WIENER-HOPF OPERATORS ON THE POSITIVE SEMIGROUP OF A HEISENBERG GROUP

by

A.NICA*)

November 1988

*) Department of Mathematics, The National Institute for Scientific and Technical Creation, Bd.Păcii 220, 79622 Bucharest, Romania.

This is an expanded version of the author's talk at the 12th International Conference on Operator Theory, Timişoara, June 6-16, 1988

WIENER-HOPF OPERATORS ON THE POSITIVE SEMIGROUP OF A HEISENBERG GROUP

1. Introduction The classical Wiener-Hopf operators are obtained by compressing the left-convolution operators on $L^2(R)$ to the subspace $L^2([0,\infty))$. One can make such a compression in the general context of a locally compact group, with $[0,\infty)$ replaced by a semigroup which is the closure of its interior. The most often considered examples of generalized Wiener-Hopf operators obtained in this manner are the euclidian ones, where the group is R^n and the compression is made to a closed convex cone with non-void interior.

An interesting non-euclidian example is given by the Heisenberg group $H_n \subseteq Mat_n(R)$ of upper-triangular matrices having 1 on the diagonal and its "positive semigroup" P_n, obtained by intersection with the set of matrices with non-negative entries. This is the example studied in the present paper in the cases n=3 and n=4 (we note that n=2 gives the classical Wiener-Hopf operators).

The suggestion of considering Wiener-Hopf operators on the Heisenberg groups was given to us in 1985 by Dan Voiculescu; we express him our most profound gratitude.

The instrument we use in our study is the groupoid theory. The observation that Wiener-Hopf operators can be derived from groupoids was made by P.Muhly and J.Renault in [1]; we shall use here the groupoid construction made in [3], which is briefly recalled in section 2 of the paper.

It is known that the C^*-algebra generated by the classical Wiener-Hopf operators contains the compact operators on $L^2([0,\infty))$ and it is natural to ask under what conditions is this assertion true in more general situations. P.Muhly and J.Renault show in [1] that this is the case when the semigroup is pointed and the set of units of the groupoid involved is a regular compactification of the semigroup (see section 3.1). In proposition 3.2.1 we present two "nice" conditions on the order relation induced by the semigroup which imply together "regular compactification" and which are satisfied by the positive semigroup of any H_n. It is noteworthy that these conditions are also satisfied in any pointed euclidian case. Hence, in all these cases, the C^*-algebra of the Wiener-Hopf operators contains the compact operators.

Section 4 is devoted to H_3. Using groupoid techniques, we find without difficulty a composition series of the C^*-algebra of the Wiener-Hopf operators on P_3, which has easily tractable quotients between consecutive ideals. In particular, this C^*-algebra is found to be of type I.

Finally, in section 5, we make the same discussion for H_4. The corresponding C^*-algebra is also of type I, and this result is also obtained by exhibiting a composition series. The general idea is the same as in section 4; but some new complications occur, which indicate that the generalization to the positive semigroup of an arbitrary H_n is not immediate.

<u>2. The groupoid construction</u> In this section we recall the constrution made in [3] of a groupoid whose associated C^*-algebra is isomorphic to the one generated by the Wiener-Hopf operators.

Our setting is as follows: let G be a locally compact second countable unimodular group and let μ be a fixed Haar measure on G. We shall call a subset A of G "solid" if $A \neq \emptyset$ and $A = \mathrm{clos}\,\mathring{A}$; it is easy to see that if A is solid th

$=A$ (so it is worth considering $L^2(\mu|A)$). Let P be a solid semigroup of G. For any f in $C_c(G)$ we define the Wiener-Hopf operator with symbol f on P to be: $W_P(f)=pL(f)j\in\mathcal{L}(L^2(\mu|P))$, where $L(f)\in\mathcal{L}(L^2(\mu))$ is the left-convolution operator with f, $p:L^2(\mu)\to L^2(\mu|P)$ is projection and $j=p^*:L^2(\mu|P)\to L^2(\mu)$ is inclusion. The C^*-subalgebra of $\mathcal{L}(L^2(\mu|P))$ generated by $\{W_P(f)\mid f\in C_c(G)\}$ is called the C^*-algebra of Wiener-Hopf operators on P and is denoted by $\mathcal{W}(P)$.

We say that P satisfies condition (M) if every element of $w^*\text{-clos}\{\chi_{tP^{-1}}\mid t\in P\}\subseteq L^\infty(\mu)$ is of the form χ_A with A a solid subset of G. (Remark: A is uniquely determined by χ_A, i.e. A,B solid and $\chi_A=\chi_B$ μ-a.e. imply A=B -see observation 2.3.3 of [3].) If condition (M) is satisfied, we can construct a groupoid $\mathcal{G}$ having $C^*(\mathcal{G})\approx\mathcal{W}(P)$ in the following manner (for details, see section 2 of [3]):

a) the set of units of $\mathcal{G}$ is $U=w^*\text{-clos}\{\chi_{tP^{-1}}\mid t\in P\}\subseteq L^\infty(\mu)$.

b) the set of arrows of $\mathcal{G}$ is given by left translations with elements of G; that is, whenever $\chi_A\in U$ and $t\in G$ are such that $\chi_{tA}\in U$ (this is shown to be equivalent to $t\in A^{-1}$), we have an arrow $x=(t,A)\in\mathcal{G}$ with $d(x)=\chi_A$ and $r(x)=\chi_{tA}$.

c) the multiplication on $\mathcal{G}$ is defined by: $(s,tA)\,(t,A)=(st,A)$; the identity at χ_A is (e,A), with e the unit of G, and the inverse of (t,A) is (t^{-1},tA).

d) the topology on $\mathcal{G}$: since $\mathcal{G}\subseteq G\times L^\infty(\mu)$, we can take the product between the topology of G and the w^*-topology on $L^\infty(\mu)$ and reduce it to $\mathcal{G}$. We remark that the groupoid topology induced on U coincides with the w^*-topology; it is compact, as we see from the Alaoglu theorem. U is also metrizable, because we assumed G second countable.

e) the Haar system: for any $\chi_A\in U$, the set of arrows leaving χ_A is $\{(t,A) \mid t \in A^{-1}\}$, canonically isomorphic to A^{-1}; we take on it the measure obtained from $\mu|A^{-1}$. In this way we get a right Haar system on $\mathcal{G}$.

