EMBEDDED PROJECTIVE VARIETIES OF SMALL INVARIANTS. III

by

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Introduction

Several years ago we have started a program aiming at a classification of embedded smooth projective varieties (over $\mathbb{C}$) following the values of their numerical invariants, assumed to be small enough (see [8], [9], [10], [11], [12]). Although we were primarily interested in the classification according to the degree d, consideration of other invariants (namely the sectional genus g and the Δ-genus Δ) became necessary. The basic tool of our investigation was the adjunction mapping, whose properties were recently understood completely (cf. [24], [25], [14]). We gradually found out the following limitations, inherent to the method employed: $d\leq 8$ (cf. [13]) $g\leq 7$, $\Delta\leq 5$ (cf.[14]). On the other hand, the classification problem naturally splits into two parts. The first task is to obtain a maximal list; secondly, each case has to be investigated in order to decide whether or not it really occurs. Thus, for $d\leq 6$ the list was effective, cf.[8], [9]. For d=7 (see [10], [11]) the existence of four types was left open, while for d=8 the undecided cases were more numerous (cf.[12]). This paper, which is the last in this series, settles the existence problem in all these situations. Thus, the list given in [11] for d=7 turns out to be effective, while from the list given in [12] for d=8 three types have to be excluded (see the table below). We have thus completed the classification of smooth projective varieties up to degree 8. In contrast to [11] where we used mainly ad hoc methods, this time we took the opportunity to present systematically the few general methods available for proving the existence of embedded manifolds.

Finally, let us point out those cases which, in the meantime, were settled by other authors, namely: A.Buium (cf.[2]) first proved the existence of a certain surface of degre 8 in $\mathbb{P}^5$; C. Okonek ([20], [21]) proved the existence of a certain 3-fold of degree 7 in $\mathbb{P}^5$ and of two types of surfaces of degree 8 in $\mathbb{P}^4$; finally, as we can judge from [15], recently J. Alexander showed the existence of a rational surface with d=8, g=5 in $\mathbb{P}^4$, a seemingly subtle case.

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Conventions. Basically we employ the same definitions and notations as in the first two parts [11] and [12]. Let us recall from [12] that the term " linear fibration " used in [11] was replaced by "scroll"; thus, the term "scroll" from [11] became "scroll over a curve". For convenience we recall some of the notations:

- $X \subset \mathbb{P}^n_{\mathbb{C}}$ is a smooth, connected, linearly normal and non-degenerate closed subvariety; dim X=r, codim X=s, degree of X=d.
- H is a (smooth) hyperplane section of X.
- g is the sectional genus of X.
- Δ is the Δ-genus of X.
- $q=h^1(\mathcal{O}_X)$
- $E^{\vee}$ denotes the dual of a vector bundle E.
- $T_X(\Omega^1_X)$ is the tangent (cotangent) bundle of X.
- ω_X or $\mathcal{O}_X(K)$ is the canonical bundle of X.
- $p_g = h^0(\omega_X)$
- $D_1=D_2$ (resp. $D_1 \equiv D_2$) denotes linear equivalence (resp. numerical equivalence) of divisors.
- If $Y \subset X$ is a subvariety, $D|_Y$ denotes restriction of a divisor (class).
- I_Y denotes the sheaf of ideals of Y.
- A smooth projective variety is also called a manifold.

The following table presents the list of (linearly normal, non-degenerate) submanifolds $X \subset \mathbb{P}^n_{\mathbb{C}}$ of degree 8. Notation $\sigma_Z: X \to Y$ means that X is the blowing-up of Y with center Z; E denotes the exceptional locus of σ_Z. L is a line in $\mathbb{P}^2$.

s	r	Abstract structure of X	H or $\mathcal{O}_X(H)$
7	1	$\mathbb{P}^1$	$\mathcal{O}(8)$
7	2-8	scroll over $\mathbb{P}^1$	
6	1	$g=1$	
6	2	$-\mathbb{P}^1 \times \mathbb{P}^1$ $-\sigma_p : X \to \mathbb{P}^2$	$\mathcal{O}(2,2)$ $\sigma^*(3L)-E$
6	3	$\mathbb{P}^3$	$\mathcal{O}(2)$
5	1	$g=2$	
5	2	$-\sigma_{P_0,\dots,P_4} : X \to \mathbb{P}^2$ - scroll over an elliptic curve	$\sigma^*(4L)-2E_0-E_1-\dots-E_4$ $e=0$ $H \equiv C_0+4F$ $e=2$ $H \equiv C_0+5F$
5	3	$X \subset \mathbb{P}^1 \times Q^3$ as a hyperplane section, $Q^3 \subset \mathbb{P}^4$ the hyperquadric	
5	4	$\mathbb{P}^1 \times Q^3$	Segre embedding
4	1	$g=3$	
4	2	$-\sigma_{P_1,\dots,P_8} : X \to F_e$ $e \leq 3$ $-\sigma_{P_1,\dots,P_8} : X \to \mathbb{P}^2$ $-f : X \to \mathbb{P}^2$ double covering	$\sigma^*(H_e)-E_1-\dots-E_8$ $H_e=2C_0+(4+e)F$ $\sigma^*(4L)-E_1-\dots-E_8$ $H=-2K$
4	3	- scroll over an elliptic curve $-\mathbb{P}^1 \times \mathbb{P}^3 \cap Q^6$, $Q^6 \subset \mathbb{P}^7$ a hyperquadric $-f : X \to Z \subset \mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$ double covering, Z a hyperplane section of $\mathbb{P}^1 \times \mathbb{P}^3$ $-\mathbb{P}(E)$, E rank-2 vector bundle on $\mathbb{P}^2$, given by $0 \to \mathcal{O}_{\mathbb{P}^2} \to E \to I^{(4)}_{\{P_1,\dots,P_8\}} \to 0$	tautological
4	4	$f : X \to \mathbb{P}^1 \times \mathbb{P}^3$ double covering, discriminant divisor $D \in \lvert\mathcal{O}(2,2)\rvert$	$f^*\mathcal{O}(1,1)$
	2	- scroll, $e=-2$, $g=2$ $-C \times \mathbb{P}^1$, $C \subset \mathbb{P}^2$ curve of degree 4 - geometrically ruled elliptic surface, $e=-1$	$H \equiv C_0+3F$ Segre embedding $H \equiv 2C_0+F$
	1	$g=4$	

