A SINGULAR PERTURBATION PROBLEM FOR THE HEAT EQUATION IN TWO PHASES MEDIA

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Abstract. The heat propagation equation in a medium containing two phases, one with a much larger thermal conductivity than the other is considered. It is proved that the solution converges to the solution of the heat propagation problem in a body in contact with a well stirred fluid.

1. INTRODUCTION. The problem of heat propagation in a body in contact with a well stirred fluid was considered by many authors (see for instance Carlsaw and Jaeger [1] and the references given there) and in several particular cases the solution is given. As far as we know a general existence result for this problem is not proved till now.

In [1] it is asserted that the solution of this problem yields a good approximation for the solution of the heat equation in a two phases body, one with much larger conductivity then the other. The purpose of this paper is to prove this assertion. In order to do that we have to consider a singular perturbation problem for the heat equation in two phases body which (as far as we know) cannot be included in the framework of general abstract results for singular perturbation evolution problems (see for instance Halanay [2], Friedman [3], Kurtz [4], Lions [5], Krein and Hazan [6]).

In section 2 the problems are stated and in section 3 same notations and preliminaries are given. In section 4 we prove an existence and uniqueness result for the heat propagation problem in a body in contact with a well stirred fluid. In the last section we prove the convergences results.

2. PROBLEM STATEMENT

Let us consider $\Omega \subset R^N$ an open bounded set with a smooth (say C^1) boundary . We shall suppose that $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$; $\Omega_1 \cap \Omega_2 = \emptyset$, where Ω_i a open subsets of Ω with smooth boundary. We denote by $\Gamma_o = \partial\Omega_1 \cap \partial\Omega_2 \neq \emptyset$, $\Gamma_1 = \partial\Omega_1 \cap \Gamma$ and $\Gamma_2 = \partial\Omega_2 \cap \Gamma$. We shall consider the heat equation in the case of a two phases body i.e. we shall suppose that the conductivity is 1 in Ω_1 and $1/\varepsilon > 0$ in Ω_2. Hence we have to determine the temperature field $u : R_+ \times \Omega \to R$ such that:

(2.1) $$\frac{\partial u_\varepsilon}{\partial t} = \operatorname{div} q_\varepsilon \qquad \text{in} \quad \Omega$$

(2.2) $$q_\varepsilon = \nabla u_\varepsilon \quad \text{in } \Omega_1, \quad q_\varepsilon = \frac{1}{\varepsilon} \nabla u_\varepsilon \quad \text{in } \Omega_2$$

(2.3) $$q_\varepsilon \cdot n_{|\Gamma} = 0 \qquad \text{for } t > 0$$

(2.4) $$u(0) = u_o.$$

If ε is small then (2.1-4) describe the heat propagation in an isolated body composed of two phases one of them having a much larger conductivity then the other. Carlsaw and Jaeger suggested ([1] p.22) that the solution of (2.1-4) converges for $\varepsilon \to 0$ to the solution of the heat propagation problem in a body in contact with a well stirred fluid.

This problem consists in finding the temperature field $u : R_+ \times \Omega_1 \to R$ in the body and the temperature $y : R_+ \to R$ of the

(2.5) $$\frac{\partial u}{\partial t} = \Delta u \qquad \text{in } \Omega_1 ,$$

(2.6) $$\frac{dy}{dt} = -\alpha^{-1} \int_{\Omega_1} \Delta u ,$$

(2.7) $$u_{|\Gamma_0} = y , \quad \frac{\partial u}{\partial n}\Big|_{\Gamma_1} = 0 , \qquad \text{for } t>0$$

(2.8) $$u(0) = v_0 , \qquad \text{in } \Omega_1, \ y(0) = y_0$$

where α is the measure of Ω_2.

