DYNAMIC PROCESSES FOR A CLASS OF ELASTIC-VISCOPLASTIC MATERIALS

by

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Abstract. An initial and boundary value problem describing dynamic processes for a class of rate-type elastic viscoplastic materials is considered. The mechanical problem is reduced to a semilinear hyperbolic equation in a Hilbert space and the existence and the uniqueness of the solution is proved. In the linear viscoplastic case a singular perturbation problem is considered. It is proved that linear elasticity is a proper asymptotic theory for viscoelastic materials.

Key words: elastic-viscoplastic, semilinear hyperbolic equations, semigroups of linear operators, singular perturbation.

AMS (MOS) subject classifications: 73F15, 35Q20, 35L15, 35B25.

1. INTRODUCTION

An initial and boundary value problem describing dynamic processes for materials with a rate-type constitutive equation of the form[1]:

$$\dot{\sigma} = E\dot{\varepsilon} + F(\sigma,\varepsilon) \tag{1.1}$$

is considered. Various results and mechanical interpretations concerning this constitutive law may be found for instance in Freudenthal and Geiringer [6], Cristescu and Suliciu [3], Gurtin, Williams and Suliciu [7], Suliciu [15] and Podio-Guidugli and Suliciu [13].

If $F(\sigma,\varepsilon) = G(\sigma)$ depends only on σ then equation (1.1) may be reduced to some classical models used in viscoplasticity. Existence and uniqueness results for dynamic or quasistatic problems involving (1.1) for different forms of G where obtained using

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*) Everywhere in this paper the dot represents the derivative with respect to the time variable.

different methods by Duvaut and Lions [5, Ch. 5], Suquet [17], [18], [19], Djaoua and Suquet [4], Anzelloti [1], Anzelloti and Giaquinta [2], Necas and Kratochvil [11], Laborde [10], Sofonea [14], [15] and others. Almost all this methods are making use of the monotony properties of G [illegible]. If F depends both on σ and ε (examples of (1.1) involving the full coupling in stress and strain are given for instance in Cristescu, Suliciu [3]) the monotony arguments used in the above mentionated papers do not work. For this reason a different technique is used here based on the equivalence between the studied problem and a semilinear evolution equation in a Hilbert space. To be more specific the mechanical problem is reduced to a Lipschitz perturbation of a linear evolution equation and semigroups techniques are used.

Existence results for quasistatic processes involving the full coupling in stress and strain were obtained by Ionescu and Sofonea [8] by reducing the problem to an ordinary differential equation in a Hilbert space.

In section 2 of this paper the mechanical problem is stated and in the next section some preliminaries and notations are given. In section 4 an existence and uniqueness result is proved (theorem 4.1). The same technique as in the proof of theorem 4.1 is used in order to obtain an existence result (theorem 4.2) for problems with hardening (the hardening parameter used is the equivalent ireversible strain).

In the papers of Suliciu [15] and Podio-Guidugli and Suliciu [13] it is assumed that $F(\sigma,\varepsilon) = -\mathcal{K}(\sigma,\varepsilon)(\sigma - R(\varepsilon))$ where $\mathcal{K}(\sigma,\varepsilon)\tau \cdot \tau \geq k|\tau|^2$ and G is a monotone function. For isolated bodies assuming the existence and smoothness of the solution and constructing an energy function in [13], [16] it is obtain the folowing inequality $\int_0^t (\int_\Omega |\sigma(s) - R(\varepsilon(s))|^2) ds \leq C/k$. This inequality shows that for large values of the viscosity coefficient k the solution of the viscoelastic problem almost obeys an elastic law. That is why one can expect that the study of the asymptotic behaviour of the viscoelastic solution upon k will show that elasticity is a proper asymptotic theory for viscoelastic materials. This fact was proved by Ionescu and Sofonea [8] in the quasistatic case. The dynamic case of this singular perturbation problem is studied in section 5 of the present work. Using the energy function constructed by Suliciu [16], and assuming that $\mathcal{R}$ is linear it is proved (theorem 5.1) that the solution of a linear viscoelastic problem converges to the solution of a linear elastic problem for k large.

2. PROBLEM STATEMENT

Let $\Omega \subset R^N$ be a bounded domain with a smooth (say C^1) boundary $\Gamma = \partial\Omega$ and let Γ_1 be an open subset of Γ and $\Gamma_2 = \Gamma - \bar{\Gamma}_1$. Let us consider the following mixt problem on the time interval $[0,T]$.

