AMENABLE ACTIONS OF KATZ ALGEBRAS ON VON NEUMANN ALGEBRAS

by

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1. INTRODUCTION

The theory of Katz algebras was introduced to give a natural framework for the duality theory of locally compact groups. Moreover, this theory permitted the extension of the results concerning the actions of locally compact groups on a von Neumann algebra and many concepts from harmonic analysis were naturally generalized to Katz algebras.

Following this method, in [5] was defined the concept of amenable Katz algebra and given the first equivalent conditions of amenability for Katz algebras; a detailed study of this problem was done by M. Enock and J.M. Schwartz in [3].

In this paper, we define the concept of *amenable action* - respectively *weakly amenable* - of a Katz algebra - respectively of a Katz algebra $K = (M,\Gamma,k,\phi)$ such that M has minimal projections - on a von Neumann algebra and we study their connection with the amenability of K. We mention that when the Katz algebra is $G = (\ell^{\infty}(G),\pi_G,k_G,\phi_G)$ with G a discrete group, using the correspondence between continuous actions of G on a von Neumann algebra A and actions of G on A, the concept of weakly amenability of an action of G on A is a natural extension of the one of amenable action of G on A defined in [1] by C. Anantharaman-Delaroche.

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2. PRELIMINARIES AND NOTATIONS

2.1. In the following $K = (M, \Gamma, k, \phi)$ will denote a Katz algebra ($\Gamma : M \to M \times M$ the comultiplication, $k : M \to M$ the coinvolution, ϕ the left Haar weight), H_ϕ the Hilbert space associated to the weight ϕ.

2.2. An action of the Katz algebra K on a von Neumann algebra A is an injective, unital, normal $*$-homomorphism $\alpha : A \to A \otimes M$ such that:

$$(i \otimes \Gamma) \circ \alpha = (\alpha \otimes i) \circ \alpha$$

where i denotes the identity mapping.

2.3. If G is a locally compact group and A a von Neumann algebra, there exists a bijective correspondence between the continuous actions of G on A and the actions of the Katz algebra $G = (L^\infty(G), \pi_G, k_G, \phi_G)$ on A (for the definition of G see. e.g. [4]. 18.5); namely, for a continuous action $\sigma : G \to \mathrm{Aut}(A)$, the corresponding action $\pi_\sigma : A \to A \otimes L^\infty(G)$ is given by:

$$\langle \pi_\sigma(x), \tau \otimes \omega \rangle = \int \tau(\sigma_g^{-1}(x)) \omega(g) dg$$

for $\tau \in A_*$ and $\omega \in L^1(G)$. If $A \subset B(H)$ and $A \otimes L^\infty(G) \subset B(L^2(G, H))$ are realized as von Neumann algebras, then:

$$(\pi_\sigma(x)\xi)(g) = \sigma_g^{-1}(x)\xi(g), \quad \text{for } \xi \in L^2(G.H) \text{ and } g \in G$$

(see, e.g. [4]. 18.6).

2.4. It is known ([3], [5]) that K is called amenable if and only if there exists a state m of M such that

$$m((i \otimes \omega)\Gamma(x)) = m(x)\omega(1). \quad \text{for all } \omega \in M_* \text{ and } x \in M.$$

2.5. Let G be a locally compact group, A a von Neumann algebra and $\sigma: G \to \mathrm{Aut}(A)$ a continuous action of G on A. σ is called amenable ([1], definition 3.4) if there exists a conditional expectation $P : A \otimes L^{\infty}(G) \to A$ (identified with the subalgebra $A \otimes \mathbb{C}$) such that:

$$\sigma_g \circ P = P \circ (\sigma_g \otimes \tau_g), \quad \text{for all } g \in G,$$

where τ_g is the automorphism of $L^{\infty}(G)$ defined by

$$(\tau_g f)(h) = f(g^{-1}h) \quad (f \in L^{\infty}(G),\ h \in G)$$

2.6. We recall that for the Katz algebra $\mathbb{G} = (L^{\infty}(G), \pi_G, k_G, \phi_G)$, π_G is defined by

$$(\pi_G f)(s,t) = f(ts) \quad (f \in L^{\infty}(G), s, t, \in G).$$

Then $\pi_G = \pi_\tau$ (where τ was defined in 2.5 and π_τ in 2.3). Indeed, for every $\phi, \psi \in L^1(G)$ and $f \in L^{\infty}(G)$ we have:

$$\langle \pi_\tau(f), \phi \otimes \psi \rangle = \int \phi(\tau_g^{-1}(f))\psi(g)dg$$

$$= \int\int (\tau_g^{-1}(f))(t)\phi(t)\psi(g)dtdg$$

$$= \int\int f(gt)\phi(t)\psi(g)dtdg,$$

respectively

$$\langle \pi_G(f), \phi \otimes \psi \rangle = \int\int \pi_G(f)(t,g)\phi(t)\psi(g)dtdg$$

$$= \int\int f(gt)\phi(t)\psi(g)dtdg,$$

so the equality is proved.

2.7. A Katz algebra $\mathbb{K} = (M, \Gamma, k, \phi)$ is said to be of discrete type ([2], definition 7.3.1) if the Banach algebra M_* is unital. For every Katz algebra of discrete type, M is atomic ([2], theorem 7.3.2).

For every locally compact group G, the Katz algebra $\mathbf{G}$ is of discrete type if and only if G is a discrete group ([2], corollary 8.1.2).

2.8. If M is a von Neumann algebra, we will denote by $Z(M)$ the center of M; for $x \in M$, $z(x)$ will be the central support of x and for $\omega \in M_*^+$, $s(\omega)$ will denote the support of ω.

3. AMENABLE ACTIONS OF A KATZ ALGEBRA ON A VON NEUMANN ALGEBRA

3.1. DEFINITION. An action α of the Katz algebra $K = (M, \Gamma, k, \phi)$ on the von Neumann algebra A is called *amenable* if there exists a conditional expectation $P : A \otimes M \to A \otimes \mathbb{C}$ such that:

(1) $\quad [((i \otimes \omega)\alpha) \otimes ((i \otimes \omega)\Gamma)] \circ P = P \circ [((i \otimes \omega)\alpha) \otimes ((i \otimes \omega)\Gamma)]$, for all $\omega \in M_*$.

or, equivalently, identifying $A \otimes \mathbb{C}$ with A.

(1') $\quad \omega(1)[(i \otimes \omega)\alpha] \circ P = P \circ [((i \otimes \omega)\alpha) \otimes ((i \otimes \omega)\Gamma)]$, for all $\omega \in M_*$

3.2. DEFINITION. If the von Neumann algebra M has minimal projections, the action α of the Katz algebra $K = (M, \Gamma, k, \phi)$ on A is called *weakly amenable* if there exists a conditional expectation $P : A \otimes M \to A \otimes \mathbb{C}$ such that

(2) $\quad [((i \otimes \omega)\alpha) \otimes ((i \otimes \omega)\Gamma)] \circ P = P \circ [((i \otimes \omega)\alpha) \otimes ((i \otimes \omega)\Gamma)]$, for all $\omega \in M_*^+$

such that $z(s(\omega))$ is a minimal projection in $Z(M)$.

Clearly, every amenable action is weakly amenable; moreover, for a discrete group G we have the following connection between amenable actions of G on A and weakly amenable actions of the Katz algebra $\mathbf{G}$ on A:

3.3. PROPOSITION. Let G be a discrete group. $A \subset \mathcal{B}(H)$ a von Neumann algebra, σ a continuous action of G on A and π_σ the corresponding action of G on A. Then σ is amenable if and only if π_σ is weakly amenable (and then π_σ amenable implies that σ is amenable).

PROOF. First, we suppose that π_σ is weakly amenable, so there exists a conditional expectation $P : A\otimes \ell^\infty(G) \to A$ which verifies (2), that is

$$\omega(1)[(i\otimes\omega)\pi_\sigma]\circ P = P\circ[((i\otimes\omega)\pi_\sigma)\otimes((i\otimes\omega)\pi_\tau)]. \quad \text{for all } \omega\in\ell^1(G)^+$$

such that $z(s(\omega))$ is a minimal projection in $Z(\ell^\infty(G))$.