The equations (2.5-8) are also governing the diffusion of a dyestuff in a yarn situated in a well stirred dyeing bath (cf. Peters [7]). In this case u is the dyestuff concentration fiel in the yarn and y is the dyestuff concentration in the dyeing bath

3. NOTATIONS AND PRELIMINARIES

Let us denote by $L=L^2(\Omega)$, $L_1=L^2(\Omega_1)$, $L_2=L^2(\Omega_2)$, $\mathcal{L}=[L^2(\Omega)]^N$, $\mathcal{L}_1=[L^2(\Omega_1)]^N$, $\mathcal{L}_2=[L^2(\Omega_2)]^N$, $H=H^1(\Omega)$, $H_0=H^{1/2}(\Gamma_0)$ with the inner products and norms denoted by: $((\ ,\ );\|\ \|)$, $((\ ,\ )_1;\|\ \|_1)$, $((\ ,\ )_2, \|\ \|_2)$, $(((\ ,\ ));|||\ |||)$, $(((\ ,\ ))_1, |||\ |||_1)$, $(((\ ,\ ))_2, |||\ |||_2)$, $((\ ,\ )_H, \|\ \|_H)$, $((\ ,\ )_0, \|\ \|_0)$, respectively.

We denote by $\gamma_i : H^1(\Omega_i) \to H^{1/2}(\partial\Omega_i)$ $i=1,2$ the trace maps. We shall denote by $u_{|\Gamma_0^+}$ and by $u_{|\Gamma_0^-}$ the restriction of $\gamma_1(u)$ and $\gamma_2(u)$ to Γ_0. We notice that $u \in H^1(\Omega)$ iff $u \in H^1(\Omega_1) \cap H^1(\Omega_2)$ and $u_{|\Gamma_0^+} = u_{|\Gamma_0^-}$.

We denote by $\gamma_{\nu i} : H(\mathrm{div}, \Omega_i) \to H^{-1/2}(\partial\Omega_i)$ the normal trace map given by

(3.1) $$\langle \gamma_{\nu i}(v), \gamma_i(u) \rangle = (\mathrm{div}\ v, u)_i + ((v, \nabla u))_i ,$$

for all $u \in H^1(\Omega_i)$ $i=1,2$. We denote by $v.\nu_{|\Gamma'}$ (where Γ' is a subset of $\partial\Omega_i$) the restriction of $\gamma_{\nu i}$ on the set of all $v \in H^{1/2}(\partial\Omega_i)$ with

$v=0$ on $\partial\Omega_i / \Gamma^1$. Let us denote by $v\cdot\nu_{|\Gamma_0^+}$, $v\cdot\nu_{|\Gamma_0^-}$ the restriction of $\gamma_{\nu 1}(v)$ and $\gamma_{\nu 2}(v)$ on Γ_0.

We notice that $v\in H(div,\Omega)$ iff $v\in H(div,\Omega_1)\cap H(div,\Omega_2)$ and $v\cdot\nu_{|\Gamma_0^+}+v\cdot\nu_{|\Gamma_0^-}=0$.

If we denote by

$$(3.2)\quad D(A_\varepsilon)=\left\{u\in H^1(\Omega)/q_\varepsilon(u)\in H(div,\Omega),\ q_\varepsilon(u)\cdot\nu_{|\Gamma}=0\right\},$$

where $q_\varepsilon : H^1(\Omega)\to \mathcal{L}$ is given by

$$(3.3)\qquad q_\varepsilon(v)=\begin{cases}\nabla v & \text{in } \Omega_1\\ \frac{1}{\varepsilon}\nabla v & \text{in } \Omega_2\end{cases}$$

and if we put $A_\varepsilon : D(A_\varepsilon)\subset L\to L$ given by

$$(3.4)\quad A_\varepsilon(u)=div\ q_\varepsilon(u) \qquad \text{for all } u\in D(A_\varepsilon)$$

then problem (2.1-4) is equivalent with

$$(3.5)\qquad \frac{du_\varepsilon}{dt}=A_\varepsilon u_\varepsilon, \qquad u_\varepsilon(0)=u_0.$$

It is well known that $A_\varepsilon=A_\varepsilon^*\leq 0$ hence A_ε is the infinitesimal generator of an analitic semigroup T (t) and for all $u_0\in L$ the problem (3.5) has an unique solution $u\in C^0(R_+,L)\cap C^\infty(0,+\infty),L)$.