Find the displacement function $u : [0,T] \times \Omega \to R^N$ and the stress function $\sigma : [0,T] \times \Omega \to S$ such that

(2.1) $$\rho\ddot{u}(t) = \operatorname{div}\sigma(t) + \rho b(t)$$

(2.2) $$\dot{\sigma}(t) = \mathcal{E}\varepsilon(\dot{u}(t)) + F(t,\sigma(t),\varepsilon(u(t))) \qquad \text{in } \Omega$$

(2.3) $$u(t)|_{\Gamma_1} = g(t) \qquad \sigma(t)\cdot\nu|_{\Gamma_2} = f(t) \quad \text{for } t \in (0,T]$$

(2.4) $$u(0) = u_0, \qquad \dot{u}(0) = v_0, \qquad \sigma(0) = \sigma_0 \quad \text{in } \Omega$$

where $S = R_S^{N\times N}$ is the set of second order symmetric tensors in R^N, $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla^T u)$ defines the strain tensor of small deformations and ν is the exterior unit normal at Γ. In the first equation $\rho : \Omega \to R_+$ is the density and $b : [0,T] \times \Omega \to R^N$ is the given body force. Equation (2.2) represents a rate type viscoelastic or viscoplastic constitutive law in which $\mathcal{E}$ is a forth order tensor and $F : [0,T] \times \Omega \times S \times S \to S$ is a constitutive function. The functions u_0, v_0, σ_0 are the initial data and f, g are the given boundary data.

REMARK 2.1. The function F depends usually on time by mean of the temperature field which can be easily obtained from the heat equation in the uncoupled thermomechanical processes.

3. NOTATIONS AND PRELIMINARIES

By $\cdot$, $|\ |$ the inner product and the euclidian norm in R^N and S will be denoted. The following Hilbert spaces: $L = [L^2(\Omega)]_S^{N\times N}$, $L = [L^2(\Omega)]^N$, $H = [H(\mathrm{div},\Omega)]_S^N$, $H = [H^1(\Omega)]^N$, $H_\Gamma = [H^{\frac{1}{2}}(\Gamma)]^N$ are used and the cannonical inner products and norms are denoted by $((,),\|\ \|)$, $(((,)),\ |||\ |||)$, $((,)_d,\ \|\ \|_d)$, $((,)_H,\ \|\ \|_H)$ and $((,)_\Gamma,\ \|\ \|_\Gamma)$ respectively. Let

(3.1) $$V = \{u \in H/\gamma_0(u) = 0 \text{ on } \Gamma_1\}$$

be a closed subspace of H endowed with the norm of H where $\gamma_0 : H \to H_\Gamma$ is the trace map. The operator $\varepsilon : H \to L$ given by

(3.2) $$\varepsilon(v) = \tfrac{1}{2}(\nabla v + \nabla^T v) \qquad \text{for all } v \in H$$

is linear and continuous and the deformation coercivity inequality holds[1)]

(3.3) $$\|\varepsilon(v)\|^2 + |||v|||^2 \geq C_1\|v\|_H^2 \qquad \text{for all } v \in H$$

If mes $\Gamma_1 > 0$ then the Korn inequality holds

(3.4) $$\|\varepsilon(v)\|^2 \geq C_2\|v\|_H^2 \qquad \text{for all } v \in V .$$

If $\tau \in H$ then there exists $\gamma_\nu(\tau) \in H_\Gamma^* = [H^{-\frac{1}{2}}(\Gamma)]^N$ (the strong dual of H_Γ) such that

(3.5) $$\langle\gamma_\nu(\tau), \gamma_0(v)\rangle = (\tau,\varepsilon(v)) + ((\mathrm{div}\,\tau, v))$$

for all $\tau \in H$, $v \in H$. The operator $\gamma_\nu : H \to H_\Gamma^*$ is linear continuous and surjective. By $\tau \cdot \nu|_{\Gamma_2}$ we shall understand the restriction of $\gamma_\nu(\tau)$ on $E = \gamma_0(V)$. Let

(3.6) $$V = \{\tau \in H/\tau \cdot \nu|_{\Gamma_2} = 0\}$$

be a closed subspace of H endowed with the norm of H.

4. AN EXISTENCE AND UNIQUENESS RESULT

In order to prove the existence and uniqueness of the solution for problem

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1) Everywhere in this paper C, C_i, $i \in N$ will represent strictly positive generic constants that depend on E, F, Ω, Γ_1, Γ_2 and do not depend on time or on input data.

(2.1-4) the following assumptions will be used:

(4.1) $\qquad E_{ijk\ell} \in L^{\infty}(\Omega),\ E(x)\tau \cdot \varepsilon = E(x)\varepsilon \cdot \tau \quad$ for all $x \in \Omega,\ \tau, \varepsilon \in S$

(4.2) $\qquad E(x)\tau \cdot \tau \geq d\,|\tau|^{2}$ with $d > 0$ for all $x \in \Omega,\ \tau \in S$.

