ if

$$\sup\{\inf\{\|x-y\| \mid y \in C_2, \|y\| \leq 1\} \mid x \in C_1, \|x\| \leq 1\} < \varepsilon$$

and $d(C_1, C_2)$ is defined by

$$d(C_1,C_2) = \inf\{\varepsilon > 0 \mid C_1 \subset^{\varepsilon} C_2 \text{ and } C_2 \subset^{\varepsilon} C_1\}.$$

We shall use the following approximation results

1.1. If C_1, C_2 are C^*-subalgebras of C, C_1 is finite dimensional and $\varepsilon > 0$, then there is $\delta > 0$ depending only on ε and $\dim C_1$ such that $C_1 \subset^{\delta} C_2 \Rightarrow \exists u \in \mathcal{U}(C)$ (the unitaries of C) with $C_1 \subset \mathrm{Ad}u(C_2)$ and $\|u-1\| < \varepsilon$.

1.2. (Christensen) If C_1, C_2 are C^*-subalgebras of C, C_1 finite dimensional, $0 \leq \gamma < 10^{-4}$ and $C_1 \subset^{\gamma} C_2$, then there is $u \in \mathcal{U}(C)$ such that $C_1 \subset \mathrm{Ad}u(C_2)$ and $\|u-1\| < 64\gamma^{1/2}$.

In his paper [6], Voiculescu considers inductive limit and almost inductive limit automorphisms of AF-algebras. We recall the definitions.

1.3. DEFINITION. We say that $\alpha \in \mathrm{Aut}(A)$ is an inductive limit automorphism if there exists a nest of finite dimensional C^*-subalgebras $(A_n)_{n\geq 0}$ of A such that $\alpha(A_n) = A_n$ for every n. $\alpha \in \mathrm{Aut}(A)$ is called almost inductive limit automorphism if there exists a nest $(A_n)_{n\geq 0}$ of A with $\lim_{n\to\infty} d(\alpha(A_n), A_n) = 0$.

It is natural to consider another class of automorphisms

1.4. DEFINITION. $\alpha \in \mathrm{Aut}(A)$ will be called (s)-automorphism if there exists a nest $(A_n)_{n\geq 0}$ of A with $\alpha(A_n) \subset A_{n+1}$ for every n.

1.5. REMARK. Let $A=\overline{\cup A_n}$, $\alpha \in \mathrm{Aut}(A)$ arbitrary and $\varepsilon > 0$. It is easy to see that there exists an increasing sequence $(n_k) \subset \mathbb{N}$ with $\alpha(A_{n_k}) \overset{\varepsilon/2^k}{\subset} A_{n_{k+1}}$ for every k. Thus every automorphism of an AF-algebra is an 'almost' (s)-automorphism.

In his paper, Voiculescu shows that almost inductive limit automorphisms are approximable by inductive limit automorphisms (Proposition 2.3. in [6]). In analogy we prove the following result which shows how large is the class of (s)-automorphisms.

1.6. THEOREM. Let A be an unital AF-algebra, $\alpha \in \mathrm{Aut}(A)$ and $\varepsilon > 0$. Then for every fixed nest of A there exists $u \in \mathcal{U}(A)$ with $\|u-1\| < \varepsilon$ such that $\mathrm{Ad}u \circ \alpha$ is (s)-automorphism with respect to a subnest of the fixed nest.

Proof. Let (B_m) a nest of A such that $\alpha(B_m) \overset{\varepsilon_m}{\subset} B_{m+1}$ with $\varepsilon_m \searrow 0$ (see 1.5.). We shall construct inductively a subnest (A_n) of the nest (B_m), automorphisms α_n and unitaries u_n such that

$$A_0 = \mathbb{C}\cdot 1,\ A_1 = B_1,\ u_1 = 1,\ \alpha_1 = \alpha$$

$$\alpha_{n+1} = \mathrm{Ad}u_{n+1} \circ \alpha_n,\ \|u_n - 1\| < \varepsilon \cdot 2^{-n} \text{ and } \alpha_n(A_j) \subset A_{j+1},\ 0 \leq j \leq n-1.$$

Having constructed A_n, u_n, α_n, the proof will be concluded since $u = \lim_{n\to\infty} u_n \ldots u_1$ is well defined, $\|u-1\| < \varepsilon$ and $(\mathrm{Ad}u \circ \alpha)(A_n) \subset A_{n+1}$ for every n.

