THE L PROBLEM OF MOMENTS IN TWO DIMENSIONS

by

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1. Introduction

The L problem of moments consists in characterizing the sequence of moments

$$(1) \qquad a_n = \int_{\mathbb{R}} t^n f(t)dt, \qquad n \in \mathbb{N},$$

of a measurable function f (with prescribed support in $\mathbb{R}$) which satisfies $0 \leq f \leq L$ a.e..This problem was formulated and completely solved by Akhiezer and Krein in the thirties,[2] and [3].Moreover,these authors analyzed in detail several ramifications of the problem,as for instance the localization of the support of the function f in terms of the sequence (1),the generalization to functions with unbounded support,the description of the extremal cases,and so on.A part of these results were independently obtained later by Verblunsky [16],see [3] for full details.

In a previous paper [15] we have characterized the moments

$$(2) \qquad a_{mn} = \int_{\mathbb{C}} z^m \bar{z}^n g(z)d\mu(z), \qquad m,n \in \mathbb{N},$$

of a compactly supported function g defined on $\mathbb{C}$,which satisfies $0 \leq g \leq L$ μ-a.e.,where μ stands for the planar Lebesgue measure.

The aim of the present paper is to continue the study of the two-dimensional L-problem of moments begun in [15],by giving analogues of some of the classical results of Akhiezer and Krein.As a matter of fact we unify the apparently different approaches used in solving the above mentioned moment problems.

The ingredient which lies at the heart of the two methods is the phase shift of M.G.Krein.This well-understood object,coming from the perturbation theory of self-adjoint operators, provides some illuminating geometrical interpretations of the sequences (1) and (2). To be more precise, after the inessential normalization L=1, the following bijections can be established, see Section 2:

dim.	functions	operators	moments
1	$f \in L^1_{comp}(\mathbb{R})$ $0 \leq f \leq 1$	$A, A' \in L(H)$ $A^* = A$ $A' = A + \xi \otimes \xi$ $H = \bigvee A^k \xi \quad , k \geq 0$	$a_n = (n+1)^{-1} Tr((A')^{n+1} - A^{n+1})$
2	$g \in L^1_{comp}(\mathbb{C})$ $0 \leq g \leq 1$	$A, A' \in L(H)$ $A^* = A, A'^* = A'$ $2i[A, A'] = \xi \otimes \xi$ $H = \bigvee A^k A'^l \xi, k, l \geq 0$	$a_{mn} = \pi^{-1} (m+1)^{-1} (n+1)^{-1} \cdot$ $\cdot Tr[(A - iA')^{m+1}, (A + iA')^{n+1}]$

Here H denotes a fixed separable complex Hilbert space, and $\bigvee E$ stands for the closed linear span of the subset $E \subset H$.

In the above table f is the phase shift of the perturbation problem $A \to A'$ and g is the principal function of the pair of self-adjoint operators (A, A'), cf. Section 2. After some exponential transformations of the moments (formulae (10) and (12) in this text), the above dictionary between operators, functions and their moments becomes effective.

This is, we believe, a proper way of solving the two L problems of moments and to read on the moments sequences properties of the respective functions. Of course, finally one can drop the operatorial picture without affecting the results.

Among the two-dimensional counterparts of the results of Akhiezer and Krein presented in this paper we mention: the localization of the support of the function g in terms of its moments (Theorem 4.1); the rigidity of those functions g with extremal (degenerate) kernels of moments (Theorem 5.1); a necessary condition for a double sequence like (2) to represent the moments of a function g as above without restrictions on the support (Theorem 4.6).

The paper is organized as follows. Section 2 is mainly descriptive and recalls, for the convenience of the nonexpert reader, the required basic properties of the phase shift and its two dimensional analogue, the principal function. Though not new, the results discussed in Section 3 are interpretations of Akhiezer and Krein's theorems within the framework of perturbation theory.

Section 4 contains the main two-dimensional results and Section 5 is devoted to an analysis of the extremal L-problems.

The paper ends with some comments and open questions.

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2. Preliminaries from perturbation theory

This paragraph is intended to guide the non-familiarized reader through the basic theory of the phase shift and some of its applications.

Let A be a self-adjoint operator acting on a finite dimensional Hilbert space H, and let $\lambda_1(A) \leq \lambda_2(A) \leq \ldots \leq \lambda_n(A)$ denote its eigenvalues counted with multiplicities. An application of Courant's minimax principle shows that the eigenvalues of a rank-one perturbation $A' = A + \xi\otimes\xi$ of the operator A separate the points $\lambda_j(A)$. In other terms one has the inequalities:

$$\lambda_1(A) \leq \lambda_1(A') \leq \ldots \leq \lambda_{n-1}(A') \leq \lambda_n(A) \leq \lambda_n(A').$$

Thus the characteristic function $\varphi = \sum_{j=1}^{n} \chi_{[\lambda(A_j),\lambda(A_j')]}$ contains complete spectral information about the perturbation $A \to A'$. Then it is immediate to derive the following identity:

$$\text{(3)} \qquad \mathrm{Tr}(p(A')-p(A)) = \int_{\mathbb{R}} p'(t)\varphi(t)dt,$$

for an arbitrary polynomial $p \in \mathbb{C}[t]$.

When A and A' are self-adjoint operators acting on an infinite dimensional Hilbert space H, and $A'-A=\xi\otimes\xi$ is a rank-one operator, a remarkable theorem of M.G.Krein [12] asserts that there exists a function $\varphi\in L^1_{comp}(\mathbb{R})$ $0\leq\varphi\leq 1$, called the phase shift and denoted $\varphi=\varphi(A\rightarrow A')$, such that relation (3) holds. In fact only the assumption $Tr|A'-A|<\infty$ is necessary for the validity of this result, see [12], [10] and [17] for proofs and related questions. It is worth mentioning that throughout this paper all operators are supposed to be bounded.

If $A'=A+\xi\otimes\xi$ is a perturbation as above, a power expansion of the resolvent functions near infinity shows that (3) is equivalent with:

$$(4)\qquad \det((A'-z)(A-z)^{-1}) = \exp(\int_{\mathbb{R}}\varphi(t)(t-z)^{-1}dt), \qquad z\in\mathbb{C}\setminus\mathbb{R}.$$

Here the determinant is extended to infinite matices of the form I+C, $Tr|C|<\infty$, see [10].