As it is shown in section 3 of [3], any closed convex cone with non-void interior in R^n satisfies condition (M), so the groupoid construction is available in the euclidian case.

We consider now the positive semigroup P_n of H_n. We shall identify H_n with $R^{\frac{(n-1)n}{2}}$. It is easy to see that the Lebesgue measure on $R^{\frac{(n-1)n}{2}}$ is both left and right invariant with respect to the multiplication on H_n; this is the Haar measure we are going to work with. By our identification P_n becomes $[0,\infty)^{\frac{(n-1)n}{2}}$. It is clear that H_n and P_n are situated in the above considered setting. We prove that, in addition, the condition (M) is satisfied, so that the groupoid construction can be used to describe $\mathcal{W}(P_n)$:

<u>Proposition 2.1</u> For any $n \geq 2$, P_n satisfies condition (M).

<u>Proof</u> By corollary 3.4.5 of [3], the set

$T = \{\chi_A \mid [0,\infty)^{\frac{(n-1)n}{2}} \subseteq A \subseteq R^{\frac{(n-1)n}{2}}$, A closed and convex$\}$ is w^*--compact. But for any t in P_n, $P_n t^{-1}$ is closed and convex and contains $[0,\infty)^{\frac{(n-1)n}{2}}$; so $w^*\text{-clos}\{\chi_{P_n t^{-1}} \mid t \in P_n\} \subseteq T$. Using unimodularity we obtain that any element of $w^*\text{-clos}\{\chi_{tP_n^{-1}} \mid t \in P_n\}$ is of the form χ_B with B^{-1} closed and convex with non-void interior (this clearly implies B solid).

3. Sufficient conditions for $\mathcal{W}(P)\supseteq\mathcal{K}(L^2(\mu|P))$

3.1. The condition of "regular compactification". We shall assume that, in the setting of section 2, the semigroup P is pointed, i.e. $P\cap P^{-1}=\{e\}$. Then the map $h:P\to U$ defined by $h(t)=\chi_{tP^{-1}}$ is one-to-one, because $tP^{-1}=sP^{-1}\Rightarrow t^{-1}s\in P\cap P^{-1}\Rightarrow t=s$. h is easily seen to be continuous, and has dense range by the very definition of U. But U is compact, so (h,U) is a compactification of P.

Let us denote by V the range of h. We recall that the compactification (h,U) is called regular if V is open in U and if $h:P\to V$ is a homeomorphism. It was remarked by P.Muhly and J.Renault in corollary 3.7.2 of [1] that "(h,U) regular" implies that $\mathcal{W}(P)$ contains the compact operators. As a matter of fact, they use a groupoid construction different from ours, but in order to make their proof work it is sufficient to show that:

a) V is an invariant set of units; indeed, for any t in P, the arrows leaving $\chi_{tP^{-1}}$ are of the form (a,tP^{-1}) with a in Pt^{-1}, hence $a=st^{-1}$ for some s in P, and the range of (st^{-1},tP^{-1}) is $\chi_{sP^{-1}}\in V$.

b) the reduced groupoid $\mathcal{G}|V$ is transitive and principal, i.e. for any s,t in P there exists an unique arrow from $\chi_{sP^{-1}}$ to $\chi_{tP^{-1}}$; this arrow is (ts^{-1},sP^{-1}).

c) $(t,s)\to(ts^{-1},sP^{-1})$ is a groupoid isomorphism between the groupoid $\mathcal{R}$ of the trivial equivalence relation on P and $\mathcal{G}|V$. If $h:P\to V$ is a homeomorphism, then this isomorphism is topological. In addition, one can easily check that it transforms the Haar system induced by $\mu|P$ on $\mathcal{R}$ into the Haar system inherited from $\mathcal{G}$ on $\mathcal{G}|V$.

We mention that in the proof of corollary 3.7.2 of [1],

$\mathcal{K}$ is found as the ideal of $\mathcal{W}(P)$ corresponding by the canonical isomorphism $C^*(\mathcal{G}) \approx \mathcal{W}(P)$ to the ideal of $C^*(\mathcal{G})$ produced by the open invariant subset $\mathcal{V}$ of U; this fact will be used in propositions 4.3 and 5.2.

We also note that as a corollary to the (strong) theorem 3.1 of [2], the C^*-algebra of a transitive and principal groupoid is always isomorphic to $\mathcal{K}$; so even if $h: P \to V$ is not a homeomorphism, the fact alone that $\mathcal{V}$ is open in U implies the existence of an ideal of $\mathcal{W}(P)$ which is isomorphic to $\mathcal{K}$.

<u>3.2 Conditions on the order relation.</u> The relation induced by P on G is defined by $x \leq y \Leftrightarrow x^{-1}y \in P$. We still assume that P is pointed, and this implies that $\leq$ is antisymmetric. The semigroup properties of P imply that $\leq$ is reflexive and transitive, so that it is an order relation on G. The "strict order relation associated to $\leq$ " is defined by $x < y \Leftrightarrow x^{-1}y \in \mathring{P}$. Let us record some simple properties of $\leq$ and $<$ which will be used in the sequel:

1° if $x \leq y < z$ or if $x < y \leq z$, then $x < z$; this happens because $P\mathring{P} \subseteq \mathring{P}$ and $\mathring{P}P \subseteq \mathring{P}$, being open subsets of P.

2° for any x in G we have $x \nless x$, because $P \cap P^{-1} = \{e\}$ implies $e \notin \mathring{P}$.

3° for any x in G the sets $\{y \in G \mid y \geq x\}$ and $\{y \in G \mid y \leq x\}$ are closed, being in fact xP and xP^{-1} respectively. Similarly, $\{y \in G \mid y > x\} = x\mathring{P}$ and $\{y \in G \mid y < x\} = x\mathring{P}^{-1}$ are non-void open sets.