From the results remainded at the begining of the section one can see that u_ε is the solution of (2.1-4) or (3.5) iff

$$(3.6)\qquad \frac{\partial u_\varepsilon}{\partial t}=\Delta u_\varepsilon \ \text{ in } \Omega_1,\quad \varepsilon\frac{\partial u_\varepsilon}{\partial t}=\Delta u_\varepsilon \quad \text{in } \Omega_2$$

$$(3.7)\qquad u_{\varepsilon|\Gamma_0^+}=u_{\varepsilon|\Gamma_0^-},\quad \varepsilon\frac{\partial u_\varepsilon}{\partial\nu}\Big|_{\Gamma_0^+}+\frac{\partial u_\varepsilon}{\partial\nu}\Big|_{\Gamma_0^-}=0,\quad \frac{\partial u_\varepsilon}{\partial\nu}\Big|_{\Gamma}=0,$$

for $t>0$ and

(3.8) $$u_\varepsilon(0)=u_o.$$

4. THE REDUCED PROBLEM

In this section we shall study problem (2.5-7) (which will be called the reduced problem) by writting it as a Cauchy problem in a Hilbert space. Let $X=L_1 \times R$ endowed with the following scalar product:

(4.1) $$\langle [v_1,z_1], [v_2,z_2]\rangle_X=(v_1,v_2)_1 +\alpha z_1z_2 \quad \text{for all } [w_i,z_i]\in X$$

Let D(A) be the following subspace of X:

(4.2) $$D(A)=\left\{[u,y]/u\in H^2(\Omega_1),\ u_{|\Gamma_o^+}=y,\ \frac{\partial u}{\partial \nu}\Big|_{\Gamma_1}=0\right\},$$

and $A:D(A)\subset X \to X$ defined by

(4.3) $$A[u,y]=[\Delta u,\ -\alpha^{-1}\int_{\Omega_1}\Delta u].$$

<u>Lemma 4.1.</u> The operator A is the infinitesimal generator of an analytic semigroup T(t) of contractions acting on X.

<u>Proof.</u> The conditions D(A) is dense in X and A is a closed operator are easely verified and a simple calculation yields $\langle A[u,y], v,z\rangle_X=-((\nabla u,\nabla v))_1$ for all $[u,y], [v,z]\in D(A)$ hence A is a symmetric and negative operator. In order to prove that A is a self-adjoint operator it is enough to check the surjectivity of $\lambda I - A$ for all $\lambda>0$. Indeed if $[f,\delta]\in X$ then we can see that $(\lambda I-A)[u,y]= [f,\delta]$ iff

(4.4) $$w=R(\lambda,C)(g+\alpha^{-1}\int_{\Omega_1} Cw)\,;$$

where $g=f-\delta$, $w=u-y$, $D(C)=\left\{u\in H^2(\Omega_1)/u_{|\Gamma_0^+}=0,\ \frac{\partial u}{\partial\nu}\Big|_{\Gamma_1}=0\right\}$, $C:D(C)\subset L_1\to L_1$, $Cu=\Delta u$ and $R(\lambda,C)=(\lambda I-C)^{-1}$ is the resolvent of C. The function w is the solution of (4.4) iff $w=R(\lambda,C)(g+z_0)$ where z_0 is the fixedpoint of $S:R\to R$ given by:

(4.5) $$S(z)=\alpha^{-1}\int_{\Omega_1} CR(\lambda,C)(g+z)\,.$$

If we make some calculations we get that

$$S(z)=z\left(\alpha^{-1}\lambda\int_{\Omega_1} R(\lambda,C)(1)-1\right)+G$$

where $G=\alpha^{-1}\int_{\Omega_1} R(\lambda,C)g-\alpha^{-1}\int_{\Omega} g$. If we have in mind that $\|R(\lambda,C)v\|_2\le\frac{\|v\|_2}{\lambda+\omega_0}$ with $\omega_0>0$ then we deduce that $\alpha^{-1}\int_{\Omega_1}\lambda R(\lambda,C)(1)<1$. hence the slope of S is negative and there exists a unique fix point of S.