Suppose we have found $A_j = B_{m_j}$, α_j, u_j with the desired properties for $1 \leq j \leq n$. Let

$$\gamma_0 = \left(\frac{\varepsilon}{(10^3(n+1))^{n+2}}\right)^{2^{n+1}}$$

and assume $\varepsilon < 10^{-4}$ which is no loss of generality. Since $\alpha(B_{m_n}) \overset{\varepsilon_{m_n+p}}{\subset} B_{m_n+p+1}$ for every $p \geq 0$ and

$$\lim_{p\to\infty} d(\mathrm{Ad}u_n(B_{m_n+p}), B_{m_n+p}) = 0,$$

we can find $m_{n+1} > m_n$ with $\alpha_n(B_{m_n}) \overset{\gamma_0}{\subset} B_{m_{n+1}}$, therefore $\alpha_n(A_n) \overset{\gamma_0}{\subset} A_{n+1}$

with $A_{n+1} := B_{m_{n+1}}$. By 1.2. there exists an unitary $v_o \in A$ with $(\mathrm{Ad}v_o \circ \alpha_n)(A_n) \subset A_{n+1}$ and $\| v_o - 1 \| < \gamma_1 := 2(n+1)10^2 \gamma_o^{1/2}$.

Putting $\gamma_j := (2(n+1)10^2)^j \gamma_o^{1/2}$, we shall find inductively unitaries $v_j \in A_{n+2-j}$, $1 \leq j \leq n$ such that

$$(\mathrm{Ad}v_j \circ \ldots \circ \mathrm{Ad}v_o \circ \alpha_n)(A_{n-j}) \subset A_{n+1-j} \quad (1 \leq j \leq n)$$

and $\| v_j - 1 \| < \gamma_{j+1}$.

Indeed, assume we have found v_j for $j \leq k < n$. Then

$$(\mathrm{Ad}v_k \circ \ldots \circ \mathrm{Ad}v_o \circ \alpha_n)(A_{n-k-1}) \overset{\delta_k}{\subset} A_{n-k},$$

where $\delta_k \leq 2\| v_k \ldots v_o - 1 \| \leq 2(\gamma_1 + \ldots + \gamma_{k+1}) \leq 2(n+1)\gamma_{k+1}$. We have

$$(\mathrm{Ad}v_k \circ \ldots \circ \mathrm{Ad}v_o \circ \alpha_n)(A_{n-k-1}) \subset (\mathrm{Ad}v_k \circ \ldots \circ \mathrm{Ad}v_o \circ \alpha_n)(A_{n-k}) \subset A_{n+1-k},$$

therefore there is a $v_{k+1} \in \mathcal{U}(A_{n+1-k})$ with

$$(\mathrm{Ad}v_{k+1} \circ \ldots \circ \mathrm{Ad}v_o \circ \alpha_n)(A_{n-k-1}) \subset A_{n-k}$$

and

$$\| v_{k+1} - 1 \| < 10^2 (2(n+1)\gamma_{k+1})^{1/2} \leq \gamma_{k+2},$$

which concludes the proof of the existence of the v_j's $(1 \leq j \leq n)$.