Let $(\varepsilon_g)_{g\in G}$ be the canonical orthonormal base of $\ell^2(G)$. Then:

(α) $\quad \{\omega\in\ell^1(G)^+ \mid z(s(\omega)) \text{ is a minimal projection in } Z(\ell^\infty(G))\} = \{\omega_{\varepsilon_g} \mid g\in G\}$

and we have the following relation:

(β) $\quad (i\otimes\omega_{\varepsilon_g})\pi_\sigma = \sigma_g^{-1}, \quad$ for all $g\in G$

Indeed, if $\xi,\eta\in H$ and $x\in A$:

$$((i\otimes\omega_{\varepsilon_g})\pi_\sigma(x)\xi\,|\,\eta) = (\pi_\sigma(x)\xi\otimes\varepsilon_g\,|\,\eta\otimes\varepsilon_g)_{\ell^2(G,H)}$$

$$= \sum_{h\in G}((\pi_\sigma(x)(\xi\otimes\varepsilon_g))(h)\,|\,(\eta\otimes\varepsilon_g)(h))_H$$

$$= \sum_{h\in G}(\sigma_h^{-1}(x)(\xi\otimes\varepsilon_g)(h)\,|\,(\eta\otimes\varepsilon_g)(h))_H$$

$$= (\sigma_g^{-1}(x)\xi\,|\,\eta), \quad \text{for all } g\in G.$$

Then:

$$\omega_{\varepsilon_g}(1)[(i\otimes\omega_{\varepsilon_g})\pi_\sigma]\circ P = \sigma_g^{-1}\circ P, \quad \text{for all } g\in G$$

and

$$P[((i\otimes\omega_{\varepsilon_g})\pi_\sigma)\otimes((i\otimes\omega_{\varepsilon_g})\pi_\tau)] = P\circ[\sigma_g^{-1}\otimes\tau_g^{-1}], \quad \text{for all } g\in G$$

so $\sigma_g\circ P = P\circ(\sigma_g\otimes\tau_g)$, for all $g\in G$, which means that σ is amenable (cf. 2.5).

Conversely, the proof is immediate, using 2.5, 3.2, (α) and (β).

3.4. REMARK. Using Proposition 3.3, 2.7 and ([1], Remark 3.7), we obtain that there exist non-amenable Katz algebras of discrete type which may act amenable on certain von Neumann algebras.

3.5. The connection between amenable Katz algebras and theirs amenable (respectively weakly amenable) actions on von Neumann algebras is given by the following.

THEOREM. Let K be a Katz algebra, $A \subset \mathcal{B}(H)$ a von Neumann algebra and $\alpha : A \to A \otimes M$ an action of K on A. Then:

a) K is amenable if and only if α is amenable and there exists a state m of A such that

$$m((i \otimes \omega)\alpha(x)) = m(x)\omega(1), \text{ for all } x \in A \text{ and } \omega \in M_*$$

b) If moreover, M is atomic, then K is amenable if and only if α is weakly amenable and there exists a state m of A such that

$$m((i \otimes \omega)\alpha(x)) = m(x)\omega(1), \text{ for all } x \in A \text{ and } \omega \in M_*$$

PROOF. a) and b) "$\Longrightarrow$" First we suppose that K is amenable and we will prove that α is amenable (and then weakly amenable) and the existence of a state m of A which satisfies the required condition.

Because K is amenable, there exists (cf 2.4) a state $\tilde{m}$ of M such that

(1) $$\tilde{m}((i \otimes \omega)\Gamma(x)) = \tilde{m}(x)\omega(1) \text{ for all } x \in M \text{ and } \omega \in M_*$$

We will obtain first the existence of a state m of A such that

(2) $$m((i \otimes \omega)\alpha(x)) = m(x)\omega(1) \text{ for all } x \in A \text{ and } \omega \in M_*$$

(we mention that this existence has been already proved - but not directly - in [3]; in this paper we give another proof).