4° if for some $s, t \in G$ it is true that "$x < s \Rightarrow x \leq t$", then $s \leq t$. Indeed, we have $x < s \Leftrightarrow x \in s\mathring{P}^{-1}$, $x \leq t \Leftrightarrow x \in tP^{-1}$, so the hypothesis becomes $s\mathring{P}^{-1} \subseteq tP^{-1}$ and implies in turn: $sP^{-1} = \text{clos } s\mathring{P}^{-1} \subseteq tP^{-1} \Rightarrow Ps^{-1}t \subseteq P \Rightarrow s^{-1}t \in P \Rightarrow s \leq t$.

Our result is the following:

Proposition 3.2.1 Let us assume that (besides the conditions imposed above) we have that for any x in P:

(i) the set $\{y \in P \mid y \leq x\}$ is compact, and

(ii) there exists a continuous path $\lambda : [0,1] \rightarrow P$ such that $\lambda(0)=e$, $\lambda(1)=x$ and $0 \leq a \leq b \leq 1$ implies $\lambda(a) \leq \lambda(b)$.

Then (h,U) is a regular comactification of P, and hence $\mathcal{W}(P)$ contains $\mathcal{K}(L^2(\mu|P))$.

Proof In order to prove (h,U) regular it suffices to show that whenever t and $(t_n)_{n=1}^{\infty}$ of P are such that $h(t_n) \xrightarrow[n\to\infty]{} h(t)$, there exists a compact subset of P which contains every t_n. This is a general fact from topology, and we leave its proof to the reader. (N.B.: The proof makes use of the metrizability of U.)

So, for the rest of the proof, we fix t and $(t_n)_{n=1}^{\infty}$ of P such that $\chi_{t_nP^{-1}} \xrightarrow[n\to\infty]{} \chi_{tP^{-1}}$. We want to exhibit a compact subset of P which contains every t_n. In order to do this, we also fix a t' in $t\overset{\circ}{P}$ ($\subseteq \overset{\circ}{P}$), i.e. such that $t < t'$. We shall prove that $t_n < t'$ for sufficiently large n; this fact, together with the compactness of $\{s \in P \mid s \leq t'\}$, ensured by the hypothesis (i), will clearly finish the proof.

We first prove a related statement:

Lemma 1 Let $s \in G$ be such that $s \nleq t$. Then $s \nleq t_n$ for sufficiently large n.

Proof of lemma 1 We define $D = \{x \in G \mid x < s,\ x \nleq t\}$; D is obviously open and it is non-void because otherwise the implication "$x < s \Rightarrow x \leq t$" would hold, leading to $s \leq t$ by observation 4° above. We have $D \cap tP^{-1} = D \cap \{x \in G \mid x \leq t\} = \emptyset$. On the other hand, for any n satisfying $s \leq t_n$ we clearly have $D \subseteq \{x \in G \mid x \leq t_n\} = t_nP^{-1}$. So if we consider an open relatively compact non-void subset D_0 of D we see that $\int_{tP^{-1}} \chi_{D_0}\, d\mu = 0$, while $\int_{t_nP^{-1}} \chi_{D_0}\, d\mu = \mu(D_0) > 0$

for any n satisfying $s \leq t_n$. But we know that $\int_{t_n P^{-1}} \chi_{D_0}\, d\mu \xrightarrow[n\to\infty]{}$

$\int_{tP^{-1}} \chi_{D_0}\, d\mu$, and this makes the statement of the lemma clear.

Now we consider the compact subset $K=\{x\in P \mid x\leq t', x\not< t'\}$ of P and prove the following lemma concerning it:

Lemma 2 For any y in G such that $y \not< t'$ there exists an x in K such that $x \leq y$.

Proof of lemma 2 If $y\leq t'$ we may take x=y, so we shall assume that $y \not\leq t'$. We define $D=\{z\in P \mid z<t'\}=P\cap t'\mathring{P}^{-1}$. D is open in P and we have $\mathrm{clos}_P D= \mathrm{clos}_G D\subseteq P\cap t'P^{-1}=\{z\in P \mid z\leq t'\}$, hence the boundary of D relatively to P is contained in K. Let $\lambda:[0,1]\longrightarrow P$ be a continuous path connecting e and y, which is increasing relatively to $\leq$ (hypothesis (ii) of the theorem). We have $\lambda(0)\in D$, $\lambda(1)\notin \mathrm{clos}_P D$, so by the connectedness of $[0,1]$ there must exist an a in $(0,1)$ such that $x=\lambda(a)\in \partial_P D\subseteq K$. But $\lambda(a)\leq\lambda(1)$ means exactly that $x\leq y$, and this ends the proof of the lemma.

Observe now that $\bigcup_{s\in G,\, s\not\leq t}\{x\in G \mid x>s\}$ is an open cover of K; indeed, if $x\in G$ does not belong to this union, then it is true that "$s\not\leq t \Rightarrow s\not< x$" which is equivalent to "$s<x\Rightarrow s\leq t$" and hence leads to $x\leq t$, by observation 4° above; but $x\leq t<t'$ implies $x<t'$, so x cannot be in K.

Let us take a finite subcover of this open cover of K; that is, we pick $s_1,\dots,s_m$ of G such that $s_j\not\leq t$ for any $1\leq j\leq m$ and such that $K\subseteq\bigcup_{j=1}^{m}\{x\in G \mid x>s_j\}$. Using lemma 2 it is clear that $\{x\in G \mid x\not< t'\}\subseteq\bigcup_{j=1}^{m}\{x\in G \mid x>s_j\}$, hence that $s_1\not< x, \dots ,$

$s_m \not< x$ imply together $x < t'$. Using lemma 1 we find n_o with the property that $s_j \not\leqslant t_n$ for any $n \geqslant n_o$ and $1 \leqslant j \leqslant m$. Then $n \geqslant n_o$ implies $t_n < t'$ and the proof is over.

<u>Proposition 3.2.2</u> The hypothesis of proposition 3.2.1 are satisfied:

a) by any closed convex pointed cone with non-void interior in an euclidian space.

b) by the positive semigroup P_n of any H_n.