Defining $u_{n+1} = v_n \ldots v_o$ we have

$$\| u_{n+1} - 1 \| \leq \gamma_1 + \ldots + \gamma_{n+1} \leq (n+1)\gamma_{n+1} < \varepsilon \cdot 2^{-n}$$

and for $0 \leq j \leq n$

$$(\mathrm{Ad}u_{n+1} \circ \alpha_n)(A_{n-j}) = (\mathrm{Ad}v_n \circ \ldots \circ \mathrm{Ad}v_{j+1})(\mathrm{Ad}v_j \circ \ldots \circ \mathrm{Ad}v_o \circ \alpha_n)(A_{n-j}) \subset$$
$$\subset (\mathrm{Ad}v_n \circ \ldots \circ \mathrm{Ad}v_{j+1})(A_{n+1-j}) = A_{n+1-j}.$$

1.7. COROLLARY. Any automorphism of an AF-algebra A is of the form $\mathrm{Ad}v \circ \beta$, where $v \in \mathcal{U}(A)$ and $\beta(A_n) \subset A_{n+1}$ $\forall n$ for a certain nest (A_n) of A.

1.8. REMARK. Let A be an AF-algebra and (A_n), (B_k) two nests of A. By lemma 2.6. [1] there exists an approximately inner automorphism $\sigma \in \mathrm{Aut}(A)$ such that for every k there is a n with $\sigma(B_k) \subset A_n$ and $A_k \subset \sigma(B_n)$. We obtain two strictly increasing sequences (n_p) and (k_p) of positive integers with

$$A_{n_1} \subset \sigma(B_{k_1}) \subset A_{n_2} \subset \sigma(B_{k_2}) \subset \ldots$$

Now if $\alpha \in \mathrm{Aut}(A)$ is such that $\alpha(A_n) \subset A_{n+1}$ $\forall n$, we have

$$(\sigma^{-1}\alpha\sigma)(B_{k_p}) \subset \sigma^{-1}\alpha(A_{n_{p+1}}) \subset \sigma^{-1}(A_{n_{p+1}+1}) \subset B_{k_{p+2}} ,$$

therefore $\sigma^{-1}\alpha\sigma$ is a (s)-automorphism relatively to a subnest of (B_k).

§2°. THE AF-ALGEBRA $A(\alpha)$

2.1.DEFINITION. Let $\alpha \in \mathrm{Aut}(A)$ be a (s)-automorphism relatively to the nest (A_n) of A. We can consider the inductive system

$$A_o \hookrightarrow A_1 \xrightarrow{\alpha} A_2 \hookrightarrow A_3 \xrightarrow{\alpha} A_4 \hookrightarrow \ldots$$

where $\hookrightarrow$ denotes the inclusion map. Of course, α commutes with the inclusion maps. Define $A(\alpha)$ to be the C^*-limit of this system. A priori, $A(\alpha)$ depends on the nest (A_n), but we shall see later that it is not the case.

2.2.LEMMA. Let $(n_k) \subset \mathbb{N}$ be a strictly increasing sequence with $n_k \geqslant 2k$ for every k and denote by $A_{n_k} \xrightarrow{\alpha} A_{n_{k+1}}$ the composition

$$A_{n_k} \xrightarrow{\alpha} A_{n_k+1} \hookrightarrow \ldots \hookrightarrow A_{n_{k+1}} .$$

Then $\varinjlim(A_{n_k}, \alpha) \simeq A(\alpha)$.

Proof. Inductively we shall construct a commutative diagram

$$\begin{array}{ccccccccc}
A_o & \longrightarrow & A_2 & \longrightarrow & \ldots & \longrightarrow & A_{2m_1} & \longrightarrow & \ldots \longrightarrow \ldots \\
\| & & \downarrow & \nearrow & & & \downarrow & \nearrow & \downarrow \\
A_o & \longrightarrow & A_{n_1} & \longrightarrow & \ldots & \longrightarrow & A_{n_q} & \longrightarrow & \ldots \longrightarrow \ldots
\end{array}$$

Suppose $A_{2p} \hookrightarrow A_{n_k}$ with $p \leqslant k$. Let m with $m-p \leqslant 2m-n_k$, therefore $m \geqslant n_k-p$. We define $\alpha^{m-p}: A_{n_k} \longrightarrow A_{2m}$ and the diagram

$$\begin{array}{ccc}
A_{2p} & \xrightarrow{\alpha^{m-p}} & A_{2m} \\
\big\downarrow & \nearrow{\scriptstyle \alpha^{m-p}} & \\
A_{n_k} & &
\end{array}$$

is commutative. But $n_{k+m-p} \geqslant 2(k+m-p) \geqslant 2m$ and the diagram

$$\begin{array}{ccc} & & A_{2m} \\ & \overset{\alpha^{m-p}}{\nearrow} & \Big\downarrow \\ A_{n_k} & \xrightarrow[\alpha^{m-p}]{} & A_{n_{k+m-p}} \end{array}$$

is commutative etc.