s	r	Abstract structure of X	H or $\mathcal{O}_X(H)$
3	2	$-\sigma_{P_1,\dots,P_{10}}: X \to Q^2 \subset \mathbb{P}^3$ $-\sigma_{P_1,\dots,P_4}: X \to S \subset \mathbb{P}^3$ S cubic surface $-\sigma_{P_1,\dots,P_{12}}: X \to F_e,\ e \leq 4$	$\sigma^*(3H_Q)-E_1-\dots-E_{10}$ $\sigma^*(2H_S)-E_1-\dots-E_4$ $\sigma^*(H_e)-E_1-\dots-E_{12}$ $H_e=2C_o+(5+e)F$
	3	$\mathbb{P}(E)$, E rank-2 vector bundle on the quadric Q, given by $0 \to \mathcal{O}_Q \to E \to I^{(3,3)}_{\{P_1,\dots,P_{10}\}} \to 0$	tautological
	1	g=5	
	2	K3 surface	
	≥3	complete intersection (2,2,2)	
2	1	g=5	
	2	$\sigma_{P_o,\dots,P_{10}}: X \to \mathbb{P}^2$	$\sigma^*(7L)-E_o-2E_1-\dots-2E_{10}$
	1	g=6	
	2	$-\sigma_p: X \to S$, S K3 surface $-\sigma_{P_1,\dots,P_{16}}: X \to \mathbb{P}^2$	 $\sigma^*(6L)-E_1-\dots-E_{12}-$ $-2E_{13}-\dots-2E_{16}$
	1	g=7	
	2	$f_{\lvert K \rvert}: X \to \mathbb{P}^1$, X minimal, elliptic, q=0	
	3	$f_{\lvert H+K \rvert}: X \to \mathbb{P}^1$ with fibres complete intersections (2,2)	
	≥1	complete intersections (2,4)	
1	≥1		

1. The Mumford-Fujita criterion

The following result due to Mumford [18] and Fujita [3] generalizes the familiar fact that on a curve of genus g, a divisor of degree $\geq 2g+1$ is very ample.

Theorem A (Mumford-Fujita). *Let H be an ample divisor on a smooth, projective variety X. Assume that |H| has finitely many base-points, $\Delta\leq g$ and $d\geq 2\Delta+1$. Then H is very ample (and $\Delta=g$).*

(1.1) Corollary. *If $q(X)=0$, |H| is ample with finitely many base-points and $d\geq 2g+1$, then H is very ample.*

Indeed, one may find a reduced, irreducible curve C got by intersecting dim X-1 generic members of |H|. We have $\Delta(X,H)=\Delta(C, H|_C)\leq g=g(C)$, so the Theorem applies.

(1.2) Let $f:X\to\mathbb{P}^1\times\mathbb{P}^3$ be a double covering ramified along a smooth member of $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^3}(2,2)|$ and let $H\in|f^*\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^3}(1,1)|$. Then H is very ample and $d=8$, $g=\Delta=3$. Note that the analogous case of a double covering of $\mathbb{P}^1\times\mathbb{P}^2$, having $d=6$, $g=\Delta=2$ was treated in [11], Prop.7.1.

(1.3) Proposition. *Let c_1, c_2 be two integers such that $c_1\geq 4$, $c_1^2/4<c_2\leq 3(c_1-1)$ and $4c_2\neq c_1^2+4$. Then there exists a 3-dimensional scroll over $\mathbb{P}^2$ with invariants $d=c_1^2-c_2$ and $g=\frac{1}{2}(c_1-1)(c_1-2)$.*

Proof. There is an ample and spanned stable rank-2 vector bundle E on $\mathbb{P}^2$ with Chern numbers c_1, c_2. This follows from [17] Prop.7.6 and Prop.6.5. If $X=\mathbb{P}(E)$ and $H\in|\mathcal{O}_X(1)|$, we find $d=c_1^2-c_2$, $g=1/2(c_1-1)(c_1-2)$. The result follows from (1.1) since $d\geq 2g+1$. In particular, taking $c_1=4$, $9\geq c_2\geq 6$, we find $g=3$, $7\leq d\leq 10$.
We need a modification of the Mumford-Fujita criterion to cover also the case $d=2g$. It is given by the following.

(1.4) Proposition. *Let |H| be an ample linear system without base-points on a smooth projective variety X. In particular there is a smooth curve C, got by intersecting dim X-1 generic members in |H|. Assume that:*

i) $q(X)=0$, $d=2g$;

ii) $|H|_C-K_C|=\emptyset$; *as we shall see, this condition ensures that $H|_C$ is very ample, embedding C into a certain projective space, say $\mathbb{P}^a$;*

iii) *the restriction map $H^0(\mathcal{O}_{\mathbb{P}^a}(2))\to H^0(\mathcal{O}_C(2))$ is onto. Then H is very ample.*

Proof. A result due to Iitaka [7] shows that $H|_C$ is indeed very ample. On the other hand, by [3] Prop.1.10, the map $H^0(\mathcal{O}_C(H))\otimes H^0(\mathcal{O}_C(tH))\to H^0(\mathcal{O}_C((t+1)H))$ is onto for $t\geq 2$. Condition iii) ensures it is onto also for $t=1$, so the proof of the Mumford-Fujita criterion applies.

