INDECOMPOSABLE COHEN-MACAULAY MODULES AND IREDUCIBLE MAPS

by

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November 1988

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Indecomposable Cohen-Macaulay modules and irreducible maps

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Introduction

Let (R,m) be a local CM-ring and M a finitely generated (shortly f.g.) R-module. M is a maximally CM module (shortly MCM R-modules). The isomorphism classes of indecomposable MCM R-modules form the vertices of the Auslander-Reiter quiver $\Gamma(R)$ of R. Section 3 studies the behaviour of $\Gamma(R)$ under base change; best results (cf. (3.10), (3.14)) being partial answers to the conjectures from [Sc] (7.3). Unfortunately, the proofs use the difficult theory of Artin approximation (cf. [Ar], or [Po1]).

A different easier method is to use the so-called CM-reduction ideals as we did in [Po2] (4.9) or have in (3.2), (3.3). This procedure is very powerful in proving results describing how large is the set of those positive integers which are multiplicities of the vertices of $\Gamma(R)$, in fact the first Brauer-Thrall conjecture (cf. [Di],[Yo],[Po2] or here (4.2), (4.3)). However the Corollary (3.3) obtained by this method is much weaker than (3.10) which uses Artin approximation theory. The reason is that the conditions under which we know the existence of CM-reduction ideals are still too complicated. Trying to simplify them we see that the difficulty is just to prove some bound properties on MCM modules (cf. Section 2) which we hope to hold for every excellent henselian local CM-ring. Our Theorem (4.4) and Corollary (4.6) give sufficient conditions when the second Brauer-Thrall conjecture holds, and our Corollary (4.7) is a nice application to rational double points (inspirated by [Yo](4.1)).

We would like to thank L. Badescu and J. Herzog for helpful conversations on (4.7), (3.10), (3.16) respectively.

1. Bound properties on MCM modules

(1.1) Let (R,m) be a local CM-ring, $k := R/m$, $p :=$ char k and $\operatorname{Reg} R = \{q \in \operatorname{Spec} R \mid R_q \text{ regular}\}$. Suppose that Reg R is open (this happens for instance when R is quasi-excellent). Then $I_s(R) = \bigcap_{q \notin \operatorname{Reg} R} q$ defines the singular locus of R. We say that R has bound properties on MCM modules if the following conditions hold:

i) there exists a positive integer r such that $I_s(R)^r \operatorname{Ext}^1_R(M,N) = 0$ for every MCM R-module M and for every f.g. R-module N, i.e. $I_s(R)$ is in the radical of the Dieterich Ext-annihilating ideal of R (cf. [Di] § 2),

ii) for every ideal $a \subset R$ and every element $y \in I_s(R)$ there exists a positive integer e such that $(aM : y^e)_M := \{z \in M \mid y^e z \in aM\} = (aM : y^{e+1})_M$ for every MCM R-module M.

Clearly it is enough to consider in i), ii) only indecomposable M. Let M be a MCM R-module and

$A_M := \{x \in R \mid x\, \mathrm{Ext}^1_R(M,N) = 0$ for every f.g. R-module $N\}$

(1.2) <u>Lemma</u>: $I_s(R) \subset \sqrt{A_M}$

<u>Proof:</u> Let $x \in I_s(R)$. Then R_x is a regular ring and so M_x is projective over R_x. Indeed, if $q \in \mathrm{Spec}\, R$ with $x \notin q$ then M_q is free over R_q by [He](1.1), M_q being still MCM by [Ma2] (17.3) and R_q is regular. Thus

$$R_x \otimes_R \mathrm{Ext}^1_R(M,N) = \mathrm{Ext}^1_{R_x}(M_x,N_x) = 0$$

and so a certain power of x kills $\mathrm{Ext}^1_{R}(M,N)$, i.e. $x \in \sqrt{A_M}$.

(1.3) <u>Remark:</u> The above Lemma shows that for each MCM R-module M we are able to find a positive integer r_M such that $I_s(R)^{r_M}\mathrm{Ext}^1_R(M,N) = 0$ for every f.g. R-module N. Thus the trouble in (1.1)i) is just to show that r_M can be bounded when M runs in CM(R). Also for each f.g.R-module M by Noetheriarity we can find in (1.1)ii) a positive integer e_M such that $(aM:y^{e_M})_M =$ $= (aM:y^{e_M+1})_M$. Again the trouble is to show that e_M can be bounded when M runs in CM(R). However, if R has finite CM-type (i.e. $\Gamma(R)$ has just a finite set of vertices) then R has bound properties on MCM modules (compare with [Di] Proposition 8).

(1.4) <u>Lemma:</u> Suppose that (R,m) is reduced complete with k perfect and $\mathrm{Reg}(R/pR) = \{q/pR \mid q \in \mathrm{Reg}\ R,\ q \supset pR\}$ if $pR \neq 0$ (i.e. if $p \neq$ $\neq$ char R). Then R has bound properties on MCM modules.

<u>Proof:</u> Clearly either R contains k or R is a flat algebra over a Cohen ring of residue field k, i.e. a complete DVR (T,t) which in an unramified extension of $\mathbb{Z}_{(p)}$, $p>0$, $t = p\cdot 1 \in T$. Let $x = (x_1,\dots,x_n)$ be a system of elements from m such that (t,x) forms a system of parameters in R. By Cohen's structure Theorems the canonical map $j: T[[X]] \to R$, $X = (X_1,\dots,X_n) \mapsto x$ is finite. As R is CM we obtain R flat (thus free) over the image S_x of j.

Let I_x be the kernel of the multiplication map $R \otimes_{S_{(n)}} R \xrightarrow{\mu} R$ and $\mathcal{N}_x := \mu(\mathrm{Ann}_{R \otimes_{S_{(n)}} R} I_x)$ the Noether different of R over $S_{(n)}$. Then

1) $\mathcal{N}_x \cdot \Omega_{R/S_x} = 0$ and $I_s(R) = \sqrt{\sum_x \mathcal{N}_x}$,

where the sum is taken over all systems of elements x such that (t,x) forms a system of parameters of R (see [Po2](2.8), (2.10), the ideas come in fact from [Yo]).