For all $t \in [0,T]$, $\sigma, \varepsilon \in S$ the function $x \to F(t,x,\sigma,\varepsilon)$ is measurable and there exists $\theta \in W^{1,1}(0,T,L^{2}(\Omega))$ such that

(4.3) $\qquad |F(t_1,x,\sigma_1,\varepsilon_1) - F(t_2,x,\sigma_2,\varepsilon_2)| \leq |\theta(x,t_1) - \theta(x,t_2)| +$

$+ L(|\sigma_1 - \sigma_2| + |\varepsilon_1 - \varepsilon_2|)$ for all $x \in \Omega$, $t_i \in [0,T]$, $\sigma_i, \varepsilon_i \in S$

We shall also suppose that $F(t,x,0,0) = 0$ and

(4.4) $\qquad b \in W^{1,1}(0,T,L),\ \ \rho \in L^{\infty}(\Omega),\ \ \rho(x) \geq c > 0$ for all $x \in \Omega$

(4.5) $\qquad$ there exists $\tilde{u} \in W^{3,1}(0,T,L) \cap W^{2,1}(0,T,H)$ such that $\tilde{u}|_{\Gamma_1} = g$

(4.6) $\qquad$ there exists $\tilde{\sigma} \in W^{2,1}(0,T,L) \cap W^{1,1}(0,T,H)$ such that $\tilde{\sigma}\nu|_{\Gamma_2} = f$

(4.7) $\qquad u_o,\ v_o \in H,\ \ \sigma_o \in H,$

(4.8) $\qquad u_o|_{\Gamma_1} = g(0),\ \ v_o|_{\Gamma_1} = \dot{g}(0),\ \ \sigma_o \cdot \nu|_{\Gamma_2} = f(0)$

The following theorem is the main result of this section

THEOREM 4.1. Let (4.1-8) hold. Then there exists a unique couple of functions u, σ solution of problem (2.1-4) such that

(4.9) $\qquad u \in W^{2,\infty}(0,T,L) \cap W^{1,\infty}(0,T,H)$

(4.10) $\qquad \sigma \in W^{1,\infty}(0,T,L) \cap L^{\infty}(0,T,H)$

In order to prove theorem 4.1 we need some preliminary results.

Let us homogenize the boundary conditions by denoting with $\bar{u} = u - \tilde{u}$, $\bar{v} = u - \dot{\tilde{u}}$, $\bar{\sigma} = \sigma - \tilde{\sigma}$ and $\bar{u}_o = u_o - \tilde{u}(0)$, $\bar{v}_o = v_o - \dot{\tilde{u}}(0)$, $\bar{\sigma}_o = \sigma_o - \tilde{\sigma}(0)$. One can easily deduce the following lemma.

LEMMA 4.1. The couple u,σ is a solution of (2.1-4) such that (4.9-10) hold iff $\bar{u} \in W^{1,\infty}(0,T,V)$, $\ \bar{v} \in W^{1,\infty}(0,T,L) \cap L^{\infty}(0,T,V)$, $\ \bar{\sigma} \in W^{1,\infty}(0,T,L) \cap L^{\infty}(0,T,V)$ are the solution of the following problem

(4.11) $\dot{\bar{u}}(t) = \bar{v}(t)$

(4.12) $\dot{\bar{v}}(t) = \rho^{-1}\operatorname{div}\bar{\sigma}(t) + a(t)$

(4.13) $\dot{\bar{\sigma}}(t) = \mathcal{E}\varepsilon(\bar{v}(t)) + H(t,\bar{\sigma}(t),\varepsilon(\bar{u}(t)))$

(4.14) $\bar{u}(0) = \bar{u}_o\,,\ \bar{v}(0) = \bar{v}_o,\ \bar{\sigma}(0) = \bar{\sigma}_o$

where $a(t) = b(t) - \ddot{\tilde{u}}(t) + \rho^{-1}\operatorname{div}\tilde{\sigma}(t)$ and $H : [0,T]\times\Omega\times\mathcal{S}\times\mathcal{S} \to \mathcal{S}$ is given by $H(t,\tau,\sigma) = $ $= F(t,\tilde{\sigma}(t) + \tau, \varepsilon(\tilde{u}(t)) + \sigma) + \mathcal{E}\varepsilon(\dot{\tilde{u}}(t)) - \dot{\tilde{\sigma}}(t)$ for all $\tau,\sigma \in \mathcal{S}$.

Let us consider the Hilbert space $X = V\times L\times\mathcal{L}$ and let $D(A) = V\times V\times\mathcal{V}\subset X$, $A : D(A)\subset X \to X$ be given by

(4.15) $A[u,v,\sigma] = [v,\mathcal{S}^{-1}\operatorname{div}\sigma,\mathcal{E}\varepsilon(v)]$

for all $u,v \in V, \sigma \in \mathcal{V}$. With the above notations we have.

LEMMA 4.2. The operator A is the infinitesimal generator of a C_o semigroup S(t) of bounded linear operators in X.