Now we are going to use (1.4) for proving the existence of a 3-fold in $\mathbb{P}^6$ having invariants $d=8$, $g=4$, which is a scroll over the quadric Q (see [12]). First we need the following:

(1.5) Lemma. A smooth curve in $\mathbb{P}^4$ having $d=8$ and $g=4$ is either arithmetically normal, or hyperelliptic.

Proof. By [3], Prop.1.10, it is arithmetically normal if the map $H^0(\mathcal{O}_{\mathbb{P}^4}(2))\to H^0(\mathcal{O}_C(2))$ is onto. Assuming the contrary, using the sequence:

$$0 \to I_C(2) \to \mathcal{O}_{\mathbb{P}^4}(2)\to\mathcal{O}_C(2)\to 0$$

we find that there are three hyperquadrics Q_1,Q_2,Q_3 (with linearly independent equations) containing C.

The intersection $Q_1\cap Q_2$ must be reducible, since otherwise C would be the complete intersection of Q_1,Q_2,Q_3.

As C is non-degenerate, it must be contained in a (non-degenerate) surface of degree 3 in $\mathbb{P}^4$. Such a surface is either a cone over $\mathbb{P}^1$, which is easily seen to be impossible (cf. Lemma (6.3) below), or a scroll over $\mathbb{P}^1$. In this last case, using the notations of [6] Ch.V, we get $C\in|2C_0+6f|$, so C is hyperelliptic and we are done.

(1.6) Proposition. There is a rank-2 vector bundle E on the quadric Q such that, if we let $X=\mathbb{P}(E)\xrightarrow{\pi} Q$ and H corresponds to the tautological bundle of X, the following hold:

i) $c_1(E)\in|\mathcal{O}_Q(3,3)|$, $c_2(E)=10$;

ii) E restricted to each line from $|\mathcal{O}_Q(1,0)|$ or $|\mathcal{O}_Q(0,1)|$ is of the form $\mathcal{O}(1)\oplus\mathcal{O}(2)$; $|H|$ is ample and base-points free;

iii) $H^0(E(-1,-1))=0$.

(1.7) Assuming for the moment the truth of (1.6), let us see how it applies to give the desired example. Indeed, from i) it follows that $d(H)=8$, $g(H)=4$. On X we have two linear systems $|U|$ and $|V|$ corresponding to the pull-backs by π of the two systems of generators on Q. We have $K=-2H+U+V$, or $-K-H=H-U-V$. Since we have $H^1(\mathcal{O}_X(-U-V))=0$, we find easily that (1.6) iii) implies that (1.4) ii) holds. By the first part in (1.6) ii), a curve got by intersecting two generic members in $|H|$ is mapped isomorphically by π to a curve of the linear system $|\mathcal{O}_Q(3,3)|$; thus it is non-hyperelliptic! Now the result follows combining (1.5) and (1.4).

We divide the proof of (1.6) in three steps.

Step I. Consider $P_1,\dots,P_6$ points "in general position" on Q (as the reader will notice, the "general position" assumptions concern the linear systems $|\mathcal{O}_Q(a,b)|$, with $a,b\le 2$). Consider rank-2 vector bundles E constructed by Serre's method from extensions of type

$$(+)\qquad 0\to \mathcal{O}_Q(1,1)\to E\to I_{\{P_1,\dots,P_6\}}(2,2)\to 0.$$

We get $c_1(E)\in|\mathcal{O}_Q(3,3)|$, $c_2(E)=10$ and E restricted to any line from $|\mathcal{O}_Q(1,0)|$ or $|\mathcal{O}_Q(0,1)|$ is $\mathcal{O}(1)\oplus\mathcal{O}(2)$ (by the assumption of "general position" such a line passes through at most one point P_i). One also gets $h^0(E(-1,-1))=1$, $H^1(E)=0$ and $H^1(E(0,-1))=0$. Now, for any $V\in|\pi^*\mathcal{O}_Q(0,1)|$, $\mathcal{O}_V(H)$ is spanned by global sections. By the exact sequence

$$0\to\mathcal{O}_X(H-V)\to\mathcal{O}_X(H)\to\mathcal{O}_V(H)\to 0,$$

we find that $\mathcal{O}_X(H)$ is spanned. We shall prove that $\mathcal{O}_X(H)$ is ample by using a criterion due to Gieseker ([5], Prop.2.1). Since $\mathcal{O}_X(H)$ is spanned, it will be enough to show that for any irreducible curve C on Q, $E|_C$ has no quotient isomorphic to $\mathcal{O}_C$ (or that the dual $\check{E}|_C$ has no non-zero global sections). Moreover, we need a convenient presentation for E, obtained as follows. Since the restriction of $E(-1,-1)$ to the lines of $|\mathcal{O}_Q(1,0)|$ and $|\mathcal{O}_Q(0,1)|$ is $\mathcal{O}\oplus\mathcal{O}(1)$, we find that $E(-1,-1)$ is a quotient of two bundles of the form $\mathcal{O}(a_1,0)\oplus\mathcal{O}(a_2,0)\oplus\mathcal{O}(a_3,0)$ and $\mathcal{O}(0,b_1)\oplus\mathcal{O}(0,b_2)\oplus\mathcal{O}(0,b_3)$, respectively. Recalling that $h^0(E(-1,-1))=1$ and computing the Chern classes we find out the following two pairs of possible types of presentations of E:

$$(*)\quad 0\to\mathcal{O}_Q(-4,0)\to\mathcal{O}_Q(-2,1)\oplus\mathcal{O}_Q(0,1)\oplus\mathcal{O}_Q(1,1)\to E\to 0,$$

$$(*')\quad 0\to\mathcal{O}_Q(0,-4)\to\mathcal{O}_Q(1,-2)\oplus\mathcal{O}_Q(1,0)\oplus\mathcal{O}_Q(1,1)\to E\to 0,$$

$$(**)\quad 0\to\mathcal{O}_Q(-4,0)\to\mathcal{O}_Q(-1,1)\oplus\mathcal{O}_Q(-1,1)\oplus\mathcal{O}_Q(1,1)\to E\to 0,$$

$$(**')\quad 0\to\mathcal{O}_Q(0,-4)\to\mathcal{O}_Q(1,-1)\oplus\mathcal{O}_Q(1,-1)\oplus\mathcal{O}_Q(1,1)\to E\to 0.$$

Let $C\in|\mathcal{O}_Q(a,b)|$ be an irreducible, reduced curve. We may assume that $a>0$, $b>0$ (if, for instance, $a=0$, it follows $b=1$, so $E|_C$ is ample by ii)). There are four cases to consider, according to the possible combinations of the presentations above. Suppose that $E|_C$ is not ample and take a surjection $E|_C\to\mathcal{O}_C$.

(1) Assume that (*) and (*') occur. Then $\mathcal{O}_C(1,1)$ is mapped to zero in $\mathcal{O}_C$.

$\mathcal{O}_C(0,1)$ is either mapped to zero, or trivial (in which case $a=0$).

Thus we may assume that $\mathcal{O}_C(-2,1)\simeq\mathcal{O}_C$, giving $a=2b$. Working similarly with (*') it follows $b=2a$, so $a=b=0$, which is absurd.

(2) Assume the presentations are (*) and (**'). From (*) we deduce as before that a=2b. From (**') it follows that $\mathcal{O}_C(-1,1)$ has a non-zero global section. The exact sequence:

$$0\to\mathcal{O}_Q(-1-2b,1-b)\to\mathcal{O}_Q(-1,1)\to\mathcal{O}_C(-1,1)\to 0$$

gives b=1. Thus $C\in|\mathcal{O}_Q(2,1)|$. But now recall that E was given by (+). We have $c_1(E|_C)=9$ and, by assumption of "general position", C contains at most 5 of the points P_i. Moreover $C\simeq\mathbb{P}^1$, so we have the exact sequence:

$$0\to\mathcal{O}_C(3+\alpha)\to E|_C\to\mathcal{O}_C(6-\alpha)\to 0,$$

with $0\le\alpha\le 5$, showing that $E|_C$ is ample.

(3) The case (**), (*') is completely similar.

(4) Finally, suppose the presentations to be (**), (**'). As above $\mathcal{O}_C(1,-1)$ and $\mathcal{O}_C(-1,1)$ must have non-zero global sections. It follows that a=b=1, $C\simeq\mathbb{P}^1$, $c_1(E|_C)=6$. Again by the assumption of generality C may contain at most three of the points P_i. The exact sequence

$$0\to\mathcal{O}_C(2+\alpha)\to E|_C\to\mathcal{O}_C(4-\alpha)\to 0,$$

with $0\le\alpha\le 3$ shows that $E|_C$ is ample.
Up to now we have constructed bundles E satisfying i) and ii) of (1.6). But, as already remarked, $h^0(E(-1,-1))=1$, so iii) clearly fails. For the moment, observe that the restriction of E(0,-2) to the members of $|\mathcal{O}_Q(1,0)|$ is $\mathcal{O}\oplus\mathcal{O}(-1)$. Therefore E(0,-2) has a sub-line bundle of type $\mathcal{O}_Q(a,0)$. Thus E is an extension of line bundles and computing Chern classes we find:

(++) $$0\to\mathcal{O}_Q(-4,2)\to E\to\mathcal{O}_Q(7,1)\to 0.$$

Step II. Now consider bundles given by extensions of the form

$$0\to\mathcal{O}_Q(0,1)\to E'\to I_{\{P_1,\dots,P_7\}}(3,2)\to 0,$$

where $P_1,\dots,P_7$ are points "in general position". It follows that $h^0(E'(-1,-1))=0$. Moreover, the restriction of E' to any member of $|\mathcal{O}_Q(1,0)|$ is $\mathcal{O}(1)\oplus\mathcal{O}(2)$. So, by the preceding argument, E' too may be written as an extension of line bundles, as in (++).

Step III. As remarked in [19], the (indecomposable) rank-2 bundles which may be given by an extension as in (++) are parametrized by a projective space $\mathbb{P}^{19}$ and there is also a "versal" bundle on $Q\times\mathbb{P}^{19}$, inducing on various fibres all bundles which may be given as in (++). It is easy to see that from $c_1(E)\in|\mathcal{O}_Q(3,3)|$, $c_2(E)=10$ and $h^0(E)=7$ it follows that E is indecomposable. Now, by Step I, since $h^1(E)=0$, there is an open, non-empty set of $\mathbb{P}^{19}$ corresponding to ample bundles E such that the tautological bundle on $\mathbb{P}(E)$ is spanned. By Step II and semicontinuity there is an open, non-empty set of bundles with $h^0(E(-1,-1))=0$. Thus we may find bundles satisfying all conditions of (1.6). Finally we remark that, conversely, if X is a scroll over Q with d=8, g=4, X is isomorphic to $\mathbb{P}(E)$, where E is a bundle fulfilling all conditions of (1.6).