In fact

$$\det((A'-z)(A-z)^{-1}) = \det(I+(\xi\otimes\xi)(A-z)^{-1})$$

$$= 1 + \langle (A-z)^{-1}\xi,\xi\rangle$$

$$= 1 + \int_{\mathbb{R}}(t-z)^{-1}d\nu(t),$$

where $d\nu(\cdot)=\langle dE(\cdot)\xi,\xi\rangle$, and E denotes the spectral measure of the operator A.

By well known results, contained for instance in [4],

$$(5)\qquad F(z) = 1 + \int_{\mathbb{R}}(t-z)^{-1}d\nu(t) = \exp(\int_{\mathbb{R}}\varphi(t)(t-z)^{-1}dt)$$

are the additive and respectively the multiplicative representations of an analytic function F belonging to the Nevanlinna class $N=\{F:\mathbb{C}_+\longrightarrow\mathbb{C}_+;\ F$ analytic, $F(\infty)=1\}$, where $\mathbb{C}_+=\{z\in\mathbb{C};\ Im z>0\}$.

The preceding representations realize a bijection between the following sets

$$\left\{\begin{array}{l}\nu\text{ -Borel measure}\\ \nu\geq 0\\ \mathrm{supp}(\nu)\subset\subset\mathbb{R}\end{array}\right\} \cong \left\{\begin{array}{c}\varphi\in L^1_{comp}(\mathbb{R})\\ 0\leq\varphi\leq 1\end{array}\right\}.$$

In conclusion we can state the next.

PROPOSITION 2.1 _Let_ A _be a bounded self-adjoint operator with cyclic vector_ ξ _.The pair_ (A,ξ) _is uniquely determined up to unitary equivalence by the phase shift of the perturbation_ $A\to A+\xi\otimes\xi$.

Every function $\varphi\in L^1_{comp}(\mathbb{R}),0\leq\varphi\leq 1$ _is the phase shift of a one dimensional perturbation_ $A\to A+\xi\otimes\xi$.

The idea of the proof is to exploit in both senses formula (4) and the representation (5).Indeed,the couple (A,ξ),or equivalently (A,A'),is determined by the measure $d\nu=\langle dE\xi,\xi\rangle$,and the measure $d\nu$ is determined by the function φ appearing in (5).

Conversely,any function $\varphi\in L^1_{comp}(\mathbb{R}),0\leq\varphi\leq 1$,produces a measure $d\nu$ by means of relation (5),and this measure uniquely defines a self-adjoint operator A with distinguished cyclic vector ξ .

Let us note from the refined dictionary between $d\nu$ and φ contained in [4],that the self-adjoint operator A with cyclic vector ξ is purely singular if and only if the function φ is equivalent in $L^1(\mathbb{R})$ with the characteristic function of a Borel set.In particular,as we have already saw, the phase shift of a perturbation problem of finite dimensional operators is integer valued.

The list of the properties of the perturbation $A\to A'$ and their effect on the associated phase shift may continue,see [4],[12],[10] and [8].

In the remaining part of this section we shall discuss some aspects related to the two dimensional analogue of the phase shift,namely of the principal function.There exists an extensive literature devoted to this subject,cf. [14] , [7], [8] , [18].The essential properties of the principal function needed for our purposes are summarized in [15],and we shall not repeat them.

Let A,A' be a pair of self-adjoint operators with one dimensional commutator. After a possible change of A with A' we may assume that

$$2i[A,A'] = \xi \otimes \xi ,$$

for a nonzero vector $\xi \in H$. With these assumptions, the (hyponormal) operator $T=A+iA'$ satisfies $[T^*,T] = \xi \otimes \xi$.

An important result of Carey-Pincus [7] and Helton-Howe, see [8], gives the analogue of the trace formula (3):

$$(6) \qquad \operatorname{Tr}\left[p(T,T^*),q(T,T^*)\right] = \pi^{-1}\int_{\mathbb{C}}(\bar{\partial}p\partial q-\bar{\partial}q\partial p)g_T\,d\mu ,$$

valid for any polynomials $p,q \in \mathbb{C}[z,\bar{z}]$. The function g_T is measurable, with compact support and satisfies $0 \le g_T \le 1$ μ-a.e..

In analogy with identity (4), relation (6) implies:

$$(7) \qquad \det((T-w)^{-1}(T^*-\bar{z})^{-1}(T-w)(T^*-\bar{z})) = \exp(-\pi^{-1}\int_{\mathbb{C}}(\zeta-w)^{-1}(\bar{\zeta}-\bar{z})^{-1}g_T(\zeta)\,d\mu(\zeta)).$$

for large values of $|z|$ and $|w|$. Moreover,

$$\det((T-w)^{-1}(T^*-\bar{z})^{-1}(T-w)(T^*-\bar{z})) = \det(1+(T-w)^{-1}(T^*-\bar{z})^{-1}[T,T^*])$$

$$= 1 - \langle (T^*-\bar{z})^{-1}\xi, (T^*-\bar{w})^{-1}\xi \rangle ,$$

whence one obtains the identity:

$$(8) \qquad 1 - \langle (T^*-\bar{z})^{-1}\xi, (T^*-\bar{w})^{-1}\xi \rangle = \exp(-\pi^{-1}\int_{\mathbb{C}}(\zeta-w)^{-1}(\bar{\zeta}-\bar{z})^{-1}g_T(\zeta)\,d\mu(\zeta)).$$

This relation was exploited in [15] for solving the two-dimensional L problem of moments.

A classical by now theorem of Pincus [14] asserts that there exists a bijective correspondence between irreducible hyponormal operators T with rank one self-commutator and their principal function g_T. This bijection was originally found with the aid of the theory of the phase shift. We shall briefly recall this construction.