2.3.REMARK. If $\alpha(A_n) \subset A_{n+1}$ for every n, then for $k \geqslant 1$ fixed we have $\alpha^k(A_n) \subset A_{n+k}$ for every n and one can see that $A(\alpha^k) \simeq A(\alpha)$ because $A(\alpha^k)$ is the limit of the inductive system

$$A_0 \hookrightarrow A_k \xrightarrow{\alpha^k} A_{2k} \hookrightarrow A_{3k} \xrightarrow{\alpha^k} A_{4k} \hookrightarrow \ldots$$

2.4.LEMMA. Let A be an unital AF-algebra, (A_n) a fixed nest of A and $C_1, C_2 \in \mathcal{F}(A)$ such that $C_1 \subset C_2$ and there exists $n_1 \in \mathbb{N}$ with $C_1 \subset A_{n_1}$. Then there exist $m \in \mathbb{N}$, $m \geqslant n_1$ and $u \in \mathcal{U}(A)$ such that

(i) $uC_2u^* \subset A_m$

(ii) $uxu^* = x \quad \forall x \in C_1$.

Proof. By lemma 2.3. of [1], given $\varepsilon > 0$ we can find $u_0 \in \mathcal{U}(A)$ and $m \geqslant n_1$ such that $\|u_0 - 1\| < \varepsilon$ and $u_0 C_2 u_0^* \subset A_m$. Let $C := u_0 C_1 u_0^* \subset A_m$. We have C_1 and C finite dimensional C^*-subalgebras of A_m and $\beta : C_1 \to C$, $\beta(x) = u_0 x u_0^*$ a $*$-isomorphism with $\|\beta - \mathrm{id}|C_1\| < 1$ (we take $\varepsilon < 1/2$). By lemma 2.4. of [1] there exists $w \in \mathcal{U}(A_m)$ with $\beta(x) = wxw^*$, $x \in C_1$. Let $u := w^* u_0 \in \mathcal{U}(A)$. We have

$$uC_2u^* = w^* u_0 C_2 u_0^* w \subset w^* A_m w = A_m$$

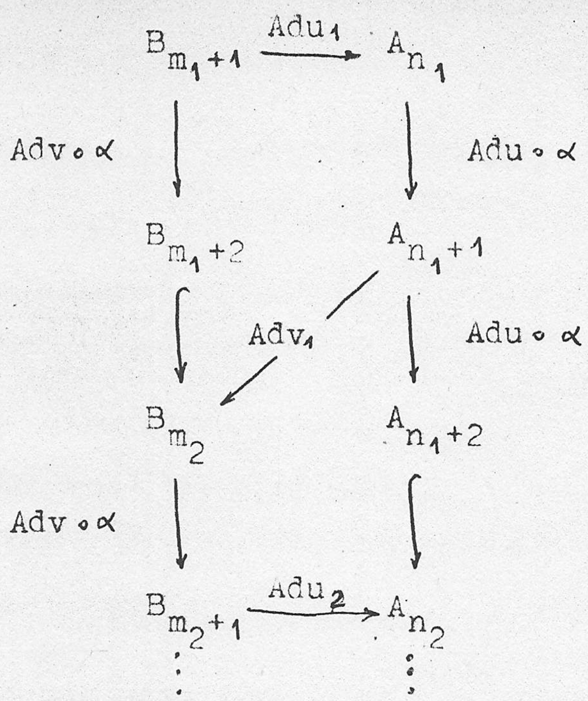
and for $x \in C_1$,

$$uxu^* = w^* u_0 x u_0^* w = w^* \beta(x) w = \beta^{-1}\beta(x) = x.$$

2.5.LEMMA. Let A be an AF-algebra, (A_n), (B_m) two fixed nests of A and $u, v \in \mathcal{U}(A)$ such that $(\mathrm{Ad}u \circ \alpha)(A_n) \subset A_{n+1}$ $\forall n$ and $(\mathrm{Ad}v \circ \alpha)(B_m) \subset B_{m+1}$ $\forall m$. Then $A(\mathrm{Ad}u \circ \alpha) \simeq A(\mathrm{Ad}v \circ \alpha)$.