<u>Proof</u> a) Let $P \subseteq R^n$ be a closed convex pointed cone with non-void interior and let us fix $x \in P$. Defining $\lambda(a)=ax$ on $[o,1]$ we see that hypothesis (ii) is satisfied ($o \leqslant a \leqslant b \leqslant 1 \Rightarrow -ax+bx= =(b-a)x \in P$). In order to verify (i) we consider the dual of P, $\hat{P}= \{\xi \in R^n \mid \langle y,\xi\rangle \geqslant o, \forall y \in P\}$, which is also a closed convex pointed cone with non-void interior. We have $sp\hat{P}=R^n$, hence we can choose n linearly independent vectors $\xi_1, \xi_2, \dots, \xi_n$ of $\hat{P}$. It is known that $\hat{\hat{P}}=P$, that is $y \in P \Leftrightarrow \langle y,\xi\rangle \geqslant o, \forall \xi \in \hat{P}$; as a consequence we see that $y \leqslant x \Leftrightarrow -y+x \in P \Leftrightarrow \langle -y+x,\xi\rangle \geqslant o, \forall \xi \in \hat{P} \Rightarrow \langle y,\xi_j\rangle \leqslant \langle x,\xi_j\rangle, \forall 1 \leqslant j \leqslant n$. Denoting $\alpha_j = \langle x,\xi_j\rangle$ $(1 \leqslant j \leqslant n)$, we obtain that $\{y \in P \mid y \leqslant x\} \subseteq \{y \in R^n \mid |\langle y,\xi_j\rangle| \leqslant \alpha_j, \forall 1 \leqslant j \leqslant n\}$; the last set is easily seen to be bounded.

b) If $x=(x_{i,j})_{i,j}$ and $y=(y_{i,j})_{i,j}$ of P_n are such that $y \leqslant x$, then $x_{i,j} \leqslant y_{i,j}$ for any $1 \leqslant i < j \leqslant n$; indeed, denoting $z=(z_{i,j})_{i,j}= y^{-1}x \in P_n$, we have $x_{i,j}=\sum_{k=i}^{j} y_{i,k}z_{k,j} \geqslant y_{i,j}z_{j,j}=y_{i,j}$. This makes clear the typothesis (i) of proposition 3.2.1.

We pass to (ii). We shall prove that for $s=(s_{i,j})_{i,j}$ and $t=(t_{i,j})_{i,j}$ of P_n placed in any of the following two situations it is true that $s \leqslant t$ and there exists an increasing continuous path connecting s with t:

α) s is obtained from t by replacing a component of the first line with o;

β) there exist $2 \leqslant p < q \leqslant n$ such that $t_{i,p}=o$ for any $1 \leqslant i \leqslant p-1$ and such that s is obtained from t by replacing $t_{p,q}$ with o. Once we have done this, it is easy to see how an arbitrary $x \in P_n$ can be connected with $e \in P_n$ by a continuous decreasing path made of $\frac{(n-1)n}{2}$ pieces.

Proof for the situation α: we can write $t=s+ce_{1,q}$ for some $c \in [o,\infty)$ and $2 \leqslant q \leqslant n$, where $e_{1,q} \in Mat_n(R)$ has the 1,q-entry equal to 1 and the others equal to o. $\lambda(a)=s+ace_{1,q}$ is then a continuous increasing path connecting s with t, because $o \leqslant a \leqslant b \leqslant 1 \Rightarrow \lambda(a)^{-1}\lambda(b)=e+(b-a)ce_{1,q} \in P_n$, (here e is the unit of H_n).

The proof for the situation β is similar to the one for α.

4. The C^*-algebra $\mathcal{W}(P_3)$

In this section we deal with the Wiener-Hopf operators on the positive semigroup P_3 of H_3. We shall explicitely describe in this particular case the unit space of the groupoid construction of section 2, and we shall use it to obtain a composition series for $\mathcal{W}(P_3)$.

We make the identification of H_3 with R^3 by writing (a,b,c) instead of $\begin{pmatrix} 1 & a & c \\ o & 1 & b \\ o & o & 1 \end{pmatrix}$; so we are in fact working with R^3 endowed with the multiplication $(a,b,c)(a',b',c')=(a+a',b+b',c+c'+ab')$. As we remarked at the end of section 2, the Haar measure coincides with the Lebesgue measure and P_3 becomes $[o,\infty)^3$.

For any t in R^3, tP_3^{-1} is easily seen to be $\{x \in R^3 \mid x_1 \leqslant t_1, x_2 \leqslant t_2, x_3+(t_2-x_2)x_1 \leqslant t_3\}$, where x_1,x_2,x_3 and t_1,t_2,t_3 are the components of x and t respectively; it is convenient to denote tP_3^{-1} by S_{t_1,t_2,t_3}. So the set of units of the groupoid construction of section 2 is $U=w^*\text{-clos}\left\{\chi_{S_{t_1,t_2,t_3}} \mid t_1,t_2,t_3\in[0,\infty)\right\}$.

We also make the following notations ($t_1,t_2,t_3 \in R$ are arbitrary)

$$S_{t_1,t_2,\cdot} = \left\{x\in R^3 \mid x_1\leq t_1,\ x_2\leq t_2\right\};$$

$$S_{\cdot,t_2,t_3} = \left\{x\in R^3 \mid x_2\leq t_2,\ x_3+(t_2-x_2)x_1\leq t_3\right\};$$

$$S_{t_1,\cdot,\cdot} = \left\{x\in R^3 \mid x_1\leq t_1\right\};$$

$$S_{\cdot,t_2,\cdot} = \left\{x\in R^3 \mid x_2\leq t_2\right\}.$$

<u>Proposition 4.1</u> U is made of six orbits, which are:

$$U_{1,2,3} = \left\{\chi_{S_{t_1,t_2,t_3}} \mid t_1,t_2,t_3\in[0,\infty)\right\};$$

$$U_{1,2} = \left\{\chi_{S_{t_1,t_2,\cdot}} \mid t_1,t_2\in[0,\infty)\right\};$$

$$U_{2,3} = \left\{\chi_{S_{\cdot,t_2,t_3}} \mid t_2,t_3\in[0,\infty)\right\};$$

$$U_1 = \left\{\chi_{S_{t_1,\cdot,\cdot}} \mid t_1\in[0,\infty)\right\};$$

$$U_2 = \left\{\chi_{S_{\cdot,t_2,\cdot}} \mid t_2\in[0.\infty)\right\};$$

$$U_o = \left\{\chi_{R^3}\right\} = \{1\}.$$

Moreover, if we place these six orbits on four levels as in the table 1, then the closure of each one consists of itself and the orbits situated (strictly) below it.