In order to prove the above lemma let us consider $Y = L\times\mathcal{L}$ with the following inner product

(4.16) $([v_1,\tau_1],[v_2,\tau_2])_Y = ((\rho v_1,v_2)) + (\mathcal{E}^{-1}\tau_1,\tau_2)$

which generates an equivalent norm on Y. Let $D(B) = V\times\mathcal{V}\subset Y$ and $B : D(B)\subset Y \to Y$ be given by

(4.17) $B[v,\sigma] = [\rho^{-1}\operatorname{div}\sigma, \mathcal{E}\varepsilon(v)]$ for all $v \in V, \sigma \in \mathcal{V}$.

LEMMA 4.3. If we consider in Y the scalar product given by (4.16) then $B^* = -B$ (B^* denotes the adjoint operator of B).

PROOF. For $\rho \equiv 1$ and $\mathcal{E} = 1_{\mathcal{S}\times\mathcal{S}}$ this lemma was proved by Sofonea [15]. However for the convenience of the reader we scketch here the proof. Let $[v,\sigma] \in D(B^*)$ and $[v^*,\sigma^*] = B^*[v,\sigma]$. For all $(u,\tau) \in D(B) = V\times\mathcal{V}$ we have

(4.18) $((\operatorname{div}\tau,v)) + (\varepsilon(u),\sigma) = ((\rho u,v^*)) + (\mathcal{E}^{-1}\tau,\sigma^*)$

If we put $\tau = 0$ in (4.18) we get $(\nabla u,\sigma) = ((u,\rho v^*))$ for all $u \in [D(\Omega)]^N$ hence $\operatorname{div}\sigma = -\rho v^* \in L$ i.e. $\sigma \in H$. We also have that $(\nabla u,\sigma) + ((\operatorname{div}\sigma,u)) = 0$ for all $u \in V$ hence $\sigma \in V$. If we put now in (4.18), $u = 0$ we get $((\operatorname{div}\tau,v)) = (\tau,E^{-1}\sigma^*)$ for all $\tau \in [D(\Omega)]_S^{N\times N}$. So we get $\varepsilon(v) \in L$ and from (3.3) we get $v \in H$ and $\varepsilon(v) = -E^{-1}\sigma^*$ hence we deduce that

$$((\operatorname{div}\tau,v)) + (\varepsilon(v),\tau) = 0 \text{ for all } \tau \in V. \qquad (4.19)$$

If $v \notin V$ then $\gamma_0(v) \notin E$ and we can construct $f \in H_\Gamma^*$ such that $f|_E = 0$ and $\langle f,\gamma_0(v)\rangle \neq 0$. Let $\tau \in H$ such that $\gamma_\nu(\tau) = f$. From $\gamma_\nu(\tau)|_E = 0$ we deduce $\tau \in V$ and from (4.19), (3.5) we obtain $\langle\gamma_\nu(\tau),\gamma_0(v)\rangle = 0$ a contradiction. Hence we proved that $[v,\sigma] \in V \times V = D(B)$ and $B^*[v,\sigma] = -B[v,\sigma]$. □

PROOF OF LEMMA 4.2. Let denote by $T(t) \in B(Y)$ the C_0 semigroup generated B (see the above lemma). Let $T_1(t) \in B(Y,L)$, $T_2(t) \in B(Y,L)$ such that $T(t)[v,\sigma] = [T_1(t)[v,\sigma], T_2(t)[v,\sigma]] \in L \times L$ for all $[v,\sigma] \in L \times L$. We shall denote by $S(t)$ the following operator

$$S(t)[u,v,\sigma] = [u + \int_0^t T_1(s)[v,\sigma]ds,\ T(t)[v,\sigma]] \qquad (4.20)$$

for all $[u,v,\sigma] \in V \times L \times L = X$. If we have in mind that $[D(B)]$ endowed with the graph norm of B is isomorph with $V \times V$ (we use (3.3), (4.2), (4.4)) and $y \to \int_0^t T(s)y ds \in B(Y,[D(B)]$ we get that $y \to \int_0^t T_1(s)y ds \in B(Y,V)$ hence $S(t) \in B(X)$. One can easily check that A is the infinitesimal generator of $S(t)$. □

In order to prove the existence and uniqueness of the solution for the problem (4.11-14) we need the following abstract result (see for instance Pazy [12 p. 189] with small modifications).

LEMMA 4.4. Let X be a reflexive Banach space, $A : D(A) \subset X \to X$ be the infinitesimal generator of a C_0 semigroup and $f : [0,T] \times X \to X$. Let us suppose that there exists $g \in L^1(0,T,R_+)$ and $L > 0$ such that

$$\| f(t_1,x_1) - f(t_2,x_2) \|_X \leq \int_{t_1}^{t_2} g(s)ds + L \| x_1 - x_2 \|_X \qquad (4.21)$$

for all $t_1 < t_2$, $t_1,t_2 \in [0,T]$, $x_1,x_2 \in X$. Then for all $x_o \in D(A)$ there exists a unique $x \in W^{1,\infty}(0,T,X) \cap L^{\infty}(0,T,[D(A)])$ strong solution of the Cauchy problem

$$\dot{x}(t) = Ax(t) + f(t,x(t)) \qquad x(0) = x_o \,. \tag{4.22}$$

PROOF OF THEOREM 4.1. We shall use Lemma 4.4 in order to prove the existence of the solution for the problem (4.11-14) and from Lemma 4.1 the statement of theorem will follow.