(1.8) Inspired by [4] we shall prove here the existence of a hyperquadric fibration of dimension 4 with d=7, g=3 (cf. [11]). Let $Y=\mathbb{P}(E)$, where $E=\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1)\oplus\mathcal{O}_{\mathbb{P}^1}(1)\oplus\mathcal{O}_{\mathbb{P}^1}(1)$.

Let $H\in|\mathcal{O}_Y(1)|$ and denote by F a fibre of the projection $\pi:Y\to\mathbb{P}^1$.

Let $S=\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1})\subset Y$ be the embedding corresponding to the surjection $E\to\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}$. We have $H|_S\in|\mathcal{O}_S(0,1)|$ and $(2H+F)|_S\in|\mathcal{O}_S(1,2)|$. Since the linear system $|2H+F|$ is base-points free, we may choose a smooth member $X\in|2H+F|$ such that $X\cap S$ is irreducible. Now we claim that $H|_X$ is ample. If not, since $|H|$ is base-points free, we may find a curve $C\subset X$ such that $(H.C)=0$. But C has to be a section for π and it follows that $C\subset S$, so $C=X\cap S$. This is absurd since $(H.C)=(H.X.S)=$ $=(H|_S.X|_S)_S=1$. We get $(H|_X)^4=7$, $g(H|_X)=3$, $q(X)=0$, so (1.1) applies and $(X, H|_X)$ has the desired properties.

Next we want to investigate projections of a manifold from one of its points. Let X' be a manifold in $\mathbb{P}^N$ with hyperplane section H' and invariants $d'=d(H')$, $g'=g(H')$. A line of X' is a curve $C\subset X'$ such that $(H'.C)=1$. If P is a point on X', let $\sigma:X\to X'$ denote the blowing-up at P, $E=\sigma^{-1}(P)$, $H=\sigma^*(H')-E$. We have $d=d(H)=d'-1$, $g=g(H)=g'$, $\Delta(X,H)=\Delta(X',H')$.

(1.9) Proposition. Assume that $q(X')=0$ and P is not contained in any line of X'. Then

i) If $d'\geq 2g'+2$, H is very ample on X;

ii) If $d'=2g'+1$ and (X',H') is not a hyperquadric fibration, H is very ample if and only if $|-K-(r-2)H|=\emptyset$, where $r=\dim X$.

Proof. The proposition is a consequence of (1.1) and (1.4), but the following "elementary" argument may also be given. We present details for ii), i) being similar and simpler. We have to show that $|H|$ separates points and tangent vectors. Once we identify elements of $|H|$

with hyperplane sections of X' through P, the problem is reduced to proving the same property for $\mathcal{O}_D(H)$, for some smooth member $D \in |H|$. This is seen by using Bertini's theorem (here it is important that there are no lines through P), the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(H) \to \mathcal{O}_D(H) \to 0,$$

and the fact that $q(X)=0$.
Inductively we are reduced to showing the very ampleness of $H|_C$, for a smooth curve C got by intersecting r-1 members of $|H|$. The theory of the adjunction mapping (see [11], Section 1) gives that $|K'+(r-1)H'|$ is base-points free (otherwise there is a line through each point of X') and either $K'+(r-1)H'=0$, or the adjunction mapping has at least a two-dimensional image. As $\sigma^*(K'+(r-1)H')=K+(r-1)H$, it follows $H^1(\mathcal{O}_X(-K-(r-1)H))=0$ by the vanishing theorem. Combining with $|-K-(r-2)H|=\emptyset$, we find inductively that $|H|_C-K_C|=\emptyset$, so we may apply Iitaka's result quoted in (1.4).

Let us see some applications:

(1.10) Take $X'=\mathbb{P}^2$, $H' \in |\mathcal{O}(a)|$, $a\geq 2$. Let p be the maximal number of projections from generic points of X', allowed by (1.9). We find $p\geq 3a-2$ for $a\geq 4$. One knows classically that for a=2, p=1 and for a=3, p=6.

(1.11) Take $X'=Q$, the quadric, $H' \in |\mathcal{O}_Q(3)|$. We have $d'=18$, $g'=4$. We can project from α generic points, where $\alpha\leq 10$. For $\alpha=10$, remark that $\dim|-K'|=8$, so we get $|-K|=\emptyset$ by choosing the ten points generically. In particular the surface thus obtained has d=8, g=4.

(1.12) Take X' to be a cubic in $\mathbb{P}^3$, $H \in |\mathcal{O}(2)|$. It follows that we may project from four generic points, the resulting surface having d=8, g=4. Here the maximal number of projections allowed is five, cf. [11] Prop.8.1.

(1.13) Let X' be the Del Pezzo surface of [11] Th.4.1 iv), $H'=-2K'$ and $P \in X'$ any point. Then, keeping the notations above, H is very ample on X. Indeed, $d'=8$, $g'=3$, $q=0$ and there are no lines on X'.

(1.14) Now take X' to be the Del Pezzo surface having $H'=-3K'$, $d'=9$, $g'=4$. This surface cannot be projected from any of its points. Indeed, we have $\dim|-K'|=1$, so $|-K|\neq\emptyset$ for any position of the point P.

(1.15) Take $X'=\mathbb{P}^3$, $H' \in |\mathcal{O}(2)|$, $P \in X'$. Then we may project one time.