Level o	$U_{1,2,3}$
Level 1	$U_{1,2}$; $U_{2,3}$
Level 2	U_1 ; U_2
Level 3	U_o

<u>Table 1</u>

<u>Proof</u> It is easier to compute $U'=w^*\text{-clos}\left\{\chi_{P_3 t^{-1}} \mid t\in P_3\right\}$; U and U' are related by the equality $U=\left\{\chi_{A^{-1}} \mid \chi_A\in U'\right\}$. For

closed convex set containing P_3.

Let $(t^{(k)})_{k=1}$ be a sequence of P_3 such that

$\chi_{P_3(t^{(k)})^{-1}} \xrightarrow[k\to\infty]{w^*} \chi_A$ for a solid set A.(In fact A must be closed and convex and must contain P_3, as we saw in the proof of proposition 2.1.) We write, for any k, $t^{(k)} = (t_1^{(k)}, t_2^{(k)}, t_3^{(k)})$. Passing to a subsequence, we shall suppose that $(t_j^{(k)})_{k=1}^{\infty}$ converges to a $c_j \in [0,\infty]$ for any $j\in\{1,2,3\}$. If $c_2=\infty$, we shall assume, in addition, that $t_2^{(k)} \neq 0$ for any k and that there exists $c=\lim_{k\to\infty} t_3^{(k)}/t_2^{(k)} \in [0,\infty]$.

There are eight possible cases, given by the following tree

- $c_2 < \infty$
  - $c_1 < \infty$
    - $c_3 < \infty$ case 1
    - $c_3 = \infty$ case 2
  - $c_1 = \infty$
    - $c_3 < \infty$ case 3
    - $c_3 = \infty$ case 4
- $c_2 = \infty$
  - $c_1 < \infty$
    - $c < \infty$ case 5
    - $c = \infty$ case 6
  - $c_1 = \infty$
    - $c < \infty$ case 7
    - $c = \infty$ case 8

In each case, the limit set A can be written explicitely. Let us take for instance the case 6. We claim that we have $A=\{x\in R^3 \mid x_1 \geq -c_1\}$. Indeed, let us denote the last set by A'. If $x\in \overset{\circ}{A'}$, that is if $x_1 > -c_1$, then $x\in P_3(t^{(k)})^{-1}$ for all sufficiently large k because: $-t_1^{(k)} \xrightarrow[k]{} -c_1 < x_1$, $-t_2^{(k)} \xrightarrow[k]{} -\infty < x_2$ and $x_3+x_1t_2^{(k)} \geq -t_3^{(k)} \Leftrightarrow x_3/t_2^{(k)}+x_1 \geq -t_3^{(k)}/t_2^{(k)}$ is valid for sufficiently large k, since the left part of the inequality tends to x_1, while its right part tends to $-\infty$. If $x\notin A'$, then $x\notin P_3(t^{(k)})^{-1}$ for all sufficiently large k, because $-t_1^{(k)} \xrightarrow[k]{} -c_1 > x_1$. Taking into account that $\partial A'$ has null Lebesgue measure, we can apply the dominated convergence theorem to see that

$\chi_{P_3(t^{(k)})^{-1}} \xrightarrow[k\to\infty]{w^*} \chi_{A'}$.

The rest of the proof is a mechanical computation. The reader may convince himself that if one will write the form of A in the other seven cases and operate the inversion $A \to A^{-1}$, then he will obtain the results stated in the proposition.

At this moment we have at our disposal a general machinery, used by P.Muhly and J.Renault in [1] (theorems 4.7 and 6.6) for some special euclidian cases, which provides a composition series for a groupoid C^*-algebra. This machinery starts with a locally compact groupoid with Haar system, $\mathcal{G}$, and with a partition of its set U of units into invariant subsets. The partition is written with double-index, $U=\bigcup_{\ell=0}^{n}(\bigcup_{j=1}^{m_\ell} U_{\ell,j})$, because its members are placed on n+1 levels (the index ℓ comes from "level"), and the following condition concerning closures is satisfied: for any ℓ and j we have clos $U_{\ell,j} \subseteq U_{\ell,j} \cup (\bigcup_{\ell'=\ell+1}^{n} (\bigcup_{j'=1}^{m_{\ell'}} U_{\ell',j'}))$. This condition is obviously weaker than the one observed for P_3 in proposition 4.1.

Under these hypothesis, we see that each $U_{\ell,j}$ is locally closed, being the difference of the two closed sets

$U_{\ell,j} \cup (\bigcup_{\ell'=\ell+1}^{n} (\bigcup_{j'=1}^{m_{\ell'}} U_{\ell',j'}))$ and $\bigcup_{\ell'=\ell+1}^{n} (\bigcup_{j'=1}^{m_{\ell'}} U_{\ell',j'})$; hence $\mathcal{G}|U_{\ell,j}$

is a locally compact groupoid, endowed with an inherited Haar system (because $U_{\ell,j}$ is invariant). The result used by P.Muhly and J.Renault is the following:

<u>Proposition 4.2</u> One can find the closed two-sided ideals $I_0 \subseteq I_1 \subseteq \ldots \subseteq I_n = C^*(\mathcal{G})$ of $C^*(\mathcal{G})$ such that I_0 is isomorphic to $\bigoplus_{j=1}^{m_0} C^*(\mathcal{G}|U_{0,j})$, and for any $1 \leq \ell \leq n$, $I_\ell/I_{\ell-1}$ is isomorphic to $\bigoplus_{j=1}^{m_\ell} C^*(\mathcal{G}|U_{\ell,j})$.