Let $T=A+iA'$ be as above an irreducible hyponormal operator with one dimensional self-commutator: $[T^*,T] = \xi\otimes\xi$. An inequality of Kato, see [8], asserts that under these assumptions the self-adjoint operators A and A' are absolutely continuous with respect to the linear Lebesgue measure. Accordingly, the space H decomposes into a direct integral which diagonalizes the operator A:

$$H = \int_{\mathbb{R}}^{\oplus} H(t)dt, \qquad A=M_t.$$

As an easy consequence of the hyponormality of the operator T, one obtains the existence of the following abstract symbols:

$$S_A^{\pm}(A') = \text{so-}\lim_{t\to\pm\infty} e^{itA}A'e^{-itA}.$$

These are self-adjoint operators which commute with A, hence they are also diagonalized by the direct integral decomposition of H:

$$S_A^{\pm}(A') = M_{s^{\pm}(t)}, \qquad s^{\pm}(t): H(t) \longrightarrow H(t), \ t\in\mathbb{R}.$$

Moreover, $s^-(t)$ is a non-negative rank-one perturbation of the operator $s^+(t)$, for all $t\in\mathbb{R}$.

The main result of [12] can be formulated as follows:

$$(9) \qquad g_T(x+iy) = \varphi(s^+(x)\rightarrow s^-(x))(y), \qquad x+iy\in\mathbb{C}, \ \mu\text{-a.e.}\ .$$

In particular this equality shows that the principal function g_T is integer valued whenever the self-adjoint operator A=ReT has finite multiplicity in almost every point of the real axis. A possibly new application of this observation is contained in the next.

LEMMA 2.2 Let T be a hyponormal operator with $[T^*,T] = \xi\otimes\xi$. If there exists a polynomial $p(z,\bar{z})$ such that $p(T,T^*)\xi = 0$, then the principal function g_T is integer valued, almost everywhere.

Proof. If $T=A+iA'$ is the Cartesian decomposition of the operator T, then a rearrangment of the terms in the polynomial p yields a new polynomial, say q, with the property $q(A,A')\xi = 0$.

We may assume without loss of generality that the operator T is irreducible, that means that $H = \bigvee_{k,l=0}^{\infty} A^k A'^l \xi$.

Let us denote $H_l = \bigvee_{k=0}^{\infty} A^k A'^l \xi$, for every fixed integer $l \geq 0$; therefore the operator $A|H_l$ is cyclic.

Let m be the degree of the polynomial q in the second variable. The hypothesis $q(A,A')\xi = 0$ insures the existence of an order $n > 0$, such that $A^p A'^m \xi$ is a linear combination of elements of the space $\bigvee_{l=0}^{m-1} H_l + \bigvee_{k=0}^{n} A^k A'^m \xi$, whenever $p > n$.

The commutator identity

$$A'A^p A'^m = \sum_{j=0}^{p-1} A^j [A',A] A^{p-j-1} A'^m + A^p A'^{m+1},$$

combined with $A'q(A,A')\xi = 0$ shows that there exists an ordered polynomial q_{m+1} of degree $m+1$ in the second variable and such that $q_{m+1}(A,A')\xi = 0$. Then by arguing as above the elements $A^p A'^{m+1}\xi$ belong to the subspace $\bigvee_{l=0}^{m} H_l + \bigvee_{k=0}^{r} A^k A'^{m+1}\xi$, for a suitable r and $p > r$.

By inductively repeating this procedure one finds integers $n(l)$, with the property that the subspaces

$$K_j = \bigvee_{l=0}^{m-1} H_l + \bigvee_{l=m}^{j} \bigvee_{k=0}^{n(l)} A^k A'^l \xi, \quad j \geq m,$$

are invariant under A, increasing and $H = \bigvee_{j\geq m} K_j$.

This decomposition shows that the multiplicity function of the self-adjoint operator A is bounded by m, except for a countable subset of $\mathbb{R}$.

In virtue of the above mentioned criterium of integrality of the principal function, the proof is finished.

Notice that Lemma 2.2 remains true, with minor modifications in the proof, in the case of hyponormal operators with finite rank self-commutator.

3. The one-dimensional L problem

Though very close to the original method of Akhiezer and Krein for solving the L problem of moments on the line, the reference to the phase shift theory brings into the field the geometry of Hilbert spaces. This a posteriori interpretation has certain advantages, a part of which we shall briefly discuss below.

In order to state the main result, we shall associate to a sequence of real numbers $(a_n)_{n=0}^{\infty}$ its exponential transform $(b_m)_{m=0}^{\infty}$, as follows:

$$\exp\left(-\sum_{n=0}^{\infty} a_n X^{n+1}\right) = 1 - \sum_{m=0}^{\infty} b_m X^{m+1} . \tag{10}$$

For further use, let us also denote

$$\exp\left(\sum_{n=0}^{\infty} a_n X^{n+1}\right) = 1 + \sum_{m=0}^{\infty} c_m X^{m+1} .$$

The above computations are carried out in the formal series ring $\mathbb{R}[[X]]$. We shall denote by $(Sb)_m = b_{m+1}$ the shift of the sequence (b_m).

THEOREM 3.1 (Akhiezer and Krein [2], [3]) Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers.

(i) There exists a measurable function $f, 0 \leqslant f \leqslant 1$, with compact support contained in the finite interval $[\alpha, \beta] \subset \mathbb{R}$, and with moments $a_n, n \geqslant 0$, if

<u>and only if the Hankel matrices</u> $(b_{m+n}),((S-\alpha)b)_{m+n}$ and $((\beta-S)c)_{m+n}$, $m,n\geq 0$, <u>are positive semi-definite</u>.

(ii) <u>Under the assumptions of</u> (i), $\det(b_{m+n})_{m,n=0}^{N} = 0$ <u>if and only if</u> f <u>is the characteristic function of at most</u> N+1 <u>intervals</u>, $N<\infty$.

(iii) <u>There exists a measurable function</u> f <u>on</u> $\mathbb{R}$, $0\leq f\leq 1$, <u>with the moments</u> a_n, $n\geq 0$, <u>if and only if the Hankel matrix</u> $(b_{m+n})_{m,n=0}^{\infty}$ <u>is positive semi-definite</u>.

The main points of a proof of this theorem, resorting to the theory of the phase shift, are the following.