Proof. We shall find inductively two increasing sequences of positive integers (m_p), (n_p) such that $m_{p+1} \geqslant m_p + 4$, $n_{p+1} \geqslant n_p + 4$, two sequences $(u_p), (v_p) \subset \mathcal{U}(A)$ such that $\mathrm{Ad}u_i(B_{m_i+1}) \subset A_{n_i}$, $\mathrm{Ad}v_i(A_{n_i+1}) \subset B_{m_{i+1}}$ and the following diagram is commutative:

(1)



Suppose that $u_1, v_1, u_2, v_2, \ldots, u_k$ have been constructed with the desired properties. We shall construct v_k. We have

$$Adu_k(B_{m_k+1}) \subset A_{n_k},$$

therefore $B_{m_k+1} \subset u_k^* A_{n_k} u_k$. Now

$$(Adv\circ\alpha)(B_{m_k+1}) \subset (Adv\circ\alpha)(u_k^* A_{n_k} u_k) = v\,\alpha(u_k^*)\,\alpha(A_{n_k})\,\alpha(u_k)v^* =$$

$$= v\,\alpha(u_k^*)u^*u\,\alpha(A_{n_k})u^*u\,\alpha(u_k)v^* \subset v\,\alpha(u_k^*)u^*A_{n_k+1}u\,\alpha(u_k)v^*.$$

Using 2.4. for $(Adv\circ\alpha)(B_{m_k+1}) \subset v\,\alpha(u_k)u^*A_{n_k+1}u\,\alpha(u_k)v^* \subset A=\overline{\bigcup B_m}$, there exist $w \in \mathcal{U}(A)$ and $m_{k+1} \geqslant m_k+4$ such that

$$wv\,\alpha(u_k^*)u^*A_{n_k+1}u\,\alpha(u_k)v^*w^* \subset B_{m_{k+1}}$$

and

$$wyw^* = y, \quad \forall y \in (Adv\circ\alpha)(B_{m_k+1})$$

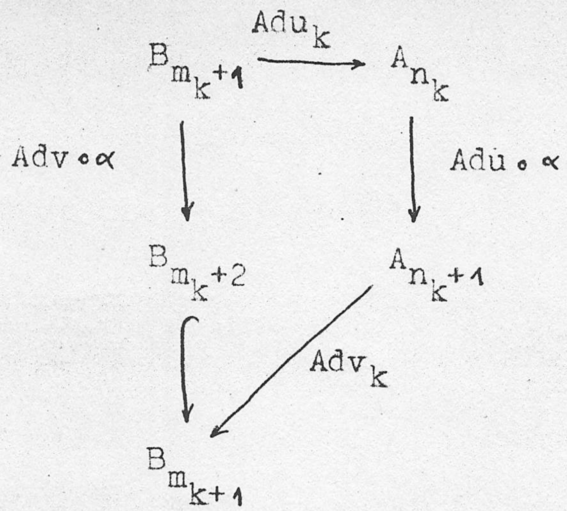
Let $v_k := wv\,\alpha(u_k^*)u^* \in \mathcal{U}(A)$. For $x \in B_{m_k+1}$ we have

$$(Adv_k\circ Adu\circ\alpha\circ Adu_k)(x) = v_k u\,\alpha(u_k x u_k^*)u^*v_k^* =$$

$$= wv\,\alpha(u_k^*)u^*u\,\alpha(u_k)\,\alpha(x)\,\alpha(u_k^*)u^*u\,\alpha(u_k)v^*w^* =$$

$$= wv\,\alpha(x)v^*w^* = w(Adv\circ\alpha)(x)w^* = (Adv\circ\alpha)(x),$$

therefore the following diagram commutes



The commutative diagram (1) defines an isomorphism between $A(\mathrm{Ad}v \circ \alpha)$ and $A(\mathrm{Ad}u \circ \alpha)$.

