INSTITUTUL
DE
MATEMATICA

INSTITUTUL NATIONAL
PENTRU CREATIE
STIINTIFICA SI TEHNICA

ISSN 0250 3638

*PERTURBATIONS OF NEST-SUBALGEBRAS*
*OF VON NEUMANN ALGEBRAS*

*by*

*Florin POP*

*PREPRINT SERIES IN MATHEMATICS*
*No. 5/1988*



BUCURESTI

# PERTURBATIONS OF NEST-SUBALGEBRAS
# OF VON NEUMANN ALGEBRAS

by

Florin POP*)

January 1988

*) *Department of Mathematics, The National Institute for Scientific and Technical Creation, Bd. Pacii 220, 79622 Bucharest, Romania.*

# PERTURBATIONS OF NEST-SUBALGEBRAS OF VON NEUMANN ALGEBRAS

FLORIN POP

In ([6]) E.C.Lance initiated the perturbation theory of nest algebras and proved that, roughly speaking, two nest algebras are close if and only if their invariant nests are close and in fact they are similar via an invertible operator close to the identity.

In ([4]) and ([5]) F.Gilfeather and D.R.Larson investigated a special class of reflexive algebras, the nest-subalgebras of von Neumann algebras.

In ([2]) K.R.Davidson suggested an analogue of Lance's results for nest-subalgebras of approximately finite von Neumann algebras.

More precisely, let H denote a Hilbert space, B(H) the algebra of bounded operators on H and $M \subset B(H)$ a von Neumann algebra. Let $L \subset M$ be a nest (i.e. a totally ordered strongly closed family) of projections and define the algebras

$$\mathrm{Alg}\, L = \left\{ x \in B(H) \; ; \; (1-p)xp = 0 \quad (\forall)\; p \in L \right\}$$

(the nest algebra with respect to L) and

$M \cap \mathrm{Alg}\, L$ the nest-subalgebra of M with respect to L.

The natural extension of Lance's result is that two nests $L_1$ and $L_2$ in M are close (i.e. there is a lattice isomorphism of $L_1$ onto $L_2$ close to the identity) if and only if the algebras $M \cap \mathrm{Alg}\, L_1$ and $M \cap \mathrm{Alg}\, L_2$ are close in the Hausdorff metric.

Unfortunately, this fails to be true if one does not take certain precautions on M.

A first necessary condition is that M must be a factor.

Indeed, suppose that the center of M is not trivial. We may consider then two nests $L_1 \subset L_2$ in the center, $L_1 \neq L_2$. It follows that

$$M \cap \mathrm{Alg}\, L_1 = M \cap \mathrm{Alg}\, L_2 = M$$

but however $L_1$ and $L_2$ cannot be close.

In this paper we show that this is the only obstruction and we obtain the desired perturbation results in arbitrary factors.

The main ingredient is that in this case nests have a reflexivity-type property (Lemma 3).

We also prove a von Neumann algebra analogue of W.Arveson's distance formula (Theorem 2), which removes the hyperfiniteness hypothesis for M, heavily used in ([2]) and ([4]).

I would like to thank Professor Şerban Strătilă for his constant support and Gabriel Nagy for several fruitful conversations.

LEMMA 1. Let M be a von Neumann algebra and

$$0 \neq p \leq q \leq r \neq 1$$ be projections in M.

If $S = (1-p)Mr$, then for every x in S

$$\mathrm{dist}(x, qS(r-p)) = \max(\|xp\|, \|(1-q)x\|)$$

Proof. Consider the operators $a = qxp$, $b = (1-q)xp$, $c = (1-q)x(r-p)$. Since $x-(a+b+c)$ belongs to $qS(r-p)$, one has

$$\mathrm{dist}(x, qS(r-p)) = \mathrm{dist}(a+b+c, qS(r-p)).$$

Note that $xp = a+b$ and $(1-q)x = b+c$.

We may clearly assume that $\|xp\| \leq 1$ and $\|(1-q)x\| \leq 1$, hence $a^*a + b^*b \leq p$ and $bb^* + cc^* \leq 1-q$.

There are contractions. $u_0 \in pMp$, $v_0 \in (1-q)M(1-q)$, u and v in M such that

$(a^*a)^{\frac{1}{2}} = u_o(p-b^*b)^{\frac{1}{2}}$ and $(cc^*)^{\frac{1}{2}} = v_o((1-q)-bb^*)^{\frac{1}{2}}$ ,

so that $a = u(p-b^*b)^{\frac{1}{2}}$ and

$c^* = v((1-q)-bb^*)^{\frac{1}{2}}$ ([3]) . Moreover,

$$u = (q-p)up \quad \text{and} \quad v = (r-p)v(1-q) .$$

Let $y = (p-b^*b)^{\frac{1}{2}} + b - b^* + ((1-q)-bb^*)^{\frac{1}{2}}$

Routine computations yield

$$(u + (1-q))y(p + v^*) = a+b+c - ubv^* ,$$

$\| u+1-q \| \leqslant 1$ , $\| p+v^* \| \leqslant 1$ and $yy^* = p+1-q$

hence $\| y \| = 1$ . Finally, since $ubv^* \in qS(r-p)$ ,

it follows that $\| a+b+c-ubv^* \| \leqslant 1$ , hence

$$\text{dist}( a+b+c , qS(r-p)) \leqslant 1$$

On the other side it is easy to see that

$$\max ( \| xp \| , \|(1-q)x \| ) \leqslant \text{dist} (x, qS(r-p))$$

and the conclusion follows.