In order to do this, we consider a state $\psi : A \to \mathbb{C}$ and, for every $x \in A$ we define $f_x : M_* \to \mathbb{C}$ by the formula

(3) $\qquad f_x(\omega) = \psi((i \otimes \omega)\alpha(x)), \quad (\omega \in M_*)$

If is easy to see that f_x is linear, continuous, so $f_x \in (M_*)^* = M$. Then we define $m : A \to \mathbb{C}$ by

(4) $\qquad m(x) = \tilde{m}(f_x), \quad (x \in A)$

Then m is a state of A and, for every $\omega, \tilde{\omega} \in M_*$, $x \in A$ we have:

$$f_{(i \otimes \omega)\alpha(x)}(\tilde{\omega}) = \psi((i \otimes \tilde{\omega})\alpha((i \otimes \omega)\alpha(x))), \text{ using (3)}$$

$$= \psi((i \otimes \tilde{\omega})(i \otimes i \otimes \omega)(\alpha \otimes i)(\alpha(x)))$$

$$= \psi((i \otimes \tilde{\omega} \otimes \omega)(i \otimes \Gamma)(\alpha(x))), \text{ using 2.2}$$

But also

$$\langle (i \otimes \omega)\Gamma(f_x), \tilde{\omega} \rangle = \langle \Gamma(f_x), \tilde{\omega} \otimes \omega \rangle$$

$$= \langle f_x, (\tilde{\omega} \otimes \omega) \circ \Gamma \rangle$$

$$= \psi((i \otimes ((\tilde{\omega} \otimes \omega) \circ \Gamma))\alpha(x)). \text{ using (3)}$$

$$= \psi((i \otimes \tilde{\omega} \otimes \omega)(i \otimes \Gamma)(\alpha(x)))$$

and therefore

(5) $\qquad f_{(i \otimes \omega)\alpha(x)} = (i \otimes \omega)\Gamma(f_x)$, for all $x \in A$ and $\omega \in M_*$.

Then, for $x \in A$ and $\omega \in M_*$

$$m((i \otimes \omega)\alpha(x)) = \tilde{m}(f_{(i \otimes \omega)\alpha(x)}), \text{ using (4)}$$

$$= \tilde{m}((i \otimes \omega)\Gamma(f_x)), \text{ with (5)}$$

$= \widetilde{m}(f_x)\omega(1)$, with (1)

$= \widetilde{m}(x)\omega(1)$, using (4).

so we have obtained (2).

Now, we will prove that α is amenable (and then weakly amenable).

Because $\widetilde{m}$ is a state of M, there exists a net $\{\xi_i\}_{i\in I} \subset H_\phi$, with $\|\xi_i\| = 1$, $(\forall)$ i, such that

(6) $\omega_{\xi_i}(x) \underset{i\in I}{\to} \widetilde{m}(x)$, for all $x \in M$.

Let LIM be any Banach limit with respect to I.

We define $P : A\otimes M \to A$ by the formula

(7) $Px = \mathrm{LIM}_i((i\otimes\omega_{\xi_i})(x))$

It is easy to verify that P is a conditional expectation. In order to obtain 3.1 (1), we may suppose that A is in standard α form on H. Then, for every $\xi,\eta \in H$ and $x \in A\otimes M$ we have:

$$(Px\xi\,|\,\eta) = \mathrm{LIM}_i((i\otimes\omega_{\xi_i})(x)\xi\,|\,\eta)$$

$$= \mathrm{LIM}_i\langle x, \omega_{\xi,\eta}\otimes\omega_{\xi_i}\rangle$$

$$= \mathrm{LIM}_i\langle(\omega_{\xi,\eta}\otimes i)(x), \omega_{\xi_i}\rangle$$

$$= \lim_{i\in I}\langle(\omega_{\xi,\eta}\otimes i)(x), \omega_{\xi_i}\rangle, \text{ using (6)}$$

$$= \widetilde{m}((\omega_{\xi,\eta}\otimes i)(x))$$

and therefore

(8) $(Px\xi\,|\,\eta) = m((\omega_{\xi,\eta}\otimes i)(x))$ for all $x \in A\otimes M$ and $\xi,\eta \in H$