Using the Hochschild cohomology we get a surjective map

$\mathrm{Hom}_R(\Omega_{R/S_x}, \mathrm{Hom}_{S_x}(M,N)) \cong \mathrm{Der}_{S_x}(R, \mathrm{Hom}_{S_x}(M,N)) \longrightarrow \mathrm{Ext}^1_R(M,N)$

for every MCM R-module M and every f.g. R-module N (see e.g. in [Di] Lemma 5). By (1) follows $\mathcal{N}_x \, \mathrm{Ext}^1_R(M,N) = 0$ and so there is $r \in \mathbb{N}$ such that

2) $I_s(R)^r \, \mathrm{Ext}^1_R(M,N) = 0$

for every MCM R-module M and every f.g. R-module N, i.e. (1.1)i) holds.

Now, let $a \subset R$ be an ideal and $y \in I_s(R)$. We show that there exists a positive integer e such that $(aM:y^e)_M = (aM:y^{e+1})_M$ for every MCM R-module M. If there exists x as above such that $y \in \mathcal{N}_x$ then it is enough to apply [Po2] (3.2) for $S_x \subset R$. Otherwise choose in $I_s(R)$ a system of elements $(n_i)_{1 \le i \le s}$ such that $I_s(R) = \sqrt{\sum_i n_i R}$ and for every i there exists $x^{(i)}$ as above such that $n_i \in \mathcal{N}_{x^{(i)}}$. Then $yn_i \in \mathcal{N}_{x^{(i)}}$ and so there exists a positive integer e_i such that

$(aM:(yn_i)^{e_i})_M = (aM:(yn_i)^{e_i+1})_M$

for every MCM R-module M. Choose a positive integer e' such that $I_s(R)^{e'} \subset \sum_{i=1}^{s} n_i^{e_i} R$. We claim that $e = e' + \max_{1 \le i \le s} e_i$ works. Indeed, let M be a MCM R-module, if $y^v z \in aM$ for a certain $z \in M$ and $v \in \mathbb{N}$ then $(yn_i)^v z \in aM$ and so $(yn_i)^{e_i} z \in aM$. Thus

$$y^e z \in y^{e-e'}\left(\sum_{i=1}^{s} n_i^{e_i} R\right)z \subset \sum_{i=1}^{s} (yn_i)^{e_i} Rz \subset aM.$$

(1.5) Lemma: Suppose that

i) (R,m) is quasi-excellent and reduced,

ii) $\mathrm{Reg}(R/pR) = \{q/pR \mid q \in \mathrm{Reg}\, R, q \supset pR\}$ if $pR \neq 0$,

iii) there exists a flat Noetherian complete local R-algebra (R',m') such that

$iii_1)$ R' is formally smooth over R,

$iii_2)$ $k' := R'/m'$ is perfect.

Then R has bound properties on MCM modules.

Proof: By André-Radu's Theorem the structural morphism $j: R \to R'$ is regular (see [An] or [BR1], [BR2]). Then

$$\mathrm{Reg}\, R' = \{q \in \mathrm{Spec}\, R' \mid j^{-1}q \in \mathrm{Reg}\, R\}$$

by [Ma1] (33.B). Hence $I_s(R') = \sqrt{I_s(R)R'}$ and $\mathrm{Reg}(R'/pR') = \{q/pR' \mid q \in \mathrm{Reg}\, R', q \supset pR'\}$. Thus R' has bound properties on MCM modules by Lemma (1.4) and it is enough to show the following

(1.6) Lemma: Let $j: R \to A$ be a flat local morphism of local CM-rings such that $I_s(A) \supset I_s(R)A$. If A has bound properties on MCM modules then R has too.

Proof: Let r be the positive integer given for A as in (1.1)i). We claim that r works also for R. Let M be a MCM R-module and N a f.g. R-module. By flatness we have

$A \otimes_R \mathrm{Ext}^1_R(M,N) \cong \mathrm{Ext}^1_A(A \otimes_R M, A \otimes_R N)$. Since

$\mathrm{depth}_A\, A \otimes_R M = \mathrm{depth}_R M + \mathrm{depth}_A\, A/mA = \mathrm{depth}\, R + \mathrm{depth}_A\, A/mA = \mathrm{depth}\, A$, the A-module $A \otimes_R M$ is a MCM. Thus $I_s(A)^r$ (so $I_s(R)^r$) kills $\mathrm{Ext}^1_A(A \otimes_R M, A \otimes_R N)$ and it follows $I_s(R)^r(A \otimes_R \mathrm{Ext}^1_R(M,N)) = 0$. By faithful flatness we get $I_s(R)^r \mathrm{Ext}^1_R(M,N) = 0$.

Now let $a \subset R$ be an ideal and $y \in I_s(R)$. Then $j(y) \in I_s(A)$, and by hypothesis there exists an $e \in \mathbb{N}$ such that

$(aP:j(y)^e)_P = (aP:j(y)^{e+1})_P$

for every MCM A-module P.

Let M be a MCM R-module. As above, $M' := A \otimes_R M$ is a MCM A-module and by flatness we obtain

$(aM':j(y)^s)_{M'} \cong A \otimes_R (aM:y^s)_M$

for every $s \in \mathbb{N}$. Thus the inclusion

(1) $(aM:y^e)_M \subset (aM:y^{e+1})_M$

goes by base change into an equality. Then (1) itself is an equality, j being faithfully flat.

(1.7) Proposition: Suppose that

i) (R,m) is quasi-excellent reduced and $[k:k^p] < \infty$ if $p > 0$,

ii) $\mathrm{Reg}(R/pR) = \{q/pR \mid q \in \mathrm{Reg}\, R,\ q \supset pR\}$ if $pR \neq 0$.

Then R has bound properties on MCM modules.

Proof: If k is perfect then apply Lemmas (1.5), (1.6), where A is the completion of (R,m). If k is not perfect, let $K := k^{1/p^\infty}$. We have $\mathrm{rank}_K \Gamma_{K/k} = \mathrm{rank}_k \Omega_{k/k^p} = [k:k^p]$ (see e.g. in [Po2] (4.4)) and so there exists a formally smooth Noetherian complete local R-algebra (R',m') such that $R'/m' \cong K$ and $\dim R' =$ $= \dim R + \mathrm{rank}\, \Gamma_{K/k}$ (see EGA (22.2.6), or [NP] Corollary (3.6)). Now apply Lemmas (1.5), (1.6).