Let now $L \subset M$ be a nest and $A = M \cap \text{Alg } L$ be the corresponding nest-subalgebra.

THEOREM 2. For every x in M

$$\text{dist}(x,A) = \sup_{p \in L} \|(1-p)xp\|$$

Proof. By a slight variation of ([1] Lemma 1) we may assume that

$$L = \left\{ 0 \neq p_1 \leqslant \dots \leqslant p_n \neq 1 \right\} \qquad n \geqslant 1 .$$

For any x in M we note that

$$(1-p_1)Ap_n = (p_2-p_1)A(p_n-p_1) \oplus (1-p_2)A(p_n-p_1) =$$

$$= (p_2-p_1)M(p_n-p_1) \oplus (1-p_2)A(p_n-p_1) , \qquad \text{hence}$$

$$\text{dist}(x,A) = \text{dist} ((1-p_1)xp_n, (1-p_1)Ap_n) \leqslant$$

$$\leqslant \text{dist}((1-p_1)xp_n-(1-p_2)a(p_n-p_1) , (p_2-p_1)M(p_n-p_1) )$$

for any a in A.

We apply Lemma 1 and obtain

$$\mathrm{dist}(x,A) \leqslant \max\left\{ \|(1-p_1)xp_1\| \ , \ \|(1-p_2)xp_n-(1-p_2)a(p_n-p_1)\| \right\}$$

for any $a$ in $A$ ,hence

$$\mathrm{dist}(x,A) \leqslant \max\left\{ \|(1-p_1)xp_1\| \ , \mathrm{dist}((1-p_2)xp_n,(1-p_2)Ap_n) \right\}$$

since $(1-p_2)A(p_n-p_1) = (1-p_2)Ap_n$ .

Suppose now that for some $k < n$ $\mathrm{dist}(x,A) \leqslant$

$$\leqslant \max\left\{ \max_{i=1,k-1} \|(1-p_i)xp_i\| \ , \mathrm{dist}((1-p_k)xp_n,(1-p_k)Ap_n) \right\}$$

Note that $(1-p_k)Ap_n = (p_{k+1}-p_k)M(p_n-p_k) \oplus (1-p_{k+1})A(p_n-p_k)$

By taking into account Lemma 1, it follows again that

$$\mathrm{dist}((1-p_k)xp_n,(1-p_k)Ap_n) \leqslant$$

$$\leqslant \mathrm{dist}((1-p_k)xp_n-(1-p_{k+1})a(p_n-p_k),(p_{k+1}-p_k)M(p_n-p_k)) \leqslant$$

$$\leqslant \max\left\{ \|(1-p_k)xp_k\| \ , \mathrm{dist}((1-p_{k+1})xp_n,(1-p_{k+1})Ap_n) \right\} .$$

At the last step one simply has

$$\mathrm{dist}((1-p_n)xp_n,(1-p_n)Ap_n) \leqslant \|(1-p_n)xp_n\| \quad \text{,hence}$$

$$\mathrm{dist}\,(x,A) \leqslant \max_{p \in L} \|(1-p)xp\| \quad \text{by induction.}$$

Since the opposite inequality is immediate, it follows that $\mathrm{dist}(x,A) = \max_{p \in L} \|(1-p)xp\|$ ,which concludes the proof.

COROLLARY. For every $x$ in $M$

$$\mathrm{dist}(x,\mathrm{Alg}\ L) = \mathrm{dist}(x,A)$$

( A similar but different result is Lemma 4.8 in ([4]) ).

If $M = B(H)$, Theorem 2 is W.Arveson's distance formula ([1]). We note that, excepting slight variations, the outline of the above proof is due, in the case $M = B(H)$ , to S.C.Power ([7]).

For any algebra $A \subset B(H)$ define

$$\text{Lat } A = \left\{ p = p^2 = p^* \in B(H) \; ; \; (1-p)xp = 0 \quad (\forall) x \in A \right\}$$

LEMMA 3 . Let M be a factor and $L \subset M$ be a nest of projections. Then

$$M \cap \text{Lat}(\, M \cap \text{Alg } L\,) = L$$

Proof. If p belongs to $M \cap \text{Lat}(M \cap \text{Alg } L)$ then p commutes with every projection in L. Suppose that there are projections $q \leq r$ , $q \neq r$ in L such that

$$q - pq = p_o \neq 0 \qquad \text{and} \qquad p(r-q) = q_o \neq 0$$

Since M is a factor, there is a partial isometry $x \in M$ , $x \neq 0$ such that $xq_o = x = p_o x$ .