The proof of proposition 4.2 is based on the fact that open invariant sets of units give rise to ideals of $C^*(\mathcal{G})$ and other related results (see for instance the proof of theorem 4.7 of [1]).

As a consequence of the propositions 4.1 and 4.2, we see that $\mathcal{W}(P_3)$ has a sequence of ideals $I_o \subseteq I_1 \subseteq I_2 \subseteq I_3 = \mathcal{W}(P_3)$, such that:

$$I_o \approx C^*(\mathcal{G}|U_{1,2,3});$$
$$I_1/I_o \approx C^*(\mathcal{G}|U_{1,2}) \oplus C^*(\mathcal{G}|U_{2,3});$$
$$I_2/I_1 \approx C^*(\mathcal{G}|U_1) \oplus C^*(\mathcal{G}|U_2);$$
$$I_3/I_2 \approx C^*(\mathcal{G}|U_o).$$

What remains to be done is the description of the C^*-algebras of the six reduced groupoids. Reviewing the discussion of 3.1 we see that $C^*(\mathcal{G}|U_{1,2,3}) \approx \mathcal{K}$, and that in fact $I_o = \mathcal{K}(L^2(\mu_3|P_3))$ with μ_3 the Lebesgue measure. At the other extreme, we have $C^*(\mathcal{G}|U_o) \approx C^*(H_3)$, because U_o has a single element, with isotropy group H_3, hence $\mathcal{G}|U_o$ is in fact H_3. In what concernes the other four groupoids, each of them can be seen to be isomorphic (as a locally compact groupoid with Haar system) to the product of a group and a trivial equivalence relation on a certain set. For example, if $\mathcal{R}$ denotes the trivial equivalence relation on $[o,\infty)^2$ considered with the Lebesgue measure μ_2, then

$$(a, (t_2,t_3), (s_2,s_3)) \longrightarrow ((a, t_2-s_2, t_3-s_3-as_2), S_{\circ,s_2,s_3})$$

establishes an isomorphism between $R \times \mathcal{R}$ and $\mathcal{G}|U_{2,3}$, which implies $C^*(\mathcal{G}|U_{2,3}) \approx C_o(R) \otimes \mathcal{K}(L^2(\mu_2|[o,\infty)^2))$. In a similar way we find that $C^*(\mathcal{G}|U_{1,2})$ is isomorphic to $C_o(R) \otimes \mathcal{K}$, too, and that $C^*(\mathcal{G}|U_1)$and $C^*(\mathcal{G}|U_2)$ are isomorphic to $C_o(R^2) \otimes \mathcal{K}$. Let us make the remark that if we use theorem 3.1 of [2], then we are exempt from establishing the groupoid isomorphisms, and we only need to compute the isotropy groups of the groupoids involved.

Finally, we have come to the following result:

Proposition 4.3 $\mathcal{W}(P_3)$ is of type I, and has a composition series of length 3, such that:

a) the first ideal is $\mathcal{K}$;

b) the last quotient is isomorphic to $C^*(H_3)$;

c) the intermediate quotients are direct sums of terms of the form $C^*(M)\otimes\mathcal{K}$, with M a subgroup of H_3 ($M=R$ or $M=R^2$).

5. The C^*-algebra $\mathcal{W}(P_4)$ Computations similar to those of the preceding section can be made in the case of $P_4 \subseteq H_4$. The main difference is that, unlike the result of proposition 4.1, U has now an infinite set of orbits. However, we can gather the orbits in a natural way into a finite number of invariant sets, such that the hypothesis of proposition 4.2 are fulfiled. After doing this, we still have the problem of describing the C^*-algebras of those reduced groupoids which are not transitive. Fortunately, all of them are seen to be isomorphic to the product of a group and a principal groupoid of the type described in the next proposition.

Proposition 5.1 Let B and Y be second countable locally compact spaces. We consider the equivalence relation defined by $(b,x,y)\sim(b',x',y')\Leftrightarrow b=b'$ on $B\times Y\times Y$, which gives a locally compact groupoid $\mathcal{H}$. If μ is a positive Radon measure on Y having supp $\mu=Y$, and $\gamma:B\to(0,\infty)$ is a continuous function, then the family of Radon measures $\Lambda=(\lambda^{(b,y)})_{(b,y)\in B\times Y}$ defined by:

$$\int_{\mathcal{H}^{(b,y)}} f(b,y,x)\, d\lambda^{(b,y)}(b,y,x) = \gamma(b)\int_Y f(b,y,x)\, d\mu(x),$$

for any f in $C_c(\mathcal{H}^{(b,y)})$, is a left Haar system on $\mathcal{H}$, and the reduced C^*-algebra associated to $\mathcal{H}$ and Λ is isomorphic to $C_0(B) \otimes \mathcal{K}(L^2(\mu))$.

<u>Remark</u> We don't need to study the amenability of $(\mathcal{H},\Lambda)$, because, using the proposition, we shall obtain the reduced C^*-algebras of some groupoids which are known to be amenable (by propositions 3.7 and 3.9 of chapter II of [4]).

<u>Proof of proposition 5.1</u> We recall that the set of units of $\mathcal{H}$ is $B \times Y$; its set of arrows is $B \times Y \times Y$, the domain and range of (b,y,x) being (b,x) and (b,y) respectively. The multiplication on $\mathcal{H}$ is given by $(b,z,y)(b,y,x) = (b,z,x)$.