2. CM-reduction ideals

(2.1) Lemma: (inspirated by [Po2] (3.1)). Let A be a Noetherian ring, M,N two f.g. A-modules, $x \in A$ an element such that $x\,\mathrm{Ext}^1_A(M,P) = 0$ for every factor A-module P of N, e a positive integer such that $(0:x^e)_N = (0:x^{e+1})_N$ and $s \in \mathbb{N}$. Then for every A-linear map $\varphi : M \to N/x^{e+s+1}N$ there exists an A-linear map $\psi : M \to N$ which makes commutative the following diagram:

$$\begin{array}{ccc} M & \xrightarrow{\ \varphi\ } & N/x^{e+s+1}N \\ \Psi \downarrow & & \downarrow \\ N & \longrightarrow & N/x^{e+s}N \end{array}$$

Proof: Let $N' := (0:x^e)_N$. We have the following commutative diagram

$$(1)\quad \begin{array}{ccccccccc} 0 & \to & N/N' & \xrightarrow{x^{e+s+1}} & N/N' & \longrightarrow & N/N'+x^{e+s+1}N & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \to & N/N' & \xrightarrow{x^{e+s}} & N/N' & \longrightarrow & N/N'+x^{e+s}N & \longrightarrow & 0 \end{array}$$

in which the lines are exact. This follows from the elementary

(2.1.1) Lemma: $x^eN \cap N' = 0$.

Indeed, if $x^{e+s}z \in N'$ for a certain $z \in N$ then $x^{e+s}z = 0$ by the above Lemma and so $z \in N'$. Applying the functor $\mathrm{Hom}_A(M,-)$ to (1) we obtain the following commutative diagram

$$(2)\quad \begin{array}{ccccc} \mathrm{Hom}_A(M,N/N') & \longrightarrow & \mathrm{Hom}_A(M,N/N'+x^{e+s+1}N) & \longrightarrow & \mathrm{Ext}_A^1(M,N/N') \\ \| & & \downarrow & & \downarrow x \\ \mathrm{Hom}_A(M,N/N') & \longrightarrow & \mathrm{Hom}_A(M,N/N'+x^{e+s}N) & \longrightarrow & \mathrm{Ext}_A^1(M,N/N') \end{array}$$

with exact lines. Since the last column is zero by hypothesis we obtain a R-linear map $\alpha: M \to N/N'$ such that the following diagram is commutative:

$$(4)\quad \begin{array}{ccccc} M & \longrightarrow & N/x^{e+s+1}N & \longrightarrow & N/N'+x^{e+s+1}N \\ \alpha \downarrow & & & & \downarrow \\ N/N' & & \longrightarrow & & N/N'+x^{e+s}N \end{array}$$

Note that in the diagram

$$(4)\qquad \begin{array}{ccccc} M & \longrightarrow & & \longrightarrow & N/x^{e+s+1}N \\ \| & & & & \downarrow \\ M & \overset{\psi}{\dashrightarrow} & N/N'\cap x^{e+s}N & \longrightarrow & N/x^{e+s}N \\ \| & & \downarrow & & \downarrow \\ M & \xrightarrow{\alpha} & N/N' & \longrightarrow & N/N'+x^{e+s}N \end{array}$$

the small square is cartesian and so there exists ψ which makes (4) commutative. Now apply Lemma (2.1.1).

<u>Proof of Lemma (2.1.1)</u>: If $y \in N' \cap x^e N$ and $z \in N$ satisfy $y = x^e z$ then $0 = x^e y = x^{2e} z$ and so $z \in N'$, i.e. $y = x^e z = 0$.

(2.2) <u>Lemma:</u> Let (R,m) be a local CM-ring which has bound properties on MCM modules, $a \subset R$ an ideal and $y \in I_s(R)$. Suppose that Reg R is open. Then there exists a function $\nu : \mathbb{N} \to \mathbb{N}$, $\nu \geq 1_{\mathbb{N}}$ such that for every $s \in \mathbb{N}$, every MCM R-modules M,N and every R-linear map $\varphi : M \to N/(a,x^{\nu(s)})N$ there exists a R-linear map $\psi : M \to N/aN$ such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N/(a,x^{\nu(s)})N \\ {\scriptstyle\psi}\downarrow & & \downarrow \\ N/aN & \longrightarrow & N/(a,x^s)N \end{array}$$

<u>Proof:</u> Let r,e be the positive integers given by (1.1)i), (1.1)ii), respectively. Define $\nu(s) = r(1+\max\{e,s\})$. Let M,N,φ be given. Since $y^r \mathrm{Ext}^1_R(M,P) = 0$ for every f.g. R-module P and $(aN:y^e)_N \cong (aN:y^{e+1})_N$ we get ψ by Lemma (2.1).

(2.3) Let A be a local CM-ring and $a \subset A$ an ideal. As in [Po2], (3.6) the couple (A,a) is a CM-approximation if there exists a function $\nu : \mathbb{N} \to \mathbb{N}$, $\nu \geq 1_{\mathbb{N}}$ such that for every $s \in \mathbb{N}$, every two MCM A-modules M,N and every A-linear map $\varphi : M \to N/a^{\nu(s)}N$ there exists an A-linear map $\psi : M \to N$ such that $A/a^s \otimes_A \varphi \cong A/a^s \otimes_A \psi$, in other words: the following diagram is commutative

$$\begin{array}{ccc} M & \xrightarrow{\ \varphi\ } & N/a^{\vartheta(s)}N \\ {\scriptstyle\Psi}\,\vdots\downarrow & & \downarrow \\ N & \longrightarrow & N/aN \end{array}$$

(2.4) <u>Lemma:</u> Let (R,m) be a local CM-ring which has bound properties on MCM modules. Suppose that Reg R is open. Then for every ideal $a \subset I_s(R)$ the couple (R,a) is a CM-approximation.