Clearly $(1-p)xp \neq 0$ but $(1-q)xq = 0$ $(\forall)$ $q \in L$ .

The contradiction shows that for every $q \leq r$ in L, $q \neq r$ one has either $q \leq p$ or $p(r-q) = 0$ (1).

Suppose now that $q \leq p \leq r$ and $q \neq r$ are consecutive projections in L (i.e. $r-q$ is an atom in L).

If $p \neq q$ and $r \neq p$ , choose $x \neq 0$ in M such that $x(p-q) = x = (r-p)x$ .

Again it follows that

$(1-p)xp \neq 0$ but $(1-q)xq = 0$ $(\forall)$ $q \in L$ .

Consequently, either $p = q$ or $p = r$ (2)

(1) and (2) show that, roughly speaking, p 'has no holes' in its decomposition with respect to L and that p is trivial on every atom of L. It follows that p belongs to L, which concludes the proof.

Recall that for two subalgebras A and B in B(H), the Hausdorff distance between them is ( slightly different but equivalent to that) given by

$$\mathrm{dist}(A,B) = \max\left\{ \sup_{\substack{x\in A\\ \|x\|\leq 1}} \inf_{y\in B} \|x-y\| \; ; \; \sup_{\substack{y\in B\\ \|y\|\leq 1}} \inf_{x\in A} \|x-y\| \right\}$$

We can state now the main result.

THEOREM 4 . Let M be a factor and $L_1$ , $L_2$ be nests of projections in M. The following statements are equivalent.

i) There is a lattice isomorphism of $L_1$ onto $L_2$ close to the identity.

ii). The algebras $M \cap \mathrm{Alg}\, L_1$ and $M \cap \mathrm{Alg}\, L_2$ are close in the Hausdorff metric.

Proof. (i) $\Rightarrow$(ii) Let x belong to $M \cap \mathrm{Alg}\, L_1$ ,$\|x\| \leq 1$ . For every projection $p \in L_2$ , choose $q \in L_1$ such that $\|p-q\| \leq \varepsilon$ , hence $\|(1-p)xp\| \leq 2\varepsilon$. Theorem 2 implies now that $\mathrm{dist}(x, M \cap \mathrm{Alg}\, L_2) \leq 2\varepsilon$ . If we reverse the roles of $L_1$ and $L_2$ we obtain (ii).

(ii) $\Rightarrow$(i) is esentially ([2] Th.4.4 ii $\Rightarrow$ i) , so we shall only sketch the proof.

Let A be a subalgebra of M such that

$$\mathrm{dist}( M \cap \mathrm{Alg}\, L_1 , A) < \varepsilon \leq 10^{-2}$$

For every p in $L_1$ , one can find a unique projection $\alpha(p)$ in $M \cap \mathrm{Lat}\, A$ such that $L_3 = \{\alpha(p) \; ; \; p \in L_1\}$ is a nest and $\|p-\alpha(p)\| \leq 40\,\varepsilon$ $(\forall)\ p \in L_1$

(See ([2]) for details).

Now if $A = M \cap \mathrm{Alg}\, L_2$, Lemma 3 implies that actually $L_3$ is a subnest of $L_2$, close to $L_1$. By the previous implication, $M \cap \mathrm{Alg}\, L_1$ and $M \cap \mathrm{Alg}\, L_3$ are close, hence $M \cap \mathrm{Alg}\, L_2$ and $M \cap \mathrm{Alg}\, L_3$ are close. Since $M \cap \mathrm{Alg}\, L_2 \subset M \cap \mathrm{Alg}\, L_3$, it follows that in fact they are equal and one uses again Lemma 3 to obtain $L_2 = L_3$, which concludes the proof.

## REFERENCES


1. ARVESON, W. Interpolation problems in nest algebras, J.Functional Analysis 20(1975), 208-233
2. DAVIDSON, K.R. Perturbations of reflexive operator algebras, J.Operator Theory 15(1986), 289-305
3. DIXMIER, J. Les algèbres d'opérateurs dans l'espace hilbertien, Gauthier-Villars, Paris 1957
4. GILFEATHER, F., LARSON, D.R. Nest-subalgebras of von Neumann algebras : commutants modulo compacts and distance estimates, J.Operator Theory 7(1982), 279-302.
5. GILFEATHER, F., LARSON, D.R. Nest-subalgebras of von Neumann algebras, Advances in Math. 46(1982) 171-199.
6. LANCE, E.C. Cohomology and perturbations of nest algebras, Proc.London Math.Soc.(3) 43(1981) 334-356
7. POWER, S.C. The distance to upper triangular operators, Math.Proc.Camb.Phil.Soc. 88(1980), 327-329