Let f be a function which fulfills the conditions imposed by (i). In virtue of Proposition 2.1 there exists a self-adjoint operator A with cyclic vector ξ, such that f is the phase shift of the perturbation $A\rightarrow A'=A+\xi\otimes\xi$. In this case $\mathrm{supp}(f)\subset[\alpha,\beta]$, where $\alpha=\inf\{\langle A\eta,\eta\rangle;\ \|\eta\|=1\}$ and $\beta=\sup\{\langle A'\eta,\eta\rangle;\ \|\eta\|=1\}$.

In view of the identities (4), (5) and (10), it follows that, denoting by (a_n) the moments (1) of the function f,

$$(11)\qquad b_m=\langle A^m\xi,\xi\rangle\ ,\quad c_m=\langle A'^m\xi,\xi\rangle\qquad m\geq 0.$$

Then it is a standard matter to check that $b_{m+n}=\langle A^m\xi,A^n\xi\rangle$ is a positive semidefinite kernel. Moreover, the fact that α is a lower bound for the spectrum of A is equivalent to the positive semi-definiteness of the kernel $((S-\alpha)b)_{m+n}$, and analogously for A'.

Conversely, assume that (b_m) is a sequence of real numbers, such that the three positivity conditions in (i) are fulfilled. Then by the classical solution to the Hamburger moment problem, see [1], we infer that there exists a positive measure ν on $\mathbb{R}$, such that

$$b_m=\int_{\mathbb{R}} t^m d\nu(t).$$

The last two positivity assumptions imply the boundedness of the support of ν .Therefore there exists a bounded self-adjoint operator A with a cyclic vector ξ ,and spectral measure E,such that $d\nu = \langle dE\xi, \xi \rangle$.

According to Proposition 2.1,the moments a_n of the phase shift $f = \varphi(A \rightarrow A + \xi \otimes \xi)$ are related to the moments of the measure ν by relation (10).Moreover,the identifications (11) show that $\mathrm{supp}(f) \subset [\alpha, \beta]$ .This finishes the proof of assertion (i).

Assume,under the previous notational conventions,that $\det(b_{m+n})_{m,n=0}^{N}$ vanishes for an integer N.Then $\det\langle A^m\xi, A^n\xi\rangle_{m,n=0}^{N} = 0$,whence the vectors $\xi, A\xi, \dots, A^N\xi$ are linearly dependent.But ξ is a cyclic vector for the operator A,so that the underlying Hilbert space H is finite dimensional. As we have remarked in Section 2,in that case the function $f = \varphi(A \rightarrow A + \xi \otimes \xi)$ is integer valued,with at most N+1 connected components in its support.

Conversely,if f is the characteristic function of at most N+1 bounded intervals,then there exists a one-dimensional perturbation A' of a self-adjoint operator A acting on a Hilbert space H of dimension equal to N+1, with the property $f = \varphi(A \rightarrow A')$.Thus assertion (ii) is proved.

Next we shall prove only the necessity implication in (iii),the sufficiency requiring a deeper analysis of the extremal problem (ii),see [3] for details.

Let $f: \mathbb{R} \rightarrow [0,1]$ be a measurable function,with the property that the moments (1) are finite.Denote by $f_n, n \geq 1$,the truncations of the function f:

$$f_n(x) = \begin{cases} f(x), & |x| \leq n \\ 0, & |x| > n. \end{cases}$$

Notice that the moments $(a_j^n)_{j=0}^{\infty}$ of the functions f_n tend by the Lebesgue dominated convergence theorem to the moments $(a_j)_{j=0}^{\infty}$ of f.

If (b_j^n) denotes the exponential transform (10) of the sequence (a_j^n),then $\lim_n b_j^n = b_j$ for every $j \geq 0$,because b_j^n is a polynomial in a_k^n, $k \leq j$.

Since every matrix $(b_{j+k}^n)_{j,k=0}^{\infty}$ is positive semi-definite by point (i), it follows that the matrix $(b_{j+k})_{j,k=0}^{\infty}$ has this property,too.

The main disadvantage of the method presented in the preceding proof is that it is not suitable for the analysis of the truncated moment problem, that is for characterizing only the first N moments of a bounded, non-negative function f. However, this method seems to be more appropriate for generalizations to higher dimansions.

4. The two-dimensional L problem

In complete analogy with Section 3, the moments (2) of a measurable function $g:\mathbb{C} \to [0,1]$, with compact support, were characterized in the preceding paper [15] by certain positivity conditions imposed to an exponential transform of the moments. Let us briefly recall this construction.

Let $(a_{mn})_{m,n=0}^{\infty}$ be a double sequence of complex numbers which satisfies

$$a_{mn} = \overline{a}_{nm} \quad , m,n \geq 0.$$

The formal exponential transform of the sequence (a_{mn}) is the sequence (b_{mn}) defined by the identity:

$$(12) \quad \exp(-\pi^{-1} \sum_{m,n=0}^{\infty} a_{mn} X^{m+1} Y^{n+1}) = 1 - \sum_{m,n=0}^{\infty} b_{mn} X^{m+1} Y^{n+1} .$$

Let $\iota = (1,0)$ and $\eta = (0,1)$ denote the generators of the semigroup $\mathbb{N}^2$, and put $\theta = (0,0)$ for its neutral element.

Instead of the Hankel matrix from the one dimensional case, we shall define a more involved kernel K, according to the following rules:

(i) $K(\theta, \alpha) = K(m\eta, n\eta) = b(\alpha)$ for $\alpha = (m,n) \in \mathbb{N}^2$,

(ii) $K(\alpha, \beta) = \overline{K(\beta, \alpha)}$,

(iii) $K(\alpha + \iota, \beta) - K(\alpha, \beta + \eta) = \sum_{r=0}^{\infty} K(\alpha, r\iota) b(\beta - (r+1)\iota)$, $\alpha, \beta \in \mathbb{N}^2$,

where $b(\alpha) = b_{mn}$ for $\alpha = (m,n)$ or $b(\alpha) = 0$ if at least one of the entries of α is negative.

We define the shift S and its formal adjoint S^* acting on a function

$E : \mathbb{N}^2 \times \mathbb{N}^2 \longrightarrow \mathbb{C}$ by

$$(SE)(\alpha,\beta) = E(\alpha+\iota,\beta), \qquad (S^*E)(\alpha,\beta) = E(\alpha,\beta+\iota), \qquad \alpha,\beta\in\mathbb{N}^2.$$

Theorem 2 of [15] asserts that (2) are the moments of a measurable function $g:\mathbb{C}\longrightarrow[0,1]$, with compact support, if and only if the kernels K and $(r^2-SS^*)K$ are positive semi-definite for a suitable positive constant r. In that case it was shown that $\mathrm{supp}(g)\subset B(0,r)$.