2.6.COROLLARY. For $\alpha \in \mathrm{Aut}(A)$ arbitrary, by 1.6. there are $u \in \mathcal{U}(A)$ and a nest (A_n) of A such that $(\mathrm{Ad}u \circ \alpha)(A_n) \subset A_{n+1}$ for every n. We can define $A(\alpha)$ to be $A(\mathrm{Ad}u \circ \alpha)$ and by the previous lemma this definition is correct (it not depends on u or (A_n)). We have associated to A and α a new AF-algebra $A(\alpha)$ which in general is not isomorphic to A (we shall see an example later).

2.7.THEOREM. Let A be an unital AF-algebra and $\alpha \in \mathrm{Aut}(A)$. Then we have:

(i) $A(\alpha^k) \simeq A(\alpha)$ $\forall k \geq 1$

(ii) $A(\sigma \circ \alpha) \simeq A(\alpha)$ for any approximately inner automorphism σ; in particular $A(\alpha) \simeq A$ if there is $k \geq 1$ with α^k approximately inner

(iii) $A(\alpha) \simeq A$ for α almost inductive limit automorphism

Proof. (i) It follows from 2.3., 2.5. and 2.6.

(ii) The proof is analogous to that of 2.5. because for every n there is $u_n \in \mathcal{U}(A)$ with $\sigma \circ \alpha|_{A_n} = \mathrm{Ad}u_n \circ \alpha|_{A_n}$ ([4]).

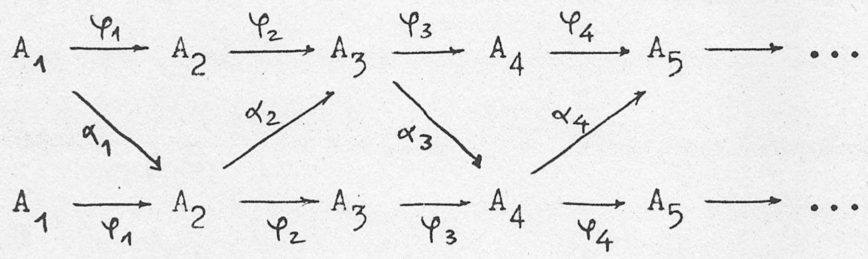
(iii) By corollary 2.6. of [6] there is an unitary $u \in A$ such that $\mathrm{Ad}u \circ \alpha$ is limit periodic i.e. there are a nest (A_n) of A and a sequence of positive integers (d_n) such that

$$(\mathrm{Ad}u \circ \alpha)(A_n) = A_n \text{ and } (\mathrm{Ad}u \circ \alpha|_{A_n})^{d_n} = \mathrm{id}_{A_n}.$$

Therefore, telescoping the system in the definition of $A(\alpha)$, we obtain a system in which appear only inclusions, so that $A(\alpha)\simeq A$.

§3°. EXAMPLES OF AUTOMORPHISMS

Given an inductive system (A_n,φ_n) with A_n finite dimensional C^*-algebras and φ_n unital $*$-monomorphisms, $\varphi_n : A_n \longrightarrow A_{n+1}$, to construct an automorphism of $A=\varinjlim(A_n,\varphi_n)$ it is sufficient to construct a commutative diagram



where α_n are unital $*$-monomorphisms.

Of course, the sequence (α_{2k+1}) determines an automorphism α, while the sequence (α_{2k}) determines its inverse.