One can easely notice that problem (2.5-7) is equivalent with the following Cauchy problem in X

(4.5) $$\frac{d}{dt}[u,y]=A[u,y]\qquad [u,y](0)=[v_0,y_0]\,.$$

A straightforward concequence of Lemma 4.1 is:

Theorem 4.1. For all $v_0\in L_1$, $y_0\in R$ there exists a unique couple of functions $[u,y]$ solution of (2.5-7) such that $u\in C^0(R_+,L_1)\cap C^\infty((0,+\ ),L^1)$, $y\in C^0(R_+,R)\cap C^\infty((0,+\infty),R)$.

5. CONVERGENCE RESULTS

The main result of this section is the following

Theorem 5.1. If $v_o = u_{o|\Omega_1} \in L_1$, and $y_o = u_o(x) \in R$ for all $x \in \Omega_2$ then for all $T>0$ we have:

(5.1) $u_\varepsilon \to u$ in $C^o([0,T], L_1)$

(5.2) $u_\varepsilon \to y$ in $C^o([0,T], L_2)$

when $\varepsilon \to 0$

Theorem 5.2. If $u_o \in L$, $u_{o|\Omega_1} = v_o$, $y_o = \alpha^{-1}\int_{\Omega_2} u_o$ then for all $T>0$

(5.3) $u_\varepsilon(t) \to u(t)$ weakly in L_1 uniformly with respect to $t \in [0,T]$,

(5.4) $u_\varepsilon \to y$ strongly in $L^2(0,T, L_2)$.

In order to prove these results several lemma are necessary.

Lemma 5.1. For all $T>0$ and $u_o \in \mathcal{D}(A_\varepsilon)$ we have

(5.5) $\int_0^T |||\nabla u_\varepsilon(s)|||_2^2 ds \leq \varepsilon/2 \, \|u_o\|^2$

(5.6) $\int_0^T |||\nabla u_\varepsilon(s)|||^2 ds \leq 1/2 \, \|u_o\|^2$

Proof. If we multiply (2.1) with u_ε and integrate the result over Ω we get $(u_\varepsilon, \frac{\partial u_\varepsilon}{\partial t}) = -|||\nabla u_\varepsilon|||_1^2 - 1/\varepsilon |||\nabla u_\varepsilon|||_2^2$.

If we integrate the above relation from 0 to T we get $1/2 \, \|u_\varepsilon(T)\|^2 + \int_0^T |||\nabla u_\varepsilon(s)|||_1^2 ds + \frac{1}{\varepsilon}\int_0^T |||\nabla u_\varepsilon(s)|||_2^2 ds \leq 1/2 \, \|u_o\|^2$ and (5.5-6) hold.

Let us denote in the following by $y_\varepsilon(t)$ the evarage of u_ε on Ω_2 i.e.

(5.7) $y_\varepsilon(t) = \alpha^{-1}\int u_\varepsilon(t)$

Lemma 5.2. There exists $C>0$ such that for all $T>0$, $u_0 \in D(A_\varepsilon)$ we have

(5.8) $$\int_0^T \|u_\varepsilon(s)-y_\varepsilon(s)\|_0^2 ds \leq \varepsilon C \|u_0\|^2.$$

Proof. If we have in mind that $\int_{\Omega_2}(u_\varepsilon - y_\varepsilon)=0$ then by using the Friedrichs-Poincaré inequality we get $\|u_\varepsilon - y_\varepsilon\|_2^2 \leq C_F |||\nabla u_\varepsilon|||_2^2$; From the continuity of the trace map γ_{02} we obtain $\|u_\varepsilon - y_\varepsilon\|_0^2 \leq C|||\nabla u_\varepsilon|||_2^2$, hence from (5.5) we can deduce (5.8).