Our first aim is to localize more accurately the support of the function g in terms of its moments, similarly to Theorem 3.1.(ii).

Let us remark that any compact subset σ of the complex plane can be defined as

$$(13) \qquad \sigma = \left\{ z\in\mathbb{C} \; ; |l_i(z)| \le 1, \; |l_j(z)| \geq 1, \; i\in I, j\in J \right\}$$

where l_i, l_j are linear functions over $\mathbb{C}$, and the sets I and J are at most countable. In other terms the compact σ is represented as an intersection of discs and complementaries of discs.

THEOREM 4.1 Let $(a_{mn})_{m,n=0}^{\infty}$ be a sequence of complex numbers which satisfies $a_{mn}=\bar{a}_{nm}$, $m,n\geq 0$, and let σ be a compact subset of $\mathbb{C}$, written as in (13).

There exists a measurable function $g:\mathbb{C}\rightarrow[0,1]$ with $\mathrm{supp}(g)\subset\sigma$ and moments a_{mn} if and only if the kernels K, $(1-l_i(S)l_i(S)^*)K$ and $(l_j(S)l_j(S)^*-1)K$ are positive semi-definite for $i\in I$, $j\in J$.

Proof. Take a measurable function $g:\mathbb{C}\rightarrow[0,1]$ with $\mathrm{supp}(g)\subset\sigma$ and consider an irreducible hyponormal operator $T\in L(H)$, with $[T^*,T]=\xi\otimes\xi$ and principal function equal to g.

In virtue of relation (8), the exponential transform (12) of the moments (a_{mn}) of the function g, has the coefficients

$$b(m,n) = \langle T^m T^{*n}\xi,\xi\rangle, \quad (m,n)\in\mathbb{N}^2.$$

Moreover, the kernel K was constructed so that

$$K(\alpha,\beta)=\langle T^m T^{*n}\xi, T^p T^{*q}\xi\rangle, \quad \alpha=(m,n),\ \beta=(p,q)\in \mathbb{N}^2.$$

In particular K is a positive semi-definite kernel, see [15] for details.

Let F denote the space of finitely supported functions $h:\mathbb{N}^2\to\mathbb{C}$, endowed with the hermitian scalar product:

$$\langle h,h'\rangle=\sum_{\alpha,\beta} K(\alpha,\beta)h(\alpha)\overline{h'(\beta)}.$$

The map

$$F\longrightarrow H, \quad h\longmapsto \sum_{m,n} h(m,n)T^m T^{*n}\xi$$

is an isometry which identifies the Hilbert space completion of F with H. In this correspondence the operator T is unitarily equivalent to the shift

$$(\tau h)(\alpha)=\begin{cases} h(\alpha-\iota), & \alpha-\iota\in\mathbb{N}^2,\\ 0, & \alpha-\iota\notin\mathbb{N}^2.\end{cases}$$

See again [15] for details.

By (13) the functions 1_i and 1_j^{-1}, $i\in I, j\in J$, are analytic in neighbourhoods of $\sigma=\sigma(T)$, whence the operators $1_i(T)$ and $1_j^{-1}(T)$ are bounded. Furthermore, a simple computation shows that these operators are still hyponormal, cf [8].

Since the spectral radius of a hyponormal operator is equal with its norm, [8], one finds $\|1_i(T)\|\leq 1$ and $\|1_j(T)^{-1}\|\leq 1$, $i\in I, j\in J$. On the Hilbert space associated to the kernel K one obtains

$$(14)\quad \varepsilon(k)\sum_{\alpha,\beta}K(\alpha,\beta)h(\alpha)\overline{h(\beta)} \geq \varepsilon(k)\sum_{\alpha,\beta}K(\alpha,\beta)(1_k(\tau)h(\alpha))\overline{(1_k(\tau)h(\beta))}$$
$$=\varepsilon(k)\sum_{\alpha,\beta}(1_k(S)1_k(S)^* K)(\alpha,\beta)h(\alpha)\overline{h(\beta)},$$

where $h\in F, k\in I$ and $\varepsilon(k)=1$ or $k\in J$ and $\varepsilon(k)=-1$.

This proves that the kernels $\varepsilon(k)(1-1_k(S)1_k(S)^*)K$, $k\in I\cup J$, are positive semi-definite, as desired.

Conversely, assume that the kernels K and $\varepsilon(k)(1-l_k(S)l_k(S)^*)K$, $k\in I\cup J$, associated to the sequence (a_{mn}) and the compact set σ are positive semi-definite.

In this case relations (14) are equivalent to

$$\|l_i(\tau)h\|_K \leq \|h\|_K, \quad i\in I$$

and

$$\|l_j(\tau)h\|_K \geq \|h\|_K, \quad j\in J,$$

for every function $h\in F$.

If $I\neq\emptyset$, then the shift τ turns out to be bounded with respect to the semi-norm $\|\cdot\|_K$. Otherwise there exists a point $c\in\mathbb{C}$, such that $(\tau-c)^{-1}$ extends to a bounded operator on the Hilbert space completion of F in the semi-norm $\|\cdot\|_K$. Then the above inequalities and the spectral mapping theorem imply $\sigma((\tau-c)^{-1})\subset(\bar{\sigma}-c)^{-1}$, whence it follows that τ is a bounded operator with respect to the semi-norm $\|\cdot\|_K$.

In conclusion, always τ is a bounded operator on F. In virtue of Theorem 2 o f [13], there exists a measurable function with compact support $g:\mathbb{C}\longrightarrow[0,1]$, with the prescribed moments (a_{mn}).

The first part of the proof shows then that $\mathrm{supp}(g)\subset\sigma$. This finishes the proof of Theorem 4.1.

The previous result may be interpreted as a higher dimensional analogue of a recent theorem of Berg and Maserick [6].