2. A Bertini-type theorem for vector bundles

Theorem B (Kleiman [16]). Let Y be a smooth, projective variety and E a vector bundle of rank a on Y spanned by global sections. Take an integer $b \leq a$ such that $\dim Y < 2(a-b+2)$. Then the dependency locus of b generic global sections of E is either empty or smooth, of pure codimension $a-b+1$ in Y.

Remark. Actually, the proof of the above is simpler and somewhat different from that of [16] (where one has to assume that $E(-1)$ is spanned) since we are in characteristic 0 and generic smoothness holds.

(2.1) Take $Y=\mathbb{P}^4$, $E=\Omega^1_Y(2)$, $b=3$. The dependency locus of three generic sections of E must be non-empty, since otherwise we find an exact sequence of the form

$$0\to \mathcal{O}_Y^{\oplus 3}\to \Omega^1_Y(2)\to \mathcal{O}_Y(3)\to 0$$

This is absurd since $H^0(\Omega^1_Y(1))=0$. Thus, by Theorem B, there is a smooth surface X (which must be connected since $Y=\mathbb{P}^4$) such that its ideal sheaf I_X has a resolution:

$$0\to \mathcal{O}_Y^{\oplus 3}\to \Omega^1_Y(2)\to I_X(3)\to 0. \tag{5}$$

Dualising we get:

$$0\to \mathcal{O}_Y(-3)\to T_Y(-2)\to \mathcal{O}_Y^{\oplus 3}\to \omega_X(2)\to 0.$$

Now it is easy to see that X must be the Veronese surface of degree 4 in $\mathbb{P}^4$.

(2.2) (C. Okonek [20]). Take $Y=\mathbb{P}^5$, $E=\Omega^1_Y(2)$, $b=4$.

The above procedure yields the existence (left open in [11]) of a 3-fold of degree 7 in $\mathbb{P}^5$ which is a scroll over the cubic surface of $\mathbb{P}^3$.

3. The results of Peskine-Szpiro for the case of codimension two

Recall that an arithmetically Cohen-Macaulay submanifold of codimension two in $\mathbb{P}^n$ is a complete intersection if $n \geq 6$. For $n \leq 5$ one has:

Theorem C (Peskine-Szpiro [22]). Let $m \geq 2$, a_i $(i=1,\dots,m-1)$, b_i $(i=1,\dots,m)$ be given positive integers such that

$\sum_{i=1}^{m-1} a_i = \sum_{i=1}^{m} b_i$ and $a_i > b_j$ for any i,j. If $n \le 5$, there is some submanifold $X \subset \mathbb{P}^n$, of codimension two, such that I_X has the resolution:

$$0 \to \bigoplus_{i=1}^{m-1} \mathcal{O}(-a_i) \to \bigoplus_{i=1}^{m} \mathcal{O}(-b_i) \to I_X \to 0.$$

Next, starting with some given submanifold Y of codimension two in $\mathbb{P}^n$, one can try to find a new one, say X, such that the union of X and Y is the complete intersection of two hypersurfaces. One says that X and Y are "linked". We have:

Theorem D (Peskine-Szpiro [22]).

i) Assume that X and Y are linked by the complete intersection of two hypersurfaces of degree a and b. If

$$0 \to E \to F \to I_Y \to 0$$

is a locally free resolution for I_Y, a resolution for I_X is given by:

$$0 \to F^{\vee}(-a-b) \to E^{\vee}(-a-b) \oplus \mathcal{O}(-a) \oplus \mathcal{O}(-b) \to I_X \to 0,$$

ii) If $n \le 5$, a,b,e are integers such that $a,b \ge e$ and $I_Y(e)$ is spanned by global sections, there are generic forms $u \in H^0(I_Y(a))$, $v \in H^0(I_Y(b))$ and a smooth subvariety $X \subset \mathbb{P}^n$ such that X and Y are linked by the complete intersection of the hypersurfaces given by u and v.

(3.1) Combining Theorem C and Theorem D i) we treated in [11] and [12] the following codimension two cases: d=5, g=2; d=6, g=3; d=7, g=5,6 and d=8, g=7.

(3.2) C. Okonek (see [21]) used Theorem D ii) for proving the existence of the two types of surfaces in $\mathbb{P}^4$ having d=8, g=6 and $p_g=0$ or 1. For convenience we give here a simplified version of his argument. Start with the Veronese surface in $\mathbb{P}^4$, denoted by Y. From (5) we see that $I_Y(3)$ is spanned. By Theorem D we may link Y to a (smooth) surface X_1 by two forms of degree 3 and 4 respectively, such that I_{X_1} has the resolution:

(6) $0 \to T(-6) \to \mathcal{O}^{\oplus 4}(-4) \oplus \mathcal{O}(-3) \to I_{X_1} \to 0$; in particular $I_{X_1}(4)$ is spanned.

Twisting (6) by $\mathcal{O}(4)$ and dualising, we get:

$$0 \to \mathcal{O}(-4) \to \mathcal{O}^{\oplus 4} \oplus \mathcal{O}(-1) \to \Omega^1(2) \to \omega_{X}(1) \to 0.$$

Now it is easy to see that X_1 has invariants d=8, g=6, $p_g=0$. Similarly we find an X_2 which is linked to X_1 by two forms of degree 4. From (6) we get:

$$0\to \mathcal{O}^{\oplus 4}\oplus \mathcal{O}(-1)\to \Omega^1(2)\oplus \mathcal{O}^{\oplus 2}\to I_{X_2}(4)\to 0 \text{ and dualising}$$

$$0\to \mathcal{O}(-4)\to T(-2)\oplus \mathcal{O}^{\oplus 2}\to \mathcal{O}^{\oplus 4}\oplus \mathcal{O}(1)\to \omega_{X_2}(1)\to 0.$$ It follows that X_2 has invariants d=8, g=6, $p_g=1$.