<u>Proof:</u> Let $y_1,\dots,y_t$ be a system of generators of a. Apply induction on t. If $t = 1$ then use Lemma (2.2) for $a = 0$. Suppose $t > 1$ and let ϑ' be the function given by induction hypothesis for $b := (y_1,\dots,y_{t-1})$. Let $\vartheta_s^{"}$ be the function given by Lemma (2.2) for $b^{\vartheta'(s)}$ and y_t, $s \in \mathbb{N}$. Then the function ϑ given by $\vartheta(s) = \vartheta'(s) + \vartheta_s^{"}(s)$ works. Indeed, let M,N be two MCM R-modules, $s \in \mathbb{N}$ and $\varphi: M \longrightarrow N/a^{\vartheta(s)}N$ a R-linear map. Then there exists a R-linear map $\alpha: M \longrightarrow N/b^{\vartheta'(s)}N$ such that $(R/y_t^sR) \otimes_R \alpha \cong$ $\cong (R/y_t^sR) \otimes_R \varphi$ (note that $a^{\vartheta(s)} \subset (b^{\vartheta'(s)}, y_t^{\vartheta_s^{"}(s)})$). Moreover, there exists a R-linear map $\Psi: M \longrightarrow N$ such that $(R/b^s) \otimes_R \Psi \cong (R/b^s) \otimes_R \alpha$. As $b^s \subset a^s$ we get $R/a^s \otimes_R \Psi \cong R/a^s \otimes_R \varphi$.

(2.5) Let b be an ideal in a local CM-ring A. Then b is a <u>CM-reduction ideal</u> if the following statements hold:

i) A MCM A-module M is indecomposable iff M/bM is indecomposable over A/b,

ii) Two indecomposable MCM A-modules M,N are isomorphic iff M/bM and N/bN are isomorphic over A/b.

(2.6) <u>Lemma:</u> Let R be a Henselian local CM-ring and $a \subset R$ an ideal such that (R,a) is a CM-approximation. Then a^r is a CM-reduction ideal for a certain positive integer r.

The proof follows from [Po2] (4.5), (4.6).

(2.7) Proposition: Let (R,m) be a Henselian local CM-ring which has bound properties on MCM modules. Suppose that Reg R is open. Then for every ideal $a \subset I_s(R)$ there exists a positive integer r such that a^r is a CM-reduction ideal.

The proof follows from Lemmas (2.4), (2.6).

(2.8) Corollary ([Po2](4.8)): Let (R,m) be a reduced quasi-excellent Henselian local CM-ring, $k := R/m$ and $p :=$ char k. Suppose that

i) $[k:k^p] < \infty$ if $p > 0$

ii) $\mathrm{Reg}(R/pR) = \{q/pR \mid q \in \mathrm{Reg}\, R,\ q \supset pR\}$ if $p \neq$ char R.

Then $I_s(R)^r$ is a CM-reduction ideal for a certain positive integer r.

The result follows from Propositions (1.7), (2.7).

3. Stability properties of Auslander-Reiten quivers under base change

(3.1) Let (R,m) be a local CM-ring, $k := R/m$, $p :=$ char k and M,N two indecomposable MCM R-modules. The R-linear map f is irreducible if it is not an isomorphism, and given any factorization $f = gh$ in CM(R), g has a section or h has a retraction. The AR-quiver $\Gamma(R)$ of R is a directed graph which has as vertices the isomorphism classes of indecomposable MCM R-modules, and there is an arrow from the isomorphism class of M to that of N provided there is an irreducible linear map from M to N. Let $\Gamma_o(R)$ be the set of vertices of $\Gamma(R)$. The multiplicity defines a map $e_R: \Gamma_o(R) \to \mathbb{N}$, $M \mapsto e_R(M)$. If R is a domain and K is its fraction field then let $\mathrm{rank}_R: \Gamma_o(R) \to \mathbb{N}$ be the map given by $M \mapsto \dim_K K \otimes_R M$. Clearly $e_R = e(R)\,\mathrm{rank}_R$ by [Ma2] (14.8).

(3.2) Proposition: Suppose that

i) R is an excellent Henselian local ring,

ii) R has bound properties on MCM modules.

Let A be the completion of R with respect to $I_s(R)$. Then the base change functor $A \otimes_R$-indces a bijection $\Gamma_o(R) \longrightarrow \Gamma_o(A)$.

Proof: Clearly A is a CM-ring because the canonical map $R \to A$ is regular by i). First we prove that a f.g. R-module M is an indecomposable MCM module iff $A \otimes_R M$ is an indecomposable MCM A-module. Using the following elementary Lemma, it is enough to show that if M is an indecomposable MCM R-module then $A \otimes_R M$ is indecomposable over A.

(3.2.1) Lemma: Let B be a local CM-ring, M a f.g. B-module and C a flat local B-algebra. Suppose that C is a CM-ring. Then

i) M is a MCM B-module iff $C \otimes_B M$ is a MCM C-module,

ii) If $C \otimes_B M$ is indecomposable then M is so.

By Proposition (2.7) $I_s(R)^r$ is a CM-reduction ideal for a certain $r \in \mathbb{N}$ and so $R/I_s(R)^r \otimes_R M$ is still indecomposable. As $R/I_s(R)^r \cong A/I_s(R)^r A$ it follows that $A \otimes_R M/I_s(R)^r A \otimes_R M$ is indecomposable and so $A \otimes_R M$ is indecomposable by Nakayama's Lemma. If M,N are two indecomposable MCM R-modules such that $A \otimes_R M \cong A \otimes_R N$ then $M/I_s(R)^r M \cong A \otimes_R M/I_s(R)^r A \otimes_R M \cong A \otimes_R N/I_s(R)^r A \otimes_R N \cong N/I_s(R)^r N$ and so $M \cong N$, because $I_s(R)^r$ is a CM-reduction ideal. Thus $A \otimes_R$ - induces an injective map $\alpha : \Gamma_o(R) \longrightarrow \Gamma_o(A)$. But α is also surjective by [E1] Theorem 3 because a MCM A-module is locally free on Spec $A \setminus V(I_s(A))$ and $I_s(A) = \sqrt{I_s(R)A}$ (see (1.2) and (1.5)), the map $R \to A$ being regular by i).

Using (1.7) we get

(3.3) Corollary: Suppose that

i) (R,m) is an excellent reduced Henselian local ring and $[k:k^p]<\infty$ if $p>0$,

ii) $\mathrm{Reg}(R/pR) = \{q/pR \mid q \in \mathrm{Reg}\, R,\ q \supset pR\}$ if $p \neq \mathrm{char}\, R$.

Let A be the completion of R with respect to $I_s(R)$. Then the base change functor $A \otimes_R -$ induces a bijection $\Gamma_o(R) \to \Gamma_o(A)$. In particular $\# \Gamma_o(R) = \# \Gamma_o(A)$.

The above Corollary can be improved if we use Artin approximation theory (see [Ar] or [Po1]).