If we denote by $f : [0,T] \times X \to X$ given by

$$f(t,[u,v,\sigma]) = [0,a(t),H(t,\sigma,\varepsilon(u))] \tag{4.23}$$

then problem (4.11-14) can be written as (4.22) where $x(t) = [\bar{u}(t),\bar{v}(t),\bar{\sigma}(t)]$ and $x_o = [\bar{u}_o,\bar{v}_o,\bar{\sigma}_o]$ and A is given by (4.15). Since from Lemma 4.2 we have that A is the infinitesimal generator of a C_o semigroup we have only to check (4.22). Indeed we have

$$\begin{aligned}
&\| f(t_1,[u_1,v_1,\sigma_1]) - f(t_2,[u_2,v_2,\sigma_2]) \|_X \le C[\, |||a(t_1) - a(t_2)||| + \|\dot{\tilde{\sigma}}(t_1) - \dot{\tilde{\sigma}}(t_2)\| + \\
&+ \|\theta(t_1) - \theta(t_2)\|_{L^2(\Omega)} + \|\varepsilon(\dot{\tilde{u}}(t_1)) - \varepsilon(\dot{\tilde{u}}(t_2))\| + L(\|\varepsilon(\tilde{u}(t_1)) - \varepsilon(\tilde{u}(t_2))\| + \\
&+ \|\tilde{\sigma}(t_1) - \tilde{\sigma}(t_2)\| + \|\varepsilon(u_1) - \varepsilon(u_2)\| + \|\sigma_1 - \sigma_2\|)] \le C[\int_{t_1}^{t_2} [\, |||\dot{b}(s) - \\
&- \dddot{\tilde{u}}(s) + \rho^{-1}\mathrm{div}\ddot{\tilde{\sigma}}(s)||| + \|\dot{\theta}(s)\|_{L^2(\Omega)} + \|\varepsilon(\ddot{\tilde{u}}(s))\| + \|\ddot{\tilde{\sigma}}(s)\| + \\
&+ L(\|\dot{\tilde{\sigma}}(s)\| + \|\dot{\tilde{u}}(s)\|_H)]ds + L(\|u_1 - u_2\|_H + \|\sigma_1 - \sigma_2\|)].
\end{aligned}$$

From (4.7) and (4.8) we deduce $x_o = [\bar{u}_o,\bar{v}_o,\bar{\sigma}_o] \in V \times V \times \mathcal{V} = D(A)$. Having in mind that $[D(A)]$ endowed with the graph norm of A is isomorph with $V \times V \times \mathcal{V}$ from Lemma 4.4 we deduce that $\bar{u} \in W^{1,\infty}(0,T,V)$, $\bar{v} \in W^{1,\infty}(0,T,L) \cap L^{\infty}(0,T,V)$, and $\bar{\sigma} \in W^{1,\infty}(0,T,\mathcal{L}) \cap L^{\infty}(0,T,\mathcal{V})$. □

The same technique can be used in order to obtain th existence of the solution in similar problem with hardening in which the rate of the hardening parameter is a function of the inversible strain rate. To be more precisely let $F : [0,T] \times \Omega \times \mathcal{S} \times \mathcal{S} \times R \to \mathcal{S}$ and let denote by $\dot{\varepsilon}^I$ the ireversible strain rate given by

(4.24) $$\dot{\varepsilon}^I = -E^{-1}F(t, \sigma(t), \varepsilon(u(t)), \chi(t))$$

Instead of (2.2) we have

(4.25) $$\dot{\sigma}(t) = E\varepsilon(\dot{u}(t)) + F(t, \sigma(t), \varepsilon(u(t)), \chi(t))$$

and we must add an equation for the hardening parameter χ

(4.26) $$\dot{\chi}(t) = R(\dot{\varepsilon}^I), \qquad \chi(0) = \chi_o$$

where $R : S \to R$. If we suppose in addition that

(4.27) F is Lipschitz continuous with respect to χ,

(4.28) $$|R(\tau_1) - R(\tau_2)| \leq L_o|\tau_1 - \tau_2| \quad \text{for all } \tau_i \in S$$

then we have the following result

THEOREM 4.2. Let (4.1-8), (4.27-28) hold and suppose that $\chi_o \in L^2(\Omega)$. Then there exists a unique couple of functions $[u, \sigma, \chi]$ solution for the problem (2.1), (2.3-4), (4.24-26) such that we have (4.9-10) and $\chi \in W^{1,\infty}(0,T,L^2(\Omega))$.