We consider and fix a positive Radon measure β on B, such that $\operatorname{supp}\beta = B$, and an element $x_0 \in X$. We have $\operatorname{supp} \beta \times \delta_{x_0} = B \times \{x_0\}$, hence the invariant support of $\beta \times \delta_{x_0}$ is $B \times Y$. This implies, by proposition 2.17 of [1], that the induced representation $\operatorname{Ind} \beta \times \delta_{x_0}$ is isometric on $C^*_{red}(\mathcal{H},\Lambda)$. The space of this representation is $L^2(\nu^{-1})$, where the Radon measure ν^{-1} on $B \times Y \times Y$ is given by:

$$\int_{B\times Y\times Y} f \, d\nu^{-1} =$$

$$= \int_{B\times Y} \left(\int_{\mathcal{H}_{(b,x)}} f(b,y,x) \, d\lambda_{(b,x)}(b,y,x) \right) d(\beta \times \delta_{x_0})(b,x) =$$

$$= \int_{B\times Y} \gamma(b) f(b,y,x_0) \, d(\beta\times\mu)(b,y), \quad \forall f \in C_c(B\times Y\times Y).$$

It is easy to see that the mapping $f \to \gamma^{1/2}(\cdot)f(\cdot,\cdot,x_0)$ from $C_c(B\times Y\times Y)$ onto $C_c(B\times Y)$ extends to a unitary $T: L^2(\nu^{-1}) \longrightarrow L^2(\beta\times\mu)$. We conjugate $\operatorname{Ind} \beta\times\delta_{x_0}$ with T and obtain an which is found to act by the formula:

$$\left[(\Pi f)\xi\right](b,y) = \gamma(b)\int_Y f(b,y,z)\,\xi(b,z)\,d\mu(z),$$

for any $f\in C_c(B\times Y\times Y)$, $\xi\in C_c(B\times Y)$, $b\in B$, $y\in Y$. If we make the identification $L^2(\beta\times\mu)\approx L^2(\beta)\overline{\otimes}L^2(\mu)$, the above formula shows that when $f=g\otimes h_1\otimes\overline{h_2}$ with $g\in C_c(B)$, $h_1,h_2\in C_c(Y)$, Πf is $M_{\gamma g}\otimes(\langle\cdot|h_2\rangle h_1)$. ($M_{\gamma g}$ is the multiplication operator with γg on $L^2(\beta)$ and $\langle\cdot|h_2\rangle h_1$ is the corresponding rank one operator on $L^2(\mu)$.) The elements of the form $g\otimes h_1\otimes\overline{h}_2$ generate $C^*(\mathcal{H},\Lambda)$ as a C^*-algebra, while the C^*-subalgebra of $\mathcal{L}(L^2(\beta)\overline{\otimes}L^2(\mu))$ generated by the operators $M_{\gamma g}\otimes(\langle\cdot|h_2\rangle h_1)$ is $\{M_g\,|\,g\in C_0(B)\}\otimes\mathcal{K}(L^2(\mu))$, isomorphic to $C_0(B)\otimes\mathcal{K}(L^2(\mu))$.

For the sake of completeness we shall present a table comprising the invariant sets of units which appear and the manner in which they are arranged on levels. As in the case of H_3, it is easier to compute $U'=w^*\text{-clos}\left\{\chi_{P_4t^{-1}}\,\middle|\,t\in P_4\right\}$ instead of $U=w^*\text{-clos}\left\{\chi_{tP_4^{-1}}\,\middle|\,t\in P_4\right\}$. But this time, because the inversion operation is more arduous, we prefer to replace the groupoid $\mathcal{G}$ of section 2 with the groupoid $\mathcal{G}'$ whose units are U' and whose arrows are given by right translations with elements of G; that is, if $\chi_B\in U'$ and $s\in G$ are such that $\chi_{Bs^{-1}}$ is still in U' (which is equivalent to $s\in B$), then we have an arrow (s,B) from χ_B to $\chi_{Bs^{-1}}$. Multiplication, topology and Haar system are defined on $\mathcal{G}'$ by symmetry with the case of $\mathcal{G}$. It is obvious that $\mathcal{G}$ and $\mathcal{G}'$ are isomorphic, so that $C^*(\mathcal{G}')\approx\mathcal{W}(P_4)$.

The identification of H_4 with R^6 is made by writing

$(x_1, x_2, \ldots, x_6)$ instead of $\begin{pmatrix} 1 & x_1 & x_4 & x_6 \\ 0 & 1 & x_2 & x_5 \\ 0 & 0 & 1 & x_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. If we write the formula of the multiplication which is obtained on R^6 and take into account that $x \in P_4 t^{-1} \Leftrightarrow xt \in P_4$, we find that, for any t in R^6, $P_4 t^{-1}$ can be expressed as $R_{t_1} \cap S_{t_2,t_4} \cap T_{t_3,t_5,t_6}$, with t_1, $t_2, \ldots, t_6$ the components of t, and where we use the following notations ($a,b,c \in R$ are arbitrary):

$$R_a = \{x \in R^6 \mid x_1 \geqslant -a\} ;$$
$$S_a = \{x \in R^6 \mid x_2 \geqslant -a\} ;$$
$$T_a = \{x \in R^6 \mid x_3 \geqslant -a\} ;$$
$$S_{a,b} = \{x \in R^6 \mid x_2 \geqslant -a,\ x_4 + ax_1 \geqslant -b\} ;$$
$$\Sigma_{a,b} = \{x \in R^6 \mid x_4 + ax_1 \geqslant -b\} ;$$
$$T_{a,b} = \{x \in R^6 \mid x_3 \geqslant -a,\ x_5 + ax_2 \geqslant -b\} ;$$
$$T_{a,b,c} = \{x \in R^6 \mid x_3 \geqslant -a,\ x_5 + ax_2 \geqslant -b,\ x_6 + bx_1 + ax_4 \geqslant -c\}.$$

With these notations, the family of sets whose characteristic functions appear in U' is listed in table 2. U' is divided into 28 invariant sets, placed on seven levels; unless otherwise stated, the coefficients which appear are arbitrary in $[0, \infty)$.