(3.4) Let $n: B \to C$ be a morphism of rings. We call n algebraically pure (see [Po1] § 3) if every finite system of polynomial equations over B has a solution in B if it has one in C. Also n is called strong algebraically pure (see [Po1](9.1)) if for every finite system of polynomials f from $B[X]$, $X=(X_1,\dots,X_s)$ and for every finite set of finite systems of polynomials $(g_i)_{1\leq i\leq r}$ in $B[X,Y]$, $Y = (Y_1,\dots,Y_t)$ the following conditions are equivalent:

1) f has a solution x in B such that for every i, $1\leq i\leq r$ the system $g_i(x,y)$ has no solutions in B,

2) f has a solution x in C such that for every i, $1\leq i\leq r$ the system $g_i(x,y)$ has no solutions in C.

Suppose that B is Noetherian and let $b\subset B$ be an ideal, C the completion of B with respect to b and n the completion map. Then (B,b) has the property of Artin approximation (shortly (B,b) is an AP-couple) if for every finite system of polynomials f in $B[X]$, $X=(X_1,\dots,X_s)$, every positive integer e and every solution x of f in C, there exists a solution x_e of f in B such that

$x_e \equiv x \bmod b^eC$. It is easy to see that (B,b) is an AP-couple iff n is algebraically pure (see [Po1] § 1). When B is local and b its maximal ideal then (B,b) is an AP-couple too iff n is strong algebraically pure (see [BNP] (5.1) where these morphisms are called T^*-existentially complete). If (B,b) is a Henselian couple and n is regular then (B,b) is an AP-couple by [Po1] (1.3).

(3.5) Lemma: Let $A \to B$ be an algebraically pure morphism of Noetherian rings and M,N two f.g. A-modules. Then $B \otimes_A M \cong B \otimes_A N$ over B iff $M \cong N$ over A.

Proof: Let $M \cong A^n/(u)$, $N \cong A^m/(v)$, $u_i = \sum_{j=1}^{n} u_{ij}e_j$, $i=1,\dots,n'$, $v_r = \sum_{s=1}^{m} v_{rs}e'_s$, $r = 1,\dots,m'$, where u_{ij}, $v_{rs} \in A$ and (e_j) resp. (e'_j) are canonical bases in A^n resp. A^m. Let $\varphi : A^n \to A^m$ be a linear A-map given by $e_j \mapsto \sum_{s=1}^{m} x_{js}e'_s$, where $x_{js} \in A$. Then φ induces a map $f: M \to N$ iff there exist (z_{ir}) in A such that $\varphi(u_i) = \sum_{r=1}^{m'} z_{ir}v_r$, i.e.

$$(1) \qquad \sum_{j=1}^{n} u_{ij}\, x_{js} = \sum_{r=1}^{m'} z_{ir}\, v_{rs}, \quad 1 \le i \le n',\ 1 \le s \le m.$$

Clearly f is an isomorphism iff there exists $\psi : A^m \to A^n$ given by $e'_s \mapsto \sum_{j=1}^{n} y_{sj}e_j$, $y_{sj} \in A$ such that $\psi(v) \subset (u)$, $(\varphi\psi - 1)(e') \subset (v)$ and $(\psi\varphi - 1)(e) \subset (u)$, i.e. there exist t_{ri}, w_{sr}, $w'_{ji} \in A$ such that

$$(2)\begin{cases} \sum_{s=1}^{m} v_{rs}y_{sj} = \sum_{i=1}^{n'} t_{ri}u_{ij}, & 1 \le r \le m',\ 1 \le j \le n \\ \sum_{s=1}^{m} x_{js}y_{sj'} - \delta_{jj'} = \sum_{i=1}^{n'} w_{ji}u_{ij'}, & 1 \le j, j' \le n \\ \sum_{j=1}^{n} y_{sj}x_{js'} - \delta_{ss'} = \sum_{r=1}^{m'} w'_{sr}v_{rs'}, & 1 \le s, s' \le m \end{cases}$$

where $\delta_{ss'}$ denotes Kronecker's symbol. Thus $M \cong N$ iff the following system of polynomial equations:

$$(*)\begin{cases} \sum_{j=1}^{n} u_{ij}X_{js} = \sum_{r=1}^{m'} Z_{ir}v_{rs}, & 1\le i\le n',\ 1\le s\le m \\ \sum_{s=1}^{m} v_{rs}\,Y_{sj} = \sum_{i=1}^{n'} T_{ri}u_{ij}, & 1\le r\le m',\ 1\le j\le n \\ \sum_{s=1}^{m} X_{js}Y_{sj'} - \delta_{jj'} = \sum_{i=1}^{n'} W_{ji}u_{ij'}, & 1\le j,j'\le n \\ \sum_{j=1}^{n} Y_{sj}X_{js'} - \delta_{ss'} = \sum_{r=1}^{m'} W'_{sr}v_{rs'}, & 1\le s,s'\le m \end{cases}$$

has a solution in A. Similarly $B\otimes_A M \cong B\otimes_A N$ as B-modules iff (*) has a solution in B. But (*) has a solution in A iff it has one in B because $A\rightarrow B$ is algebraically pure.

(3.6) <u>Lemma:</u> Let $h: A\rightarrow B$ be a morphism of Noetherian rings and M a f.g. A-module. Suppose that either

i) h is strong algebraically pure, or

ii) h is algebraically pure and B is the completion of A with respect to an ideal $a\subset A$ contained in the Jacobson radical of A.

Then $B\otimes_A M$ is an indecomposable B-module iff M is an indecomposable A-module.

<u>Proof:</u> Conserving the notations from the proof of (3.5) for M = N we note that

1) f is idempotent iff $(\varphi^2-\varphi)(e)\subset(u)$, i.e. there exist $d_{ji}\in A$ such that

$$\sum_{s=1}^{n} x_{js}x_{sr}-x_{jr} = \sum_{i=1}^{n'} d_{ji}u_{ir}, \quad 1\le j,r\le n$$

2) $f\ne 0,1$ iff the following two systems of polynomials

$$(G_1)\quad \sum_{i=1}^{n'} Q_{ji}u_{is} = X_{js} \qquad 1\le j,s\le n$$

$$(G2)\quad \sum_{i=1}^{n'} Q'_{ji}u_{is} = X_{js}-\delta_{js} \qquad 1\le j,s\le n$$

have no solutions in A with $X = x$. Thus M is decomposable iff the following system

$$(F)\begin{cases} \sum_{j=1}^{n} u_{ij}X_{js} = \sum_{r=1}^{n'} Z_{ir}u_{rs}, & 1\leq i\leq n',\ 1\leq s\leq n \\ \sum_{s=1}^{n} X_{js}X_{sr}-X_{jr} = \sum_{i=1}^{n'} \Delta_{ji}u_{ir}, & 1\leq j,r\leq n \end{cases}$$

has a solution (x,z,d) for which the systems $G_1(x,Q)$, $G_2(x,Q')$ have no solutions in A. A similar statement is true for $B \otimes_A M$ and they are equivalent if h is strong algebraically pure.