SKETCH OF PROOF. After the same homogenization of the boundary data we use also Lemma 4.4 with $X = V \times L \times L \times L^2(\Omega)$, $D(A) = V \times V \times V \times L^2(\Omega)$ and $A[u,v,\sigma,\chi] = (v, \rho^{-1}\mathrm{div}\,\sigma, E\,\varepsilon\,(v), 0)$.

REMARK 4.2. In the case $F(\sigma, \varepsilon, \chi) = -G(\sigma,\chi)$ with G a monotone function of σ existence results were obtained by Laborde [10] and by Sofonea [15] using a fix point technique.

5. APPROACH TO ELASTICITY

Let us consider now the linear viscoelastic case i.e.

(5.1) $$F(t,\sigma,\varepsilon) = -k(\sigma - A\,\varepsilon)$$

where A is a forth order tensor such that

(5.2) $$A_{ijke} \in L^\infty(\Omega),\ A(x)\varepsilon \cdot \tau = A\tau \cdot \varepsilon \quad \text{for all } x \in \Omega,\ \varepsilon,\tau \in S$$

(5.3) $$A(x)\tau \cdot \tau \geq \alpha|\tau|^2 \quad \text{for all } \tau \in S \text{ with } \alpha > 0$$

and $k > 0$ is a viscosity constant. In order to construct the energy function we shall suppose (see Suliciu [16]) that

(5.4) $\quad (E - A)\tau \cdot \tau \geq \beta |\tau|^2$ for all $\tau \in S$ with $\beta > 0$.

Let now consider the following linear elastic problem. Find the displacement function $\hat{u} : [0,T] \times \Omega \to R^N$, and the stress function $\hat{\sigma} : [0,T] \times \Omega \to S$ such that

(5.5) $\quad \rho\ddot{\hat{u}}(t) = \mathrm{div}\,\hat{\sigma}(t) + \rho b(t)$

(5.6) $\quad \hat{\sigma}(t) = A\,\varepsilon(\hat{u}(t))$ in Ω.

(5.7) $\quad \hat{u}(t)|_{\Gamma_1} = g(t),\ \hat{\sigma}(t)\nu|_{\Gamma_2} = f(t)$ for $t > 0$

(5.8) $\quad \hat{u}(0) = u_0 \quad \dot{\hat{u}}(0) = v_0$

Using standard existence theorems for linear dynamic elasticity one can easily deduce the following lemma

LEMMA 5.1. Let (4.4-8), (5.2-3) hold with $\sigma_0 = A\,\varepsilon(u_0)$. Then there exists a unique couple of functions $\hat{u}, \hat{\sigma}$ solution for (5.5-8) such that:

(5.9) $\quad \hat{u} \in W^{2,\infty}(0,T,L) \cap W^{1,\infty}(0,T,H)$

(5.10) $\quad \hat{\sigma} \in W^{1,\infty}(0,T,\mathcal{L}) \cap L^{\infty}(0,T,\mathcal{H})$

The main result of this section is the following.

THEOREM 5.1. Let $\mathrm{mes}\,\Gamma_1 > 0$, (4.1-8), (5.2-4) hold and suppose that $\sigma_0 = A\,\varepsilon(u_0)$. If we denote by $[u_k,\sigma_k]_{k>0}$ the solution of problem (2.1-4) with F given by (5.1) then for all $t \in [0,T]$ we have that $u_k(t) \to \hat{u}(t)$ in H, $\dot{u}_k(t) \to \dot{\hat{u}}(t)$ in L, $\sigma_k(t) \to \hat{\sigma}(t)$ in $\mathcal{L}$ for $k \to +\infty$.

PROOF. Let us denote by $\bar{u}_k = u_k - \hat{u}$, $\bar{v}_k = \dot{u}_k - \dot{\hat{u}}$, $\bar{\sigma}_k = \sigma_k - \hat{\sigma}$. We remark that $\bar{u}_k \in W^{1,\infty}(0,T,V)$, $\bar{v}_k \in W^{1,\infty}(0,T,L) \cap L^{\infty}(0,T,V)$, $\bar{\sigma}_k \in W^{1,\infty}(0,T,\mathcal{L}) \cap L^{\infty}(0,T,\mathcal{V})$ and the couple $[\bar{u}_k,\bar{v}_k,\bar{\sigma}_k]$ is the solution of the following problem

(5.11) $\quad \dot{\bar{u}}_k(t) = \bar{v}_k(t)$

(5.12) $\dot{\bar{v}}_k(t) = \rho^{-1}\mathrm{div}\,\bar{\sigma}_k(t)$

(5.13) $\dot{\bar{\sigma}}_k(t) = \mathcal{E}\varepsilon(\bar{v}_k(t)) - k(\bar{\sigma}_k(t) - \mathcal{A}\,\varepsilon\,(\bar{u}_k(t))) + c(t)$