Suppose φ_n are given (passing to K-theory) by matrices $\varphi_{n*}\in GL(p,\mathbb{Z})$. If we can find $x_n\in GL(p,\mathbb{Q})$ such that $x_n\varphi_{n*}$, $\varphi_{n+1*}x_n^{-1}$ have positive integer entries and $x_n\varphi_{n*}$ defines a homomorphism $A_n \longrightarrow A_{n+1}$ for every $n\geq 1$, we can take $\alpha_{n*}=x_n\varphi_{n*}$, $\alpha_{n+1*}=\varphi_{n+1*}x_n^{-1}$. The condition $x_{n+1}\varphi_{n+1*}=\varphi_{n+1*}x_n^{-1}$ implies the reccurence formula

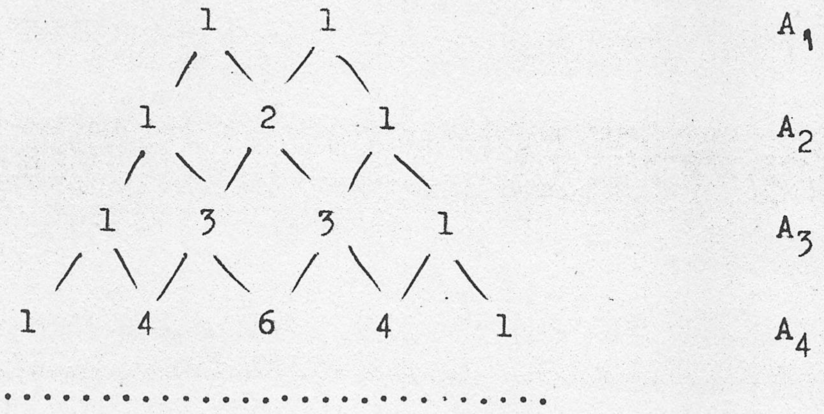
$$x_{n+1}=\varphi_{n+1*}x_n^{-1}\varphi^{-1}_{n+1*}.$$

This construction depends on the existence of a (nontrivial) x_1 with the desired properties.

3.1. EXAMPLE. Let $A=\bigotimes_{-\infty}^{\infty} M_2 = UHF(2^\infty)$, where $A_n=\bigotimes_{-n}^{n} M_2$ and $\varphi_n : A_n \longrightarrow A_{n+1}$, $\varphi_n(a)=1\otimes a\otimes 1$. The shift automorphism on A is given by $a\longmapsto 1\otimes 1\otimes a$ and its inverse by $a\longmapsto a\otimes 1\otimes 1$.

Since every automorphism of an UHF-algebra is approximately inner, it follows that $A(\alpha)\simeq A$.

3.2.EXAMPLE. Let A be the GICAR-algebra, which has the following Bratteli diagram



$\varphi_n : A_n \to A_{n+1}$ is given by a $(n+2) \times (n+1)$ matrix

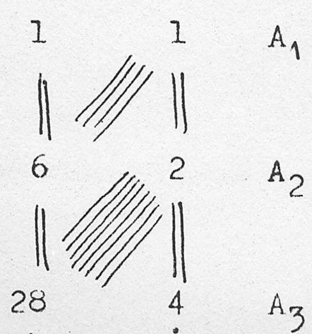
$$\begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 1 & \ldots & 0 & 0 \\ \ldots & & & & & \ldots \\ 0 & 0 & 0 & \ldots & 1 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}$$

Let α_n given by the $(n+2) \times (n+1)$ matrix

$$\begin{pmatrix} 0 & 0 & \ldots & 0 & 0 & 1 \\ 0 & 0 & \ldots & 0 & 1 & 1 \\ 0 & 0 & \ldots & 1 & 1 & 0 \\ \ldots & & & & & \ldots \\ 1 & 1 & \ldots & 0 & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 & 0 \end{pmatrix}$$

We have $\alpha_{n+1}\alpha_n = \varphi_{n+1}\varphi_n$ and α_n defines a homomorphism $A_n \to A_{n+1}$. We obtain an automorphism α of A such that $\alpha^2 = id$. Therefore $A(\alpha) \simeq A(id) \simeq A$ by 2.7.