COROLLARY 4.2 <u>In the conditions of Theorem</u> 4.1, <u>if a linear function</u> l <u>satisfies</u> $|l|\leq 1$ <u>or</u> $|l|\geq 1$ <u>on</u> σ, <u>then the kernel</u> $(1-l(S)l(S)^*)K$, <u>respectively</u> $(l(S)l(S)^*-1)K$, <u>is positive semi-definite</u>.

The proof of this corollary follows from the observation, that the statement of Theorem 4.1 is independent from the choice of the representation (13) for σ.

COROLLARY 4.3 <u>In the conditions of Thoerem 4.1, if</u> $a_{00}=\mu(\sigma)$, <u>then</u> g <u>coincides with the characteristic function of</u> σ.

Indeed, $g \leq \chi_\sigma$ and $a_{00} = \int_{\mathbb{C}} g(z)d\mu(z)$, whence $a_{00}=\mu(\sigma)$ if and only if $g=\chi_\sigma$.

DEFINITION 4.4 From now on, we shall say that a sequence $(a_{mn})_{m,n=0}^{\infty}$ is <u>admissible</u> if it is the sequence of moments (2) of a measurable function $g:\mathbb{C}\to[0,1]$, with compact support.

For an admissible sequence (a_{mn}) we denote by $K:\mathbb{N}^2\times\mathbb{N}^2 \longrightarrow \mathbb{C}$ the kernel associated to its exponential transform (12).

The moments of the characteristis function of a disc can be described by an additional positivity property. Later we shall give a second and completely independent condition.

PROPOSITION 4.5 <u>An admissible sequence</u> (a_{mn}) <u>represents the moments of the characteristic function of a disc</u> D, <u>if and only if the kernel</u>

$$M:\mathbb{N}^3\times\mathbb{N}^3 \longrightarrow \mathbb{C}, \quad M(k,\alpha;l,\beta) = S^l S^{*k} K(\alpha,\beta)$$

<u>is positive semi-definite</u>.

<u>In that case the disc</u> D <u>has centrum</u> $K(\iota,\theta)K(\theta,\theta)^{-1}$ <u>and radius</u> $K(\theta,\theta)^{1/2}$.

<u>Proof</u>. Let g be the function with moments (a_{mn}) and let $T\in L(H)$ be the irreducible hyponormal operator with rank one self-commutator, $[T^*,T]=\xi\otimes\xi$, and principal function equal to g.

A theorem of Morrel [13] states that the operator T is subnormal if and only if $T=aU_+ + b$, $a>0$, $b\in\mathbb{C}$, where U_+ is the unilateral shift of muliplicity 1. Then and only then g is the characteristic function of a disc of centrum b and radius a.

According to the celebrated Halmos and Bram criterion of subnormality, the positive semi-definiteness of the kernel

$$(k,l) \longmapsto T^{*k}T^l$$

is equivalent to the subnormality of T.

Written on a dense subspace of H, this condition turns out to be equivalent with

$$\sum_{k,l} \langle \sum_{m,n} c_{kmn} T^{*k} T^{l} T^{m} T^{*n} \xi , \sum_{p,q} c_{lpq} T^{p} T^{*q} \xi \rangle \geqslant 0,$$

for every function $c: \mathbb{N}^3 \longrightarrow \mathbb{C}$ with finite support.

The last condition is equivalent to the positive semi-definiteness of the kernel M in the statement. This proves the first assertion of Proposition 4.5.

If g is the characteristic function of the disc D and $T = aU_+ + b$, then

$$a^2 = \mathrm{Tr}[T^*, T] = \pi^{-1} \int_{\mathbb{C}} g d\mu = \pi^{-1} \mu(D) = \pi^{-1} a_{00},$$

whence D has radius $a = \pi^{-1/2} a_{00}{}^{1/2} = K(\theta, \theta)^{1/2}$.

Further,

$$b = \langle (aU_+ + b)\xi , \xi \rangle \|\xi\|^{-2} = K(\iota, \theta) K(\theta, \theta)^{-1}$$

is the center of the disc D.

To complete the analysis of the general L problem of moments in two dimensions, we give a necessary condition for a double sequence to be the sequence of moments of a function with arbitrary support.

THEOREM 4.6 <u>The sequence</u> $(a_{mn})_{m,n=0}$ <u>coincides with the moments</u> (2) <u>of a measurable function</u> $g: \mathbb{C} \longrightarrow [0,1]$, <u>only if</u> $a_{mn} = \overline{a}_{nm}$, $m,n \geqslant 0$, <u>and the kernel</u> K <u>is positive semi-definite</u>.

<u>Proof</u>. Assume that (a_{mn}) are the moments of a measurable function g as in the statement. Then obviously $a_{mn} = \overline{a}_{nm}$ for $m,n \geqslant 0$.

Let g^N denote the truncation of the function g:

$$g^N(z) = \begin{cases} g(z), & |z| \leq N, \\ 0, & |z| > N, \end{cases}$$

and consider its moments $(a^N_{mn})^{\infty}_{m,n=0}$.

The Lebesgue dominated convergence theorem implies $\lim_N a^N_{mn} = a_{mn}$ for $m, n \geq 0$. Since by its very construction the kernel K^N associated to the sequence (a^N_{mn}) has polynomial entries in the variables (a^N_{mn}), it follows that $\lim_N K^N(\alpha, \beta) = K(\alpha, \beta)$ for any pair $(\alpha, \beta) \in \mathbb{N}^2 \times \mathbb{N}^2$.

In virtue of Theorem 4.1 above, the kernels K^N are positive semi-definite, therefore K has this property, too. The proof of Theorem 4.6 is complete.

Let σ be a compact subset of $\mathbb{C}$ and $L > 0$. One might ask whether the majorization

(15) $\quad g \leq L\chi_{\sigma}$

cannot be obtained directly from the condition

(16) $\quad \int_{\mathbb{C}} (L\chi_{\sigma} - g)\left|\sum_{m,n} c_{mn} z^m \bar{z}^n\right|^2 d\mu(z) \geq 0, \quad \text{supp}(c) \text{ finite},$

imposed on its moments.

Indeed, by Weierstrass approximation theorem, relation (16) is equivalent with

$$L \int_{\sigma} |f|^2 d\mu \geq \int_{\sigma} g|f|^2 d\mu, \quad f \in L^1(\sigma),$$

which certainly implies (15). Thus (16) is a solution of the L problem of moments for functions with prescribed compact support.