Now suppose that ii) holds. Then h is faithfully flat and we obtain: M is indecomposable if $B \otimes_A M$ is so (cf. (3.2.1)). If $B \otimes_A M$ is decomposable then it has an idempotent endomorphism $\neq 0,1$ which gives a solution $(\tilde{x},\tilde{z},\tilde{d})$ of F in B. Since (A,a) is an AP-couple, h being algebraically pure (see (3.4)) there exists a solution (x,z,d) of F in A such that $(x,z,d) \equiv (\tilde{x},\tilde{z},\tilde{d})$ mod aB. Let f be the idempotent endomorphism of M given by x. Then $A/a \otimes_A f \cong B/aB \otimes_B \tilde{f}$ because $A/a \cong B/aB$. By Nakayama's Lemma $(B/aB) \otimes_B \tilde{f} \neq 0,1$ and so $f \neq 0,1$, i.e. M is decomposable.

(3.7) <u>Remark:</u> When A is a local ring, a its maximal ideal and B the completion of A with respect to a then h is strong algebraically pure if h is algebraically pure. Thus in the above Lemma ii) may be a particular case of i).

(3.8) <u>Proposition:</u> Let (A,a) be an AP-couple and B the completion of A with respect to a. Suppose that A,B are local CM-rings. Then the base change functor $B \otimes_A -$ induces an injection $\Gamma_o(A) \to \Gamma_o(B)$.

The result follows from Lemmas (3.5), (3.6).

(3.9) <u>Lemma:</u> Conserving the hypothesis of the above Proposition, let M,N be two indecomposable MCM A-modules and $f: M \to N$ an irreducible A-map. Suppose that the base change functor $B \otimes_A -$ induces a bijection $\Gamma_o(A) \to \Gamma_o(B)$. Then $B \otimes_A f$ is an irreducible B-map.

Proof: By faithfully flatness $B \otimes_A f$ is not bijective. Let $B \otimes_A f = \tilde{g}\,\tilde{h}$ be a factorization in the category of MCM B-modules, $\tilde{h} : B \otimes_A M \to \tilde{P}$, $\tilde{g}: \tilde{P} \to B \otimes_A N$. By hypothesis $\tilde{P} \cong B \otimes_A P$ for a MCM A-module P. Suppose that $\tilde{g}$ has no section and $\tilde{h}$ has no retraction. Then $B/a^sB \otimes_A \tilde{g}$ has no section and $B/a^sB \otimes_A \tilde{h}$ has no retraction for a certain $s \in \mathbb{N}$ by the following

(3.9.1) Lemma: Let (B,n) be a Noetherian local ring, $b \subset B$ an ideal and $u: M \to N$ a B-linear map. Then there exists a positive integer $s \in \mathbb{N}$ such that u has a retraction (resp. a section) iff it has one modulo b^s.

As (A,a) is an AP-couple we can find a factorization $f = g\,h$, $h: M \to P$, $g: P \to N$ such that $(B/a^sB) \otimes_A \tilde{h} \cong (B/a^sB) \otimes_A h$, $(B/a^sB) \otimes_A \tilde{g} \cong (B/a^sB) \otimes_A g$ (the idea follows the proofs of (3.5), (3.6)). Since $A/a^s \cong B/a^sB$ it follows that $(A/a^s) \otimes h$ has no retraction and $(A/a^s) \otimes g$ has no section. Thus h has no retraction and g has no section. Contradiction (f is irreducible)!

Proof of (3.9.1): As in the proof of (3.5) we see that u has a retraction (resp. a section) iff a certain linear system L of equations over B has a solution in B. Let $\hat{B}$ be the completion of B with respect to n. Then by a strong approximation Theorem (cf. e.g. [Po1] (1.5) there exists a positive integer $s \in \mathbb{N}$ such that L has solutions in $\hat{B}$ iff it has solutions in $\hat{B}/n^s\hat{B}$.

If u has a retraction (resp. a section) modulo b^s then L has a solution in B/b^s. Thus L has a solution in $\hat{B}/n^s\hat{B}$ and so a solution in $\hat{B}$. Then by faithfully flatness L has a solution in B, i.e. u has a retraction (resp. a section).

(3.10) Theorem: Suppose that (R,m) is an excellent Henselian local ring and A is the completion of R with respect to $I_s(R)$. Then the base change functor $A \otimes_R -$ induces an inclusion $\Gamma(R) \subset \Gamma(A)$ which is surjective on vertices. In particular $\# \Gamma_o(R) = \# \Gamma_o(A)$.

Proof: By hypothesis $(R, I_s(R))$ is an AP-couple (cf. [Po1](1.3)) and thus $A \otimes_R -$ induces an inclusion $\Gamma_o(R) \subset \Gamma_o(A)$ (cf. (3.8)) which is in fact an equality by [El] Theorem 3 (cf. the proof of (4.2)). Now it is enough to apply Lemma (3.9).

(3.11) Corollary: Conserving the hypothesis of Theorem (3.10) suppose that

i) R is a Gorenstein isolated singularity and p = char R (i.e. R is of equal characteristic)

ii) k is algebraically closed.

Then R is of finite CM-type iff its completion A is a simple hypersurface singularity.

Proof: Note that R is of finite CM-type iff A is a simple hypersurface singularity by [Kn],[BGS] Theorem A and [GK](1.4) since $\# \Gamma_o(R) = \# \Gamma_o(A)$.

(3.12) Corollary: Conserving the hypothesis of Theorem (3.10), suppose that $k = \mathbb{C}$ and the completion B of R with respect to m is a hypersurface. Then R has countable infinite CM-type iff B is a singularity of type A_∞, D_∞.