Proof of Theorem 5.1. Let us suppose from the moment that $u_0 \in H^2(\Omega_1)$, $u_0|_{\Gamma_0^+} = y_0$, $\frac{\partial u_0}{\partial n}\Big|_{\partial\Omega_1}=0$ and we notice that in this case $u_0 \in \bigcap_{\varepsilon>0} D(A_\varepsilon)$.

Let us denote by $\bar u_\varepsilon = u_\varepsilon - u$ in Ω_1, $\bar u_\varepsilon = u_\varepsilon - y$ in Ω_2. From (3.6-8) and (2.5-7) we can deduce that $\bar u_\varepsilon$ satisfies the following equations:

(5.9) $$\frac{\partial \bar u_\varepsilon}{\partial t} = \Delta \bar u_\varepsilon \text{ in } \Omega_1, \quad \frac{\partial \bar u_\varepsilon}{\partial t} = \frac{1}{\varepsilon}\Delta \bar u_\varepsilon + \alpha^{-1}\int_{\Omega_1}\Delta u \quad \text{in } \Omega_2$$

(5.10) $$\bar u_\varepsilon|_{\Gamma_0^+} = \bar u_\varepsilon|_{\Gamma_0^-}, \quad \frac{\partial \bar u_\varepsilon}{\partial n}\Big|_{\Gamma_0^+} + \frac{1}{\varepsilon}\frac{\partial \bar u_\varepsilon}{\partial n}\Big|_{\Gamma_0^-} = -\frac{\partial u}{\partial n}\Big|_{\Gamma_0^+}, \quad \frac{\partial \bar u_\varepsilon}{\partial n}\Big|_{\Gamma} = 0$$

(5.11) $$\bar u_\varepsilon(0)=0$$

If we multiply (5.9) by $\bar u_\varepsilon$ and we integrate the result over Ω we get $\left(\frac{\partial \bar u_\varepsilon}{\partial t}, \bar u_\varepsilon\right) = -|||\nabla \bar u_\varepsilon|||_1^2 - 1/\varepsilon |||\nabla \bar u_\varepsilon|||_2^2 - \int_{\Gamma_0^+}\frac{\partial u}{\partial n}(u_\varepsilon - y_\varepsilon) \leq \left\|\frac{\partial u}{\partial n}\right\|_{H^{-1/2}} \|u_\varepsilon - y_\varepsilon\|_0$.

By integrating the above inequality from 0 to t we deduce $1/2\,\|\bar u_\varepsilon(t)\|^2 \leq \left(\int_0^T \left\|\frac{\partial u}{\partial n}\right\|_{H^{-1/2}(\Gamma_0)}^2 ds\right)^{1/2} \left(\int_0^T \|u_\varepsilon - y_\varepsilon\|_0^2 ds\right)^{1/2}$ and from (5.8) we get (5.1-2). In order to prove (5.1-2) for all $u_0 \in L_1$ it is enough to notice that the set of all $u \in H^2(\Omega_1)$, with $u|_{\Gamma_0^+} = y_0$, $\frac{\partial u}{\partial n}\Big|_{\partial\Omega_1} = 0$ is dense in L_1.

Proof of Theorem 5.2. Having in mind that the set $D=\left\{u\in\mathcal{D}(\Omega_2) / \int_{\Omega_2} u=0\right\}$ in dense in the set $\left\{u\in L^2(\Omega_2) / \int_{\Omega_2} u=0\right\}$ and using theorem 5.1 we notice that it is enough to prove the statements of the theorem for $u_0 \in D$. In this case the solution of the reduced problem (2.5-7) is $u=0$, $y=0$. Let $w_\varepsilon(t)$ be the solution of the following elliptic problem:

$$(5.12) \qquad \Delta w_\varepsilon(t)=0 \qquad \text{in } \Omega_1,$$

$$(5.13) \qquad w_\varepsilon(t)\Big|_{\Gamma_0^+} = u_\varepsilon(t)-y_\varepsilon(t) \quad , \quad \frac{\partial w_\varepsilon}{\partial n}\Big|_{\Gamma_1} = 0 .$$