An explanation, which is valid for the one dimensional case too, of the fact that the solutions presented in Theorems 4.1 and 3.1 are preferred, is that condition (16) heavily depends on the set σ. On the contrary, the kernels constructed by the exponential transforms of the moments are universal, the support σ being determined by additional restrictions on them. This has immediate advantages when trying to solve the corresponding problems with no necessary compact support.

5. The extremal L problem

This section is devoted to the study of the measurable functions $g:\mathbb{C} \to [0, 1]$ with compact support, whose associated kernels K (see Section 4) are degenerated.

Let us try to isolate first the core of the concept of degeneracy in two dimensions.

The kernel $(b_{k+l})_{k,l}$ associated to the one-dimensional L problem was degenerated (see Section 3) if

$$\det (b_{k+l})_{k,l=0}^{N} = 0,$$

for an integer $N \geqslant 1$. The solution presented to Theorem 3.1.(ii) shows that in that case

$$\det (b_{k+l})_{k,l \in I} = 0,$$

whenever I is a subset of $\mathbb{N}$ of cardinality N+1.

Since the second vanishing condition is not automatically satisfied in the two dimensional case needed for our analysis, we are led to cosider several degrees of degeneracy.

For a kernel $K : \mathbb{N}^2 \times \mathbb{N}^2 \longrightarrow \mathbb{C}$ we introduce the following conditions:

(D1) There exists an integer N, such that $\det K(\alpha, \beta)_{|\alpha|,|\beta|=0}^{N} = 0$;

(D2) There exists an integer N, such that $\det K(m\eta, n\eta)_{m,n=0}^{N} = 0$;

(D3) For every countable subset $J \subset \mathbb{N}^2$, there exists a finite subset J_0 with the property $\det K(\alpha, \beta)_{\alpha,\beta \in J_0} = 0$.

Notice that even the kernel associated to the classical two-dimensional Hamburger problem of moments distinguighes between the above three conditions.

Not very far from Theorem 3.1.(ii) we can state the next.

THEOREM 5.1 Let $g:\mathbb{C}\to[0,1]$ be a measurable function with compact support, moments $(a_{mn})_{m,n=0}^{\infty}$ and associated kernel K.

(i) If K satisfies (D1), then g is integer valued.

(ii) If K satisfies (D2), then g is the characteristic function of a bounded open subset of $\mathbb{C}$ with real algebraic boundary.

(iii) If K satisfies (D3), then g=0.

Proof. Let $T\in L(H)$ denote an irreducible hyponormal operator with rank-one self-commutator $[T^*,T]=\xi\otimes\xi$ and principal function equal to g. As we have already remarked in the proof of Theorem 4.1,

$$(17)\qquad K(\alpha,\beta)=\langle T^mT^{*n}\xi, T^pT^{*q}\xi\rangle,$$

for $\alpha=(m,n)$, $\beta=(p,q)\in\mathbb{N}^2$.

Condition (D1) is equivalent to the linear dependence of the vectors $T^mT^{*n}\xi$, $m+n\le N$. In virtue of Lemma 2.2 we infer in that case that g is integer valued, hence g coincides with the characteristic function of a compact set $\sigma\subset\mathbb{C}$. This proves point (i).

Assume that the kernel K satisfies condition (D3). Then the sequence $(T^{*n}\xi)_{n=0}^{\infty}$ contains linearly dependent elements. In other terms there exists an integer N' with the property that the vectors $\xi, T^*\xi, \dots, T^{*N'}\xi$ are linearly dependent. By applying the same device to the sequences $(T^nT^{*k}\xi)_{n=0}^{\infty}$ one finds an integer N, so that

$$H=\bigvee_{m,n=0}^{\infty}T^mT^{*n}\xi=\bigvee_{m,n=0}^{N}T^mT^{*n}\xi,$$

that is $\dim H<\infty$.

But any finite dimensional hyponormal operator is zero, whence $g=0$. This completes the proof of assertion (iii).

In order to prove point (ii), let us remark that condition (D2) is equivalent, in view of (17), to the linear dependence of the system of vectors $\xi, T^*\xi, \dots, T^{*N}\xi$. Consider a polynomial p of degree N, such that $p(T^*)\xi = 0$, and denote

$$q(z,w) = (p(z)-p(w))(z-w)^{-1}.$$

Of course q is a polynomial in z and w. Then

$$p(\bar{z})(\bar{z}-T^*)^{-1}\xi = (p(\bar{z})-p(T^*))(\bar{z}-T^*)^{-1}\xi = q(\bar{z},T^*)\xi$$

is a polynomial in $\bar{z}$, for large values of $|z|$, with coefficients in H.

If condition (D2) holds, then point (i) of the proof shows that the function g is the characteristic function of a compact set $\sigma \subset \mathbb{C}$. Moreover, the exponential representation (8) and the preceding arguments prove that the function

$$\exp\left(-\pi^{-1}\int_\sigma |\zeta - z|^{-2} d\mu(\zeta)\right) = 1 - \langle (\bar{z}-T^*)^{-1}\xi, (\bar{z}-T^*)^{-1}\xi \rangle$$

is rational in z and $\bar{z}$, in a neighbourhood of infinity.

Thus the proof of Theorem 5.1 will be finished provided we have proved the next.

LEMMA 5.2 Let σ be a compact subset of $\mathbb{C}$. If the function $\exp(-\pi^{-1}\int_\sigma |\zeta - z|^{-2} d\mu(\zeta))$ is rational in z and $\bar{z}$, for $|z|$ large, then σ coincides, up to a set of Lebesgue measure zero, with the closure of an open subset of $\mathbb{C}$, with real algebraic boundary.

Proof. Let $f(z,\bar{z})$ denote the function of the statement. We shall work outside a ball $B(0,R)$ which contains σ compactly.