Then we deduce:

$$(P[i\otimes((i\otimes\omega)\Gamma)](x)\xi|\eta) = \tilde{m}((\omega_{\xi,\eta}\otimes i)(i\otimes((i\otimes\omega)\Gamma))(x)), \text{ using (8)}$$

$$= \tilde{m}((\omega_{\xi,\eta}\otimes((i\otimes\omega)\Gamma))(x))$$

$$= \tilde{m}((i\otimes\omega)\Gamma((\omega_{\xi,\eta}\otimes i)(x)))$$

$$= \omega(1)\tilde{m}((\omega_{\xi,\eta}\otimes i)(x)), \text{ using (1)}$$

$$= \omega(1)(Px\xi|\eta), \text{ using (8)}$$

But also

$$((i\otimes((i\otimes\omega)\Gamma))(Px)\xi|\eta) = \omega(1)(Px\xi|\eta)$$

because $Px \in A\otimes \mathbb{C} \simeq A$. and we have obtained that

(9) $\quad P\circ[i\otimes((i\otimes\omega)\Gamma)] = [i\otimes((i\otimes\omega)\Gamma)]\circ P$, for all $\omega \in M_*$

On the other side we have

$$(P(((i\otimes\omega)\alpha)\otimes i)(x)\xi|\eta) = \mathrm{LIM}_i((i\otimes\omega_{\xi_i})(((i\otimes\omega)\alpha)\otimes i)(x)\xi|\eta), \text{ using (7)}$$

$$= \mathrm{LIM}_i((((i\otimes\omega)\alpha)\otimes\omega_{\xi_i})(x)\xi|\eta)$$

$$= \mathrm{LIM}_i((i\otimes\omega)\alpha((i\otimes\omega_{\xi_i})(x))\xi|\eta)$$

$$= \mathrm{LIM}_i\langle(i\otimes\omega_{\xi_i})(x),(\omega_{\xi,\eta}\otimes\omega)\circ\alpha\rangle$$

But because we supposed A in standard form on H and $(\omega_{\xi,\eta}\otimes\omega)\circ\alpha \in A_*$, there exists $\xi',\eta' \in H$ such that $(\omega_{\xi,\eta}\otimes\omega)\circ\alpha = \omega_{\xi',\eta'}$, so we may continue in the last equality with

$$= \mathrm{LIM}_i\langle(i\otimes\omega_{\xi_i})(x),\omega_{\xi',\eta'}\rangle$$

$$= \mathrm{LIM}_i((i\otimes\omega_{\xi_i})(x)\xi'|\eta')$$

$$= (Px\xi'|\eta'), \text{ with (7)}$$

$$= \langle Px,\omega_{\xi',\eta'}\rangle$$

$$= \langle Px,(\omega_{\xi,\eta}\otimes\omega)\circ\alpha\rangle$$

$$= ((i\otimes\omega)\alpha(Px)\xi\,|\,\eta)$$

and so we have proved that for every $\omega \in M_*$

(10) $$P \circ [((i\otimes\omega)\alpha)\otimes i] = [(i\otimes\omega)\alpha] \circ P = [((i\otimes\omega)\alpha)\otimes i] \circ P$$

(in the last term of the equality we used the identification $A\otimes \mathbb{C} \simeq A$). With (9) and (10) we obtain

$$P \circ [((i\otimes\omega)\alpha)\otimes((i\otimes\omega)\Gamma)] = \omega(1)[(i\otimes\omega)\alpha] \circ P, \text{ for all } \omega \in M_*,$$

that is the action α is amenable.

a) "$\Longleftarrow$" We suppose that there exists a state m of A such that

(11) $$m((i\otimes\omega)\alpha(x)) = m(x)\omega(1), \text{ for all } x \in A \text{ and } \omega \in M_*$$

and also that the action α is amenable, so there exists a conditional expectation $P : A\otimes M \to A$ such that 3.1 (1') is verified. We shall prove that K is amenable.