If we denote by $v_\varepsilon = u_\varepsilon - w_\varepsilon$ then v_ε, y_ε is the solution of the following equations:

$$(5.14) \qquad \frac{\partial v_\varepsilon}{\partial t} = \Delta v_\varepsilon - \frac{\partial w_\varepsilon}{\partial t} \qquad \text{in } \Omega_1$$

$$(5.15) \qquad \frac{dy_\varepsilon}{dt} = -\alpha^{-1}\int_{\Omega_1} \Delta v_\varepsilon$$

$$(5.16) \qquad v_\varepsilon\Big|_{\Gamma_0^+} = y_\varepsilon \ , \ \frac{\partial v_\varepsilon}{\partial n}\Big|_{\Gamma_1} = 0 \qquad \text{for all } t>0$$

$$(5.17) \qquad v_\varepsilon(0)=0 \quad y_\varepsilon(0)=0.$$

If we consider $f_\varepsilon \in C(R_+,X)$ given by

$$(5.18) \qquad f_\varepsilon(t)=\left[w_\varepsilon(t),\ 0\right] ,$$

then we notice that (5.14-17) is equivalent with

(5.19) $\frac{d}{dt}\left[v_\varepsilon, y_\varepsilon\right] = A\left[v_\varepsilon, y_\varepsilon\right] - \frac{df_\varepsilon}{dt}$, $\left[v_\varepsilon, y_\varepsilon\right](0) = \left[0,0\right]$

From (5.19) we deduce that $\left[v_\varepsilon, y_\varepsilon\right](t) = -\int_0^t T(t-s)\frac{df_\varepsilon}{ds}(s)ds =$
$= -f_\varepsilon(t) - A\int_0^t T(t-s)f_\varepsilon(s)ds + T(t)f_\varepsilon(0)$.
Having in mind that $f_\varepsilon(0)=0$ we get

(5.20) $\left[u_\varepsilon(t), y_\varepsilon(t)\right] = -A\int_0^t T(t-s)f_\varepsilon(s)ds.$

Taking the inner product in X with $[\varphi, 0] \in \mathcal{D}(A)$, $\varphi \in \mathcal{D}(\Omega_1)$ then (5.20) becomes:

$$\left|(u_\varepsilon(t),\varphi)_1\right| = \left|\langle\int_0^T T(t-s)\, f_\varepsilon(s)ds,\ A[\varphi,0]\rangle_X\right| \le$$

$\le C_\varphi\int_0^t \|f_\varepsilon(s)\|_X ds \le C_\varphi\int_0^t \|w_\varepsilon(s)\|_1 ds$ where $C_\varphi = \|A[\varphi,0]\|_X$.

From (5.12-13) we can find that $\bar{C}>0$ such that $\|w_\varepsilon(t)\|_1 \le \bar{C}\,\|u_\varepsilon(t) - y_\varepsilon(t)\|_0$ and from the above inequality we obtain $\left|(u_\varepsilon(t),\varphi)\right| \le C_\varphi \bar{C}\sqrt{T}\left(\int_0^T\|u_\varepsilon(s) - y_\varepsilon(s)\|_0^2 ds\right)^{1/2}$.

Using now Lemma 5.2 we get:

(5.21) $\left|(u_\varepsilon(t),\varphi)_2\right| \le c.\,c_\varphi \bar{C}\sqrt{T}\sqrt{\varepsilon}\|u_0\|^2$

Since $\mathcal{D}(\Omega_1)$ is dense in L_1 we obtain that $u_\varepsilon(t)\to 0$ weakly in L_1 uniformly with respect to $t\in[0,T]$.

If we take into consideration that

$\int_{\Omega_1} u_\varepsilon(t) + \alpha y_\varepsilon(t) = 0$ for all $t\in[0,T]$ it follows that $y_\varepsilon \to 0$ in $C^0(0,T,R)$
Using now again Lemma (5.2) we get that $u_\varepsilon \to 0$ strongly in $L^2(0,T,L_2)$

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