(5.14) $\bar{u}_k(0) = \bar{v}_k(0) = 0, \quad \bar{\sigma}_k(0) = 0$

where $c \in L^{\infty}(0,T,\mathcal{L})$ is given by

(5.15) $c(t) = -\dot{\hat{\sigma}}(t) + \mathcal{E}\varepsilon(\dot{\hat{u}}(t))$

Let $X = V \times L \times \mathcal{L}$ and A be given by (4.15) and let us consider $C : X \to X$ be given by

(5.16) $C[u,v,\sigma] = [0,0,\sigma - \mathcal{A}\,\varepsilon\,(u)]$

for all $[u,v,\sigma] \in X$. If we denote by $D_k : D(A) \subset X \to X, D_k = A - kC$, and if we have in mind that C is linear and continuous then from Lemma 4.2 we can deduce that D_k is the infinitesimal generator of a C_o semigroup denoted by $U_k(t)$. We shall construct now an energetical norm in X such that $U_k(t)$ is a contraction semigroup.

Having in mind (5.4) one can easily deduce that $\mathcal{B}$ given by

(5.17) $\mathcal{B} = (\mathcal{E} - \mathcal{A})^{-1} - \mathcal{E}^{-1}$

is positive definite i.e. there exists $\gamma = \alpha\beta/(|\mathcal{E}|\,|\mathcal{E} - \mathcal{A}|^2)$ such that

(5.18) $\mathcal{B}(x)\tau \cdot \tau \geq \gamma|\tau|^2$ for all $x \in \Omega, \tau \in \mathcal{S}$.

Let us consider in X the following energetical scalar product

(5.19) $([u_1,v_1,\sigma_1],[u_2,v_2,\sigma_2])_E = ((\rho v_1,v_2)) + (\mathcal{E}^{-1}\sigma_1,\sigma_2) +$
$+ (\mathcal{B}(\sigma_1 - \mathcal{E}\varepsilon(u_1)), \sigma_2 - \mathcal{E}\varepsilon(u_2))$

for all $u_i \in V$, $v_i \in L$, $\sigma_i \in \mathcal{L}$ which generates the energy norm (which is exactly the energy constructed by Suliciu [16] in our case) given by

(5.20) $\|[u,v,\sigma]\|_E^2 = ((\rho v,v)) + (\mathcal{E}^{-1}\sigma,\sigma) + (\mathcal{B}(\sigma - \mathcal{E}\varepsilon(u)), \sigma - \mathcal{E}\varepsilon(u))$

This norm is an equivalent norm X for mes $\Gamma_1 > 0$ (see (3.4)). After some algebra one can deduce that for all $u,v \in V$, $\sigma \in \mathcal{V}$ we have

(5.21) $(D_k[u,v,\sigma],[u,v,\sigma])_E = -k((E - A)^{-1}(\sigma - A\varepsilon(u)),\sigma - A\varepsilon(u)) \leq 0$

Having in mind that $\frac{1}{2}(d/dt)\|U_k(t)[u,v,\sigma]\|_E^2 = (D_k U_k(t)[u,v,\sigma],U_k(t)[u,v,\sigma])_E \leq 0$ we obtain that $\|U_k(t)[u,v,\sigma]\|_E \leq \|[u,v,\sigma]\|_E$ for all $u,v, \varepsilon V$ $\sigma \varepsilon V$. Using the density of D(A) in X we get

(5.22) $\|U_k(t)x\|_E \leq \|x\|_E$ for all $x \varepsilon X$, $t > 0$.

Let us remark now that if we denote by $f \varepsilon L^{\infty}(0,T,X)$

(5.23) $f(t) = [0,0,c(t)]$

and by $x_k(t) = [\bar{u}_k(t), \bar{v}_k(t), \bar{\sigma}_k(t)]$ then from (5.11-14) we have that

(5.24) $\dot{x}_k(t) = D_k x_k(t) + f(t) \qquad x_k(0) = 0$

hence we can deduce

(5.25) $x_k(t) = [\bar{u}_k(t), \bar{v}_k(t), \bar{\sigma}_k(t)] = \int_o^t U_k(t - s)f(s)ds$.