4. Reider's theorem on surfaces

The following remarkable result gives an efficient way of proving the very ampleness of a linear system on a surface.

Theorem E (I.Reider [23]). Let X be a smooth, projective surface and H a divisor on X such that:

i) H-K is nef;

ii) $(H-K)^2\geq 9$;

iii) there is no effective divisor E on X such that either

$(H-K.E)=0$, $(E^2)=-1,-2$ or

$(H-K.E)=1$, $(E^2)=0,-1$ or

$(H-K.E)=2$, $(E^2)=0$ or

$H-K\equiv 3E$, $(E^2)=1$.

Then H is very ample.

(4.1) Let X be a geometrically ruled elliptic surface with invariant e=-1 and let $H\equiv 2C_o+F$ (notations as in [6] Ch.V). Then H is very ample. Indeed we have $H-K\equiv 4C_o$, so it is ample (cf.[6] loc cit). The remaining conditions in Theorem E are obvious. The existence of this type of surfaces of degree 8 was first proved by Buium in [2].

(4.2) Let X be a geometrically ruled surface over a curve of genus 2 with invariant e=-2 and let $H\equiv C_o+3F$. Then H is very ample. Indeed, $H-K\equiv 3C_o-F$ is ample and condition iii) in Theorem E is easily verified. We also remark that, conversely, any surface of degree 8 which is a scroll over a curve of genus 2 is necessarily of this type.

5. Scrolls over a curve of genus ≤1

The following is classical and easy. It was included only for perspective.

(5.1) Proposition. A scroll of dimension r and degree d over $\mathbb{P}^1$ has $\Delta=g=0$; the existence of such a scroll with given invariants d,r is equivalent to the numerical condition $d\geq r$. They are all obtained as linear sections of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^{d-1}$ in $\mathbb{P}^{2d-1}$.

(5.2) Proposition. A scroll of dimension r and degree d over an elliptic curve C has $g=1$, $\Delta=r$; the existence of such a scroll with given invariants d,r is equivalent to the numerical condition $d>2r$.

Proof. The equality $\Delta=r$ was proved in [11] Prop.3.11. Assume that X exists in $\mathbb{P}^n$, with $n=h^0(\mathcal{O}_X(H))-1$. Since $\Delta=r$ it follows $d=n+1$. If $d\leq 2r$, we would get $r-(n-r)\geq 1$ and Barth's Theorem (cf [1]) gives $q(X)=0$. This is a contradiction, so necessarily $d>2r$. Conversely, assume this last condition holds. First we show that for any integers $a,b>0$, there exists an ample vector bundle E on C having rank b and $c_1(E)=a$. If $b=1$ this is obvious. Assume we have already found an ample rank$(b-1)$ bundle F on C with $c_1(F)=a$. By Riemann-Roch there is some non-split exact sequence of the form

$$0\to \mathcal{O}_C\to E\to F\to 0.$$

By a result due to Gieseker (see [5], Th.2.2), E is ample and, obviously, $c_1(E)=a$, $rk(E)=b$. The above argument shows that we may find an ample vector bundle E_1 on C with $rk(E_1)=r$, $c_1(E_1)=d-2r$ and an exact sequence

$$0\to \mathcal{O}_C\to E_1\to F\to 0.$$

Take some $L\in Pic(C)$ with $c_1(L)=2$. Let $E=E_1\otimes L$, $X=\mathbb{P}(E)\xrightarrow{\pi} C$. Since we have $c_1(E)=d$ it will be enough to prove that $\mathcal{O}_X(1)$ is very ample on X. For any two points $P,Q\in C$ we let $L_1=L\otimes \mathcal{O}_C(-P-Q)\in Pic^0(C)$. We get $H^1(E\otimes \mathcal{O}_C(-P-Q))=H^1(E_1\otimes L_1)=H^0(E_1^\vee\otimes L_1^\vee)=0$ since E_1 is ample and L_1 has degree 0. Thus we have $H^1(\mathcal{O}_X(1)\otimes\pi^*\mathcal{O}_C(-P-Q))=0$ and the result is a consequence of the following simple lemma.

(5.3) Lemma (cf.[2] Lemma 3.4). Let X be a manifold and $\pi:X\to C$ a morphism onto some smooth curve. Let $X_P=\pi^{-1}(P)$ for $P\in C$. If M is an invertible sheaf on X such that $M|_{X_P}$ is very ample and $H^1(M\otimes\mathcal{O}_X(-X_P-X_Q))=0$ for any $P,Q\in C$, then M is very ample.

6. The effective list of manifolds of degree 7 and 8

(6.1) Theorem. The list of manifolds of degree 7 given in [11] is effective.

For a proof, apply (5.2), (1.3), (1.8) and (2.2) to show the existence of the four types left undecided in [11].