Table 2

Level 0	1. $R_{c_1} \cap S_{c_2,c_4} \cap T_{c_3,c_5,c_6}$
Level 1	2. $R_{c_1} \cap S_{c_2,c_4} \cap T_{c_3,c_5}$ 3. $R_{c_1} \cap S_{c_2} \cap T_{c_3,c_5,c_6}$ 4. $S_{c_2,c_4} \cap T_{c_3,c_5,c_6}$ 5. $R_{c_1} \cap S_{c_2,c_4} \cap \Sigma_{d_2,d_4}$, with $d_2 > c_2$ and $d_4 < c_4 + c_1(d_2 - c_2)$

Level 2	6. $R_{c_1} \cap S_{c_2} \cap T_{c_3,c_5}$ 7. $R_{c_1} \cap S_{c_2,c_4} \cap T_{c_3}$ 8. $R_{c_1} \cap T_{c_3,c_5,c_6}$ 9. $S_{c_2,c_4} \cap T_{c_3,c_5}$ 1o. $S_{c_2} \cap T_{c_3,c_5,c_6}$ 11. $R_{c_1} \cap S_{c_2} \cap \Sigma_{d_2,d_4}$, with $d_2 > c_2$ 12. $S_{c_2,c_4} \cap \Sigma_{d_2,d_4}$, with $d_2 > c_2$
Level 3	13. $R_{c_1} \cap S_{c_2} \cap T_{c_3}$ 14. $R_{c_1} \cap S_{c_2,c_4}$ 15. $R_{c_1} \cap T_{c_3,c_5}$ 16. $S_{c_2} \cap T_{c_3,c_5}$ 17. $S_{c_2,c_4} \cap T_{c_3}$ 18. T_{c_3,c_5,c_6} 19. $S_{c_2} \cap \Sigma_{d_2,d_4}$, with $d_2 > c_2$
Level 4	2o. $R_{c_1} \cap S_{c_2}$ 21. $R_{c_1} \cap T_{c_3}$ 22. $S_{c_2} \cap T_{c_3}$ 23. S_{c_2,c_4} 24. T_{c_3,c_5}
Level 5	25. R_{c_1} 26. S_{c_2} 27. T_{c_3}
Level 6	28. R^6

The reduced groupoids corresponding to the lines 5,11,12 and 19 of the table 2 are not transitive; they are handled with the aid of proposition 5.1. For example, in the case of line 5, we put $B=(0,\infty)^2$, $Y=\{x\in[0,\infty)^3 \mid x_1+x_3\geq 1\}$, μ = Lebesgue measure on Y, $\gamma(b_1,b_2)=b_2^2/b_1$ on B; we construct $\mathcal{H}$ and Λ as in proposition 5.1, and we make the product $R^3\times\mathcal{H}$. A topological isomorphism, which preserves the canonical Haar systems, from the reduced groupoid corresponding to the line 5 onto $R^3\times\mathcal{H}$ can be defined by the formula $x=(t,\ R_{c_1}\cap S_{c_2,c_4}\cap\Sigma_{d_2,d_4})\longrightarrow ((a_1(x),a_2(x),a_3(x))\in R^3,\ b(x)\in B,\ y_1(x)\in Y,\ y_2(x)\in Y)$, where:

$$a_1(x) = t_3;$$
$$a_2(x) = t_5-(c_2+t_2)t_3 ;$$
$$a_3(x) = t_6-t_5(t_1+c_1)-t_3(t_4+c_4)+t_3(c_1c_2+c_1t_2+t_1t_2) ;$$
$$b(x) = (d_2-c_2,c_4-d_4+c_1(d_2-c_2)) ;$$
$$y_1(x) = \left(\frac{(d_2-c_2)(t_1+c_1)}{c_4-d_4+c_1(d_2-c_2)},\ t_2+c_2,\ \frac{t_4+c_4+t_1c_2}{c_4-d_4+c_1(d_2-c_2)}\right) ;$$
$$y_2(x) = \left(\frac{(d_2-c_2)c_1}{c_4-d_4+c_1(d_2-c_2)},\ c_2,\ \frac{c_4}{c_4-d_4+c_1(d_2-c_2)}\right) .$$

The reduced groupoids corresponding to the rest of the lines are transitive, hence, by theorem 3.1 of [2], their C^*-algebras are determined by their isotropy groups.

We finally obtain:

<u>Proposition 5.2</u> $\mathcal{W}(P_4)$ is of type I, and has a composition series $\mathcal{K}=I_0\subseteq I_1\subseteq \dots \subseteq I_6=\mathcal{W}(P_4)$ such that:

$$I_1/I_0 \approx (\bigoplus_{j=1}^{3} C_0(R)\otimes\mathcal{K})\oplus(C_0(R^3)\otimes C_0((0,\infty)^2)\otimes\mathcal{K}) ;$$
$$I_2/I_1 \approx (\bigoplus_{j=1}^{5} C_0(R^2)\otimes\mathcal{K})\oplus(\bigoplus_{j=1}^{2} (C_0(R^3)\otimes C_0((0,\infty))\otimes\mathcal{K}));$$
$$I_3/I_2 \approx (\bigoplus_{j=1}^{4} C_0(R^3)\otimes\mathcal{K})\oplus(\bigoplus_{j=1}^{2} C^*(H_3)\otimes\mathcal{K})\oplus$$
$$\oplus(C^*(R\times H_3)\otimes C_0((0,\infty))\otimes\mathcal{K}) ;$$

$I_4/I_3 \approx (\bigoplus_{j=1}^{4} C^*(R \times H_3) \otimes \mathcal{K}) \oplus (C_0(R^4) \otimes \mathcal{K})$;

$I_5/I_4 \approx \bigoplus_{j=1}^{3} C^*(M_j) \otimes \mathcal{K}$, where for any $1 \leq j \leq 3$ M_j is the subgroup of H_4 obtained by forcing the j-th component to be zero;

$I_6/I_5 \approx C^*(H_4)$.

Bibliography

[1] P.Muhly, J.Renault, C^*-algebras of multivariable Wiener-Hopf operators, Transactions A.M.S. 274 (1982), pg. 1-44.

[2] P.Muhly, J.Renault, D.Williams, Equivalence and isomorphism for groupoid C^*-algebras, Journal of Operator Theory 17 (1987), pg.3-22.

[3] A.Nica, Some remarks on the groupoid approach to the Wiener-Hopf operators, Journal of Operator Theory 18 (1987), pg. 163-198.

[4] J.Renault, A groupoid approach to C^*-algebras, Lecture Notes in Mathematics, vol. 793, Springer Verlag, New York, 1980.