Let us suppose for the moment that $f \varepsilon C^1(0,T,X)$. From (5.25) and (5.22) we have that $\|x_k(t)\|_E \leq \int_o^T \|f(s)\|_E ds$ hence we have just obtained that

(5.26) $\bar{u}_k$ is bounded in $L^{\infty}(0,T,V)$

(5.27) $\bar{\sigma}_k$ is bounded in $L^{\infty}(0,T,L)$

Having in mind that $\dot{x}_k(t) = U_k(t)f(0) + \int_o^t U_k(t - s)\dot{f}(s)ds$ one can deduce that $\|\dot{x}_k(t)\|_E \leq \|f(0)\|_E + \int_o^T \|\dot{f}(s)\|_E ds$ hence we have

(5.27) $\dot{\bar{u}}_k = \bar{v}_k$ is bounded in $L^{\infty}(0,T,V)$

(5.28) $\dot{\bar{v}}_k = \frac{1}{\rho}\mathrm{div}\,\bar{\sigma}_k$ is bounded in $L^{\infty}(0,T,L)$

(5.29) $\dot{\bar{\sigma}}_k$ is bounded in $L^{\infty}(0,T,L)$

From (5.24) we obtain $\|D_k x_k(t)\|_E \leq \|\dot{x}_k(t)\|_E + \|f(t)\|_E < +\infty$ and if we do some computations we deduce $(B(\bar{\sigma}_k(t) - A\varepsilon(\bar{u}_k(t))), \bar{\sigma}_k(t) - A\varepsilon(\bar{u}_k(t))) \leq c/k^2$ and using (5.18) we have

(5.30) $\|\bar{\sigma}_k(t) - \mathcal{A}\varepsilon(\bar{u}_k(t))\|_E \leq c/k$

for all $t \in [0,T]$, $k > 0$. From the apriori estimates (5.26-30) we deduce that there exists $\bar{u} \in W^{1,\infty}(0,T,V)$, $\bar{v} \in W^{1,\infty}(0,T,L) \cap L^{\infty}(0,T,V)$, $\bar{\sigma} \in W^{1,\infty}(0,T,\mathcal{L}) \cap L^{\infty}(0,T,\mathcal{V})$ such that

(5.31) $\bar{u}_k \to \bar{u}$, $\dot{\bar{u}}_k \to \dot{\bar{u}}$ weak* in $L^{\infty}(0,T,V)$

(5.32) $\bar{v}_k \to \bar{v}$, $\dot{\bar{v}}_k \to \dot{\bar{v}}$ weak* in $L^{\infty}(0,T,L)$

(5.33) $\bar{\sigma}_k \to \bar{\sigma}$, $\dot{\bar{\sigma}}_k \to \dot{\bar{\sigma}}$ weak* in $L^{\infty}(0,T,\mathcal{L})$

(5.34) $\bar{\sigma}_k \to \bar{\sigma}$ weak* in $L^{\infty}(0,T,\mathcal{V})$

(5.35) $\bar{v}_k \to \bar{v}$ weak* in $L^{\infty}(0,T,V)$

(if necessary we can pass in (5.31-35) to a subsequence). From (5.11-12) and (5.31-32), (5.34-35) we deduce

(5.36) $\bar{u} = v$

(5.37) $\bar{v} = \rho^{-1}\text{div}\,\bar{\sigma}$

and from (5.30-33) we have

(5.38) $\bar{\sigma} = \mathcal{A}\,\varepsilon(\bar{u})$

Using now (5.31-33) we get that $\bar{u}_k(0) \to \bar{u}(0)$ weak in V, $\bar{v}_k(0) \to \bar{v}(0)$ weak in L and $\bar{\sigma}_k(0) \to \bar{\sigma}(0)$ weak in $\mathcal{L}$. From (5.14) we have

(5.39) $\bar{u}(0) = \bar{v}(0) = 0$, $\bar{\sigma}(0) = 0$.

If we make use of the unicity result given in Lemma 5.1 we have $\bar{u} = \bar{v} = 0$, $\bar{\sigma} = 0$ hence

(5.40) $x_k \to 0$ weak* in $L^{\infty}(0,T,X)$

Using again (5.21) from (5.24) we have

$$\tfrac{1}{2}(d/dt)\|x_k(t)\|_E^2 = (D_k x_k(t), x_k(t))_E + (f(t), x_k(t))_E \leq (f(t),x_k(t))_E$$

and after an integration we get that $\frac{1}{2}\|x_k(t)\|_E^2 \leq \int_0^t (f(s),x_k(s))_E ds$ and using (5.40) we have just proved that for all $f \in C^1(0,T,X)$, $t > 0$

(5.41) $\quad x_k(t) \to 0$ strongly in X for all $t \in [0,T]$

where $x_k(t)$ is given by (5.25).

Let us return to our case $f \in L^{\infty}(0,T,X)$. If we denote by $f_{\varepsilon} \in C^1(0,T,X)$ such that $\int_0^T \| f_{\varepsilon}(s) - f(s) \|_E ds < \varepsilon$ then we have that

$$\| x_k(t) \|_E \leq \int_0^t \| f_{\varepsilon}(s) - f(s) \|_E ds + \| \int_0^t U_k(t-s) f_{\varepsilon}(s) ds \|_E$$

and using (5.41) we obtain the statement of the theorem.

REMARK 5.1. For $g,f,b \equiv 0$ one can use a result of Kurtz [9, theorem 2.1] and the energy norm given here in order to prove the statement of the theorem 5.1.

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