(6.2.) Consider now the case when (X, H) is a hyperquadric fibration over $\mathbb{P}^1$ having invariants $d=8$, $g=\Delta=4$ (cf.[12]). Examples with dim $X=2$ are got by taking divisors of type $(4,2)$ on $\mathbb{P}^1\times\mathbb{P}^2$, embedded Segre into $\mathbb{P}^5$. Next we prove that the cases dim $X\geq 3$ are not possible. We start with a useful remark valid for hyperquadric fibrations $\varphi: X\to C$ of dimension r, over any base curve C. Let us first introduce the following notations: Q for a fibre of φ, $E=\varphi_*(\mathcal{O}_X(H))$, $Y=\mathbb{P}(E)\overset{\pi}{\to} C$, F for a fibre of π and $\mathcal{O}_Y(L)=:\mathcal{O}_Y(1)$. Then we claim that $\mathcal{O}_Y(L)$ is spanned by global sections. Indeed, consider the exact sequence

$$(7) \qquad 0\to\mathcal{O}_X(H-Q)\to\mathcal{O}_X(H)\to\mathcal{O}_Q(H)\to 0.$$

Since Q is a hyperquadric in $\mathbb{P}^r$, the restriction map $H^0(\mathcal{O}_X(H))\to H^0(\mathcal{O}_Q(H))$ is surjective, so we get
$h^0(\mathcal{O}_X(H-Q))=h^0(\mathcal{O}_X(H))-r-1$. It follows that
$h^0(\mathcal{O}_Y(L-F))=h^0(\mathcal{O}_Y(L))-r-1$ and the exact sequence
$0\to\mathcal{O}_Y(L-F)\to\mathcal{O}_Y(L)\to\mathcal{O}_F(L)\to 0$ shows that $\mathcal{O}_Y(L)$ is spanned, since $\mathcal{O}_F(L)$ is so for any fibre F.

Now return to our case when dim $X=3$, $d=2g=8$, $C=\mathbb{P}^1$.
We find $H^1(\mathcal{O}_X(H))=0$ and, using (7), $H^1(\mathcal{O}_X(H-Q))=0$. Thus we must have $E\simeq\bigoplus_{i=1}^{4}\mathcal{O}(a_i)$, with $a_i\geq 0$. Moreover, we get $X\in|2L+2F|$ and $c_1(E)=\sum_{i=1}^{4}a_i=3$.
Since at least one a_i is zero, the map $\Psi_{|L|}: Y\to\mathbb{P}^6$ maps Y onto a cone of degree 3. Thus X is contained in such a cone. Passing to hyperplane sections and using Bertini's theorem we find that some smooth sectional curve of X lies on a two-dimensional cone of degree 3. This contradicts the following lemma.

(6.3) <u>Lemma. Let C be a smooth curve of degree d contained in a surface of degree b which is a cone with vertex P over some smooth curve. Then:</u>

i) <u>if</u> $P\notin C$, b <u>divides</u> d,

ii) <u>if</u> $P\in C$, b <u>divides</u> $d-1$.

For a proof, blow-up P and compute intersection numbers on the resulting geometrically ruled surface.

(6.4) Consider now surfaces in $\mathbb{P}^4$ of degree 8 with $g=5$ and $q=1$ which are hyperquadric fibrations (cf[12]).

We show that they cannot exist. Using the notations introduced in (6.2) we find $(L^3)=4$, $X\in|2L+\pi^*B|$ for some degree-zero divisor on the

elliptic curve C. As remarked in (6.2) we have a morphism $\Psi_{|L|}:Y\to\mathbb{P}^4$. Since $(L^3)=4$, either $\Psi_{|L|}$ maps Y birationally onto a hypersurface of degree 4, say Z, or X is contained in a hyperquadric. This last possibility is absurd since otherwise X would be a complete intersection. Next we show that Z is a cone. Indeed, since the fibres F are mapped to planes, it is enough to find a curve contracted by $\Psi_{|L|}$. If there are no such curves, L is ample and it follows $H^1(E)=H^1(\mathcal{O}_X(H))=0$. But we find $h^1(\mathcal{O}_X(H))=1$. Now, if T is a curve such that $(L.T)=0$, we have $T\cap X=\emptyset$ (because $L|_X=H$ is very ample we cannot have $T\subset X$).
As a consequence, no divisor on Y is contracted by $\Psi_{|L|}$. Indeed, if, say $D\equiv\alpha L+\beta F$ is contracted, it follows:

$$0=(D.L^2)=4\alpha+\beta$$

and, since $D\cap X=\emptyset$, $(D.X.F)=2\alpha=0$, so $\alpha=\beta=0$ which is absurd. Thus we proved that if $S\in|L|$ is a generic member, the map induced by restricting $\Psi_{|L|}$ is a finite, birational morphism between S and a certain surface of degree 4 in $\mathbb{P}^3$. Since S is a geometrically ruled elliptic surface, using the notations of [6] Ch.V, we have $L|_S\equiv C_o+bf$, $4=2b-e$ and $b-e=(C_o+bf.C_o)\geq 2$.

Moreover, $e\geq -1$ implies that necessarily $e=0$, $b=2$, so $b-e=2$ and C_o is mapped two to one onto some line. From this we deduce that there are infinitely many pairs of fibres F mapped to pairs of planes intersecting in a line. If we take a hyperplane of $\mathbb{P}^4$ containing such a pair of planes, its intersection with Z containes a certain quadric, besides the two planes. Taking its pullback by $\Psi_{|L|}$, we find a geometrically ruled elliptic surface mapped birationally to a quadric, which is clearly absurd. It should be pointed out that this class of surfaces was first excluded by Okonek in [21] by a completely different argument.

(6.5) Consider now the rational surfaces of $\mathbb{P}^4$ having $d=8$, $g=5$ (cf.[12]). As we understood from [15], recently J. Alexander proved the existence of this type of surfaces. This seems to be a rather subtle case, since none of the methods described so far can be applied.

Now, looking over the maximal list proposed in [12] for degree 8 and using (5.2), (1.3), (1.2), (4.2), (1.14), (1.7), (1.12), (6.2), (6.4), (6.5) and (3.2) we see that the following result was proved.

(6.6) Theorem. The effective list of manifolds of degree 8 is as given in the table following the introduction.

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