From our assumption it follows that the function $f^{-1}\partial f$ is still rational, so that there exists a polynomial $Q(z,\bar{z}) = \sum_{k,l=0}^{N} c_{kl} z^k \bar{z}^l$, such that

$$Q(z,\bar{z})\int_{\sigma}(\zeta-z)^{-2}(\bar{\zeta}-\bar{z})\,d\mu(\zeta)$$

is a polynomial,too.The corresponding convergent series is,after an arrangment of the summation:

$$\sum_{k,l=0}^{N}\sum_{m,n=0}^{\infty}c_{kl}(m+1)z^{-m+k-2}\bar{z}^{-n+l-1}\int_{\sigma}\zeta^{m}\bar{\zeta}^{n}d\mu(\zeta)$$

$$=\sum_{p,q=-N}^{\infty}\sum_{k,l=0}^{N}c_{kl}z^{-p-1}\bar{z}^{-q-1}\int_{\sigma}\partial_{\zeta}(\zeta^{p+k}\bar{\zeta}^{q+l})d\mu(\zeta)$$

$$=\sum_{p,q=-N}^{\infty}\int_{\sigma}\partial_{\zeta}(\zeta^{p}\bar{\zeta}^{q}Q(\zeta,\bar{\zeta}))d\mu(\zeta)z^{-p-1}\bar{z}^{-q-1}.$$

The condition for the last series to be a polynomial in z and $\bar{z}$ is equivalent to the vanishing of the following coefficients:

$$\int_{\sigma}\partial_{\zeta}(\zeta^{p}\bar{\zeta}^{q}Q(\zeta,\bar{\zeta}))d\mu(\zeta)=0,\quad p,q\geqslant 0.$$

In terms of distributions,this is equivalent to $Q\,\partial\chi_{\sigma}=0$,this time on the whole complex plane.

Let Z denote the real algebraic set of zeroes of Q.Since $Q\neq 0$, $\mu(Z)=0$.

If $\lambda\in\mathbb{C}\setminus Z$,then $\partial\chi_{\sigma}$ vanishes in a connected neighbourhood V of λ.Hence the function χ_{σ} coincides in $L^{1}_{loc}(V)$ with the class of an antianalytic function h.But the function χ_{σ} takes only two values,therefore $h=\chi_{V}$ or $h=0$,in virtue of the uniqueness principle for antianalytic functions.

Denote by C_0 a connected component of $\mathbb{C}\setminus Z$.The preceding argument shows that either $[\chi_{\sigma}]=[\chi_{C_0}]$ or $[\chi_{\sigma}]=0$ in $L^{1}_{loc}(C_0)$.

In conclusion we have proved that there exists a finite union C of connected components $C_0,\ldots,C_r$ of $\mathbb{C}\setminus Z$,such that

$$[\chi_{\sigma}]=[\chi_{C}]\quad\text{in } L^{1}_{loc}(\mathbb{C}).$$

The union is finite because the algebraic curve Z has finite degree.

The proof of Lemma 5.2 is thus complete.

It is worth mentioning that the assertions (i) and (ii) give only necessary conditions. However, in both cases the function g is determined by its support, and definitively by the geometry of its support. An enumeration of all supports appearing in (i) and (ii) seems not to be at hand.

To give a simple example, let $\sigma = B(a,r)$ be a disc of center $a \in \mathbb{C}$ and radius $r > 0$. The operator with principal function χ_σ is in that case $T = rU_+ + a$, cf. the proof of Proposition 4.5. If $[U_+^*, U_+] = \xi \otimes \xi$, then $[T^*, T] = r^2 \xi \otimes \xi$, $\|\xi\| = 1$, and

$$(\bar{z} - T^*)^{-1} r\xi = (\bar{z} - \bar{a} - rU_+^*)^{-1} r\xi = (\bar{z} - \bar{a})^{-1} r\xi ,$$

for $|z|$ large. By taking into account relation (8), one obtains

$$(18) \qquad \exp\Big(-\pi^{-1} \int_{B(a,r)} |\zeta - z|^{-2} d\mu(\zeta)\Big) = 1 - r^2 |z-a|^{-2}, \qquad |z-a| > r.$$

This is another possible condensed expression of the moments of the characteristic function of a disc (compare with Proposition 4.5).

The same computation shows also that the converse of Lemma 5.2 is not true.

Indeed, let us denote $D_\pm = \{z \in \mathbb{C} \; ; \; |z| \leq 1, \; \pm \operatorname{Re} z \geq 0\}$, and $D = D_+ \cup D_-$. Suppose that the function

$$R(z) = \exp\Big(-\pi^{-1} \int_{D_+} |\zeta - z|^{-2} d\mu(\zeta)\Big)$$

is rational in z and $\bar{z}$, for $|z|$ large. Then

$$\overline{R(z)} = \exp\Big(-\pi^{-1} \int_{D_-} |\zeta - z|^{-2} d\mu(\zeta)\Big),$$

so that

$$|R(z)|^2 = \exp(-\pi^{-1}\int_D |\zeta - z|^{-2} d\mu(\zeta)) = 1 - |z|^{-2}, \quad |z| \gg 0,$$

because of the identity (18).

Since the function R(z) is real analytic by its very definition for $|z|>1$, an application of the identity principle for real analytic functions shows that R is rational for $|z|>1$ and

(19) $\quad |R(z)|^2 + |z|^{-2} = 1, \quad |z|>1.$

In view of the last equality, the function R can be analytically extended across ∂D, whence, in virtue of the same identity principle, identity (19) holds for a point $\lambda \in D$. This evidently contradicts the inequality $|\lambda|^{-2} > 1$.

In conclusion, the function R is not rational, though its boundary is real algebraic.

6. Final remarks

The above presentation of the L problem of moments in two dimensions is far from being complete. For instance, we conjecture that the reciprocal to Theorem 4.6 is also true. An answer to this question would go beyond the methods developed in the present paper. This problem could be related to the solution of the truncated L problem of moments, i.e. for sequences (a_{mn}) with (m,n) ranging over suitable (finite) subsets of $\mathbb{N}^2$. In connection with this, a characterization of the compact subsets $\sigma \subset \mathbb{C}$, with the property that the function appearing in Lemma 5.2 is rational, would be interesting.

In spite of the different nature and additional difficulties related to higher dimensional moment problems (cf. [9]), there are some striking similarities between the two L problems of moments presented in our paper. These facts suggest a common explanation, which would extend in higher dimensions, too. The multidimensional trace formulae could be the main tool in answering this question.

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