3.3.EXAMPLE. Let $A = UHF(2^\infty) \otimes \mathcal{K}^\sim$. A can be described by the following Bratteli diagram



In this case φ_n are given by

$$\varphi_{n*}=\begin{pmatrix} 2 & 2^{n+1} \\ 0 & 2 \end{pmatrix}.$$

Let α_{2n+1} given by

$$\alpha_{2n+1*}=\begin{pmatrix} 1 & 3\cdot 2^{2n+1}-1 \\ 0 & 2 \end{pmatrix}$$

and α_{2n} by

$$\alpha_{2n*}=\begin{pmatrix} 4 & 2 \\ 0 & 2 \end{pmatrix}.$$

The compatibility properties are verified, therefore we obtain an automorphism $\alpha \in \mathrm{Aut}(A)$.

Remark that $K_0(A)=\mathbb{Z}\left[\frac{1}{2}\right]\oplus\mathbb{Z}\left[\frac{1}{2}\right]$, $K_0(A)_+=\{(p,q)\mid q>0\}\cup \cup\{(p,0)\mid p\geqslant 0\}$, $[1_A]=(0,1)$ and the group of automorphisms of $(K_0(A),K_0(A)_+,[1_A])$ is isomorphic with $\mathbb{Z}$ by the map

$$\mathbb{Z}\ni m\longmapsto\begin{pmatrix} 2^m & 0 \\ 0 & 1 \end{pmatrix}\in \mathrm{Aut}(K_0(A),K_0(A)_+,[1_A]).$$

By the functor K, α goes to

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}.$$

Remark that α^k is approximately inner only for $k=0$.

We have $A(\alpha)\simeq \mathrm{UHF}(2^\infty)\otimes B$, where B is an extension

$$0\longrightarrow \mathcal{K}\longrightarrow B\longrightarrow \mathrm{UHF}(2^\infty)\longrightarrow 0$$

while $A(\alpha^{-1})\simeq \mathrm{UHF}(2^\infty)\otimes(\mathrm{UHF}(2^\infty)\otimes\mathcal{K})^{\sim}$ and therefore

$$A(\alpha)\not\simeq A\not\simeq A(\alpha^{-1})\not\simeq A(\alpha).$$

3.4.EXAMPLE. Let $A=\mathcal{C}(X)$, where X is the one point compactification of $\mathbb{Z}$ and $\alpha\in\mathrm{Aut}(A)$, $\alpha(x)(t)=x(t+1)$, $t\in\mathbb{Z}$, $\alpha(x)(\infty)=$ $=x(\infty)$. It can be shown that $A(\alpha)\simeq A$. It is known that $A\rtimes_\alpha\mathbb{Z}$ embedds into an AF-algebra.

3.5.EXAMPLE. Let $A=\mathcal{C}(X)$, where X is the two points compactification of $\mathbb{Z}$ and $\alpha\in \mathrm{Aut}(A)$, $\alpha(x)(t)=x(t+1), t\in\mathbb{Z}$, $\alpha(x)(\infty)=x(\infty)$, $\alpha(x)(-\infty)=-\infty$. Also $A(\alpha)\simeq A$, however $A\rtimes_{\alpha}\mathbb{Z}$ embedds not into an AF-algebra. ([5])

REFERENCES

[1] O.Bratteli. Inductive limits of finite dimensional C^*-algebras, Trans.AMS vol.171, Sept 1972, 195-234.

[2] E.Christensen. Near inclusion of C^*-algebras, Acta Math. 144(1980)3-4, 249-266.

[3] E.G.Effros. Dimensions and C^*-algebras, Conf.Board Math. Sci.46(1981)

[4] E.G.Effros & J.Rosenberg. C^*-algebras with approximately inner flip, Pacific J.Math.77(1978), 417-443.

[5] M.Pimsner. Embedding some transformation group C^*-algebras into AF-algebras, Ergod.Th. & Dynam.Syst.3(1983), 613-626.

[6] D.Voiculescu. Almost inductive limit automorphisms and embeddings into AF-algebras, Ergod.Th. & Dynam.Syst.6(1986) 475-484.

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