The result follows from [BGS] Theorem B and our Theorem (3.10)

(3.13) Remark: Concerning Theorem (3.10), it would be also nice to know when $\Gamma(R) = \Gamma(A)$. Unfortunately it seems that Artin approximation theory does not help because the definition of irreducible maps involves in fact an infinite set of equations corresponding to all factorizations.

(3.14) Proposition: Let A be a flat local R-algebra such that mA is the maximal ideal of A. Suppose that

i) R is an excellent Henselian local ring,

ii) A is a CM-ring,

iii) the residue field extension of $R \to A$ is strong algebraically pure (e.g. if k is algebraically closed).

Then the base change functor $A \otimes_R -$ induces an injective map $\Gamma_o(R) \to \Gamma_o(A)$. In particular $\# \Gamma_o(R) \leq \# \Gamma_o(A)$.

Proof: (R,m) is an AP-couple by [Po1] (1.3) and so the map $R \to A$ is strong algebraically pure by iii) (cf. [BNP] (5.6)). Now apply Lemmas (3.5), (3.6).

(3.15) Remark: If R is not Henselian or iii) does not hold then our Proposition does not hold in general:

(3.16) Example: i) Let $R = \mathbb{R}[X,Y]_{(X,Y)}/(X^2+Y^2)$, $A := \mathbb{C}[X,Y]_{(X,Y)}/(X^2+Y^2)$.

Then $M := (X,Y)R$ is an indecomposable MCM R-module but $A \otimes_R M \cong (X+iY)A \oplus (X-iY)A$ is not. Moreover, $\# \Gamma_o(R) = 2$ and $\# \Gamma_o(A) = 3$ by [BEH] (3.1).

ii) Let $R := \mathbb{C}[X,Y]_{(X,Y)}/(Y^2-X^2-X^3)$ and A its henselization. Clearly A contains a unit u such that $u^2 = 1+X$. Then $M := (X,Y)R$ is an indecomposable MCM R-module but $A \otimes_R M \cong (Y-uX)A \oplus (Y+uX)A$ is not. Also note that $\# \Gamma_o(A) = \Gamma_o(\hat{A}) = 3$, $\hat{A}$ being the completion of A (see (3.10)).

4. The Brauer-Thrall conjectures on isolated singularities

Let (R,m) be a Henselian local CM-ring, $k := R/m$, $p :=$ char k. We suppose that R is an isolated singularity, i.e. $I_s(R) = m$.

(4.1) Proposition: Let Γ^0 be a connected component of $\Gamma(R)$. Suppose that

i) R has bound properties on MCM modules,

ii) Γ^0 is of bounded multiplicity type, i.e. all indecomposable MCM R-modules M whose isomorphic classes are vertices in Γ^0 have multiplicity $e(M) \leq s$ for a certain constant integer $s = s(\Gamma_0)$.

Then $\Gamma(R) = \Gamma^0$ and $\Gamma(R)$ is a finite graph.

Proof: By Proposition (2.7) there is a positive integer r such that m^r is a Dieterich reduction ideal, i.e., a CM-reduction ideal which is m-primary. Now it is enough to follow [Po2] (5.4) (in fact the ideas come from [Di] Proposition 2 and [Yo] Theorem (1.1)).

(4.2) Corollary: Suppose that

i) R has bound properties on MCM modules,

ii) R has infinite CM-type.

Then there exist MCM R-modules of arbitrarily high multiplicity (or rank if R is a domain).

(4.3) Corollary: ([Po2](1.2)) Suppose that

i) R is an excellent ring and $[k:k^p] < \infty$ if $p > 0$,

ii) $Reg(R/pR) = \{q/pR \mid q \in Reg\, R,\ q \supset pR\}$ if $pR \neq 0$.

Then the first Brauer-Thrall conjecture is valid for R, i.e., if R has infinite CM-type then there exist MCM R-modulas of arbitrarily high multiplicity (or rank if R is a domain).

(4.4) Proposition: Suppose that

i) (R,m) is a two dimensional excellent Gorenstein ring

ii) R has bound properties on MCM modules,

iii) the divisor class group $Cl(R)$ of R is infinite.

Then for all $n \in \mathbb{N}$, $n \geq 1$, there are infinitely many isomorphism classes of indecomposable MCM R-modules of rank n over R. In particular, the second Brauer-Thrall conjecture holds for R, i.e., if R is of infinite CM-type then for arbitrarily high positive integers n, there exist infinitely many vertices in $\Gamma_o(R)$ with multiplicity n (or rank n if R is a domain).

Proof: Let K be the fraction field of R and Δ a Weil divisor on Spec R. Then $J_\Delta := \{x \in K \mid \operatorname{div} x \geq \Delta\}$ is a reflexive R-module of rank one and the correspondence $\Delta \mapsto J_\Delta$ defines an injective map $u: Cl(R) \to \Gamma_o(R)$.

Fix $\alpha \in \operatorname{Im} u$. Let Γ_α be the connected component of α in $\Gamma(R)$, I the set of vertices of Γ_α and M an indecomposable MCM R-module whose isomorphism class $\hat{M}$ belongs to I. If $M \not\cong R$ then there exists an almost split sequence

$$0 \to P \to E \to M \to 0$$

because R is an isolated singularity (cf. [Au3]). As R is a two dimensional Gorenstein ring we obtain $P \cong M$ by [Au1] III Proposition (1.8) (cf. also [Yo]A14 or [Re]).

Let N be an indecomposable MCM R-module. If there is an irreducible map $N \to M$ (resp. $P \cong M \to N$) then it factorizes by $E \to M$ (resp. $M \to E$) and thus N is a direct summand in E. The converse is also true (cf. e.g. [Re] (3.2)) and in particular

1) there exists an irreducible map $N \to M$ iff there exists an irreducible map $M \to N$ (so we can consider instead of Γ_α its undirected graph $|\Gamma_\alpha|$ obtained by removing all the loops and forgetting the direction of the arrows).

2) $\quad 2 \operatorname{rank}_R M = \operatorname{rank}_R E \geq \sum_{\tilde{N}} \operatorname{rank}_R \tilde{N}$,

where $\tilde{N}$ runs through the vertices of $|\Gamma_\alpha|$ which are incident to $\tilde{M}$.

For M = R the fundamental sequence (an exact sequence of this form corresponding to a nonzero element of $\mathrm{Ext}_R^2(k,R) \cong k$, (cf. [Au2] § 6 or [AR] § 1 or [Re]) shows that 2) is valid also in this case.

We proceed with a combinatorial remark (cf. also [HPR]).

Let $\Gamma = (\Gamma_o, \Gamma_1)$ be an undirected graph, Γ_o the set of vertices, $\Gamma_1 \subseteq \Gamma_o \times \Gamma_o$ the set of edges ($(i,i) \notin \Gamma_1$ for all $i \in \Gamma_o$ and $(i,j) \in \Gamma_1$ iff $(j,i) \in \Gamma_1$). $\Gamma' = (\Gamma'_o, \Gamma'_1)$ is a subgraph of Γ if $\Gamma'_o \subset \Gamma_o$ and $\Gamma'_1 = \{(i,j) \in \Gamma_1 | i,j \in \Gamma'_o\}$. A function $r: \Gamma_o \to \mathbb{N}$ is subadditive if

$$(*) \quad 2\,r(i) \geq \sum_{(i,j)\in\Gamma_1} r(j) \quad \text{for all } i \in \Gamma_o.$$

(4.5) Lemma: Assume that

i) Γ is connected,

ii) r is subadditive,

iii) r is not bounded and has minimal value 1.

Then Γ is A_∞: 1 — 2 — 3 ----- and $r(i) = i$ for all i.

Proof: Note that (*) implies:

(a) $2\,r(j) \geq d_j := \#\{i \in \Gamma_o | (i,j) \in \Gamma_1\}$, $j \in \Gamma_o$,

(b) (Monotony) Let i — j — s be a subgraph of Γ. Then $r(i) \geq r(j)$ implies $r(j) \geq r(s)$ (Indeed, by (*) we have $2\,r(j) \geq \sum_{(t,j)\in\Gamma_1} r(t) \geq r(i) + r(s)$, thus $2\,r(j) \geq r(j) + r(s)$).

The assertion of the Lemma immediately follows from

(**) For all $n \geq 1$ there is a subgraph

1 — 2 — 3 ----- n-1 — n

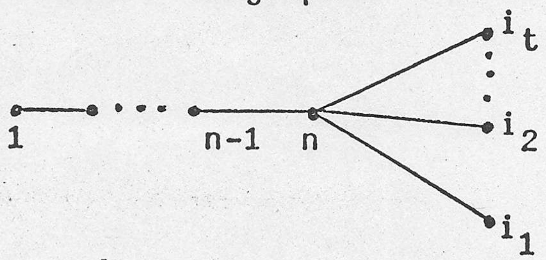
of Γ such that

(c) for $1 \leq j \leq n-1$, $d_j = 1$ if $j = 1$ and $d_j = 2$ if $j > 1$,

(d) $r(j) = j$ for $1 \leq j \leq n$.

(Then Γ contains a subgraph A_∞, which must be Γ itself by c) since Γ is connected.)

We prove (**) by induction on n. If $n = 1$, choose an element $1 \in \Gamma_o$ such that $r(1) = 1$. Now assume (**) is true for $n \geq 1$. Then we find a subgraph



where $t = \begin{cases} d_n - 1 & \text{for } n > 1, \\ d_n & \text{for } n = 1. \end{cases}$

By (*) we have

$$2n \geq n-1 + \sum_{s=1}^{t} r(i_s),$$

and one of the numbers $r(i_s)$ is $> n$ (otherwise r is bounded on Γ_o by (b)). This implies $n+1 \geq \sum_{s=1}^{t} r(i_s) > n$, i.e. $t = 1$, $d_n = 1$ for $n = 1$, respectively, $d_n = 2$ for $n > 1$. If we denote i_1 by $n+1$, we obtain $r(n+1) = n+1$, which completes the proof.

Back to our Proposition, we want to apply (4.5) to $|\Gamma_\alpha|$ for $r :=$ rank. By 2) r is subadditive. If r is bounded we obtain $\Gamma_\alpha = \Gamma(R)$ finite (cf. (5.1)) and thus Cl(R) finite, contradiction! Thus r is unbounded. By (4.5) $|\Gamma_\alpha| = A_\infty$ and $\mathrm{rank}_R(i) = i$ for every i.

Let $\alpha' \in \mathrm{Im}\, u$, $\alpha' \neq \alpha$. Then $\alpha' \notin I$ because Γ_α contains only one vertex of rank one. Thus $\Gamma_\alpha \cap \Gamma_{\alpha'} = \emptyset$ and so for each $n \in \mathbb{N}$ we find $\# \mathrm{rank}^{-1}(\{n\}) \geq \# \mathrm{Cl}(R)$.

(4.6) <u>Corollary</u>: Suppose that

i) (R,m) is a two dimensional excellent Gorenstein ring,

ii) $[k:k^p] < \infty$ if $p > 0$,

iii) $\mathrm{Reg}(R/pR) = \{q/pR \mid q \in \mathrm{Reg}\, R,\ q \supset pR\}$ if $p \neq \mathrm{char}\, R$,

iv) Cl(R) is infinite.

Then for all $n \in \mathbb{N}$, $n \geq 1$ there are infinitely many isomorphism classes of indecomposable MCM R-modules of rank n. In particular the second Brauer-Thrall conjecture holds.

The result follows from (1.7) and (4.5) .

(4.6.1) <u>Remark:</u> Dieterich shows in [Di] Theorem 20, that the second Brauer-Thrall conjecture is valid in arbitrary dimension for complete isolated hypersurface singularities over algebraically closed fields of characteristic $\neq 2$. Using our (3.10) we are able to extend it for a large class of excellent henselian local rings. However in the two dimensional case our Corollary (4.6) gives a sharper version.

(4.7) <u>Corollary:</u> Let n be a positive integer. Suppose that

i) (R,m) is a two dimensional excellent Gorenstein ring

ii) R is a k-algebra and k is algebraically closed.

Then the following conditions are equivalent:

1) R is not a rational double point,

2) There are infinitely many isomorphism classes of indecomposable MCM R-modules of rank n.

<u>Proof:</u> By a well-known result (cf. e.g. [Ba]) R is not a rational double point iff R is not rational (R is Gorenstein!) and thus iff Cl(R) is infinite (cf. [Li] (17.4), (16.2)). Thus i) $\Rightarrow$ ii) follows from the above Corollary. On the other hand, ii) implies: R is of infinite CM-type and thus R is not a rational double point by [AV] (1.11) (see also [EK]) and our (3.3) or (3.10)).

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