We define $\tilde{m} : M \to \mathbb{C}$ by

(12) $$\tilde{m}(x) = m(P(1\otimes x)), \quad (x \in M)$$

Clearly $\tilde{m}$ is a state of M. For every $\omega \in M_*$ and $x \in M$ we have

$$\tilde{m}((i\otimes\omega)\Gamma(x)) = m(P(1\otimes((i\otimes\omega)\Gamma(x))), \text{ using (12)}$$

$$= m(P[((i\otimes\omega)\alpha)\otimes((i\otimes\omega)\Gamma)](1\otimes x))$$

$$= \omega(1)m((i\otimes\omega)\alpha(P(1\otimes x))), \text{ using 3.1 (1')}$$

$$= \omega(1)m(P(1\otimes x)), \text{ using (11)}$$

$$= \omega(1)\tilde{m}(x), \text{ using (12)}$$

and so K is amenable, using 2.4.

b) "$\Longleftarrow$". We suppose M is atomic, α weakly amenable and the existence of a state m of A such that $m((i\otimes\omega)\alpha(x)) = m(x)\omega(1)$ for all $x \in A$ and $\omega \in M_*$. We must prove the amenability of K. Using the same notations and proof as in the previous implication, we obtain that there exists a state $\widetilde{m}$ of M such that

(13) $\widetilde{m}((i\otimes\omega)\Gamma(x)) = \widetilde{m}(x)\omega(1)$ for all $x\in M$ and $\omega\in M_*^+$ such that $z(s(\omega))$ is a minimal projection in $Z(M)$

In order to obtain the above equality for every $x\in M$ and $\omega\in M_*$, it is sufficient to prove that

$$\widetilde{m}((i\otimes\omega_\xi)\Gamma(x)) = \widetilde{m}(x)\omega_\xi(1), \text{ for all } x\in M \text{ and } \xi\in H_\phi$$

Because M is atomic, $1=\sum_{i\in I} e_i$, where e_i are minimal central, mutually orthogonal projections in $Z(M)$. So, if $\xi\in H_\phi$, then $\xi=\sum_{i\in I} e_i\xi$ and we may find a sequence $\{\xi_n\}_{n\in N}\subset H_\phi$, with:

$$\|\xi_n-\xi\| \xrightarrow[n\to\infty]{} 0$$

and such that for every $n\in N$

$$\xi_n = \sum_{i\in F_n} e_i\xi,$$

where $F_n\subset I$, F_n is a finite set and $e_i\xi\neq 0$ for $i\in F_n$. Then, for every $x\in M$ we have

$$\omega_{\xi_n}(x) = \sum_{i\in F_n}\omega_{e_i\xi}(x).$$

$$s(\omega_{e_i\xi})\leq e_i, \text{ for all } i\in F_n \text{ and } n\in N$$

so

$$z(s(\omega_{e_i\xi}))\leq e_i, \text{ for all } i\in F_n \text{ and } n\in N$$

and, because $0\neq z(s(\omega_{e_i\xi}))$ and e_i is a minimal projection in $Z(M)$ we obtain

$$z(s(\omega_{e_i\xi})) = e_i, \text{ for all } i\in F_n \text{ and } n\in N$$

Then, for every $x\in M$,

(14) $\widetilde{m}((i\otimes\omega_{\xi_n})\Gamma(x)) = \widetilde{m}(\sum_{i\in F_n}(i\otimes\omega_{e_i\xi})\Gamma(x))$

$$= \sum_{i \in F_n} \tilde{m}((i \otimes \omega_{e_i \xi})^{\Gamma}(x))$$

$$= \sum_{i \in F_n} \omega_{e_i \xi}(1) \tilde{m}(x), \text{ using } (13)$$

$$= \omega_{\xi_n}(1) \tilde{m}(x)$$

Because $\xi_n \xrightarrow[n \to \infty]{} \xi$, for every $\varepsilon > 0$ there exists a $n_\varepsilon \in N$ such that $\|\xi_n - \xi\| \leq \frac{\varepsilon}{2c}$, for all $n \geq n_\varepsilon$, where $c = \max\{\|\xi\|, \sup_{n \in N} \|\xi_n\|\}$. Then, for $n \geq n_\varepsilon$ and $x \in M$ we have

$$|\omega_{\xi_n}(x) - \omega_\xi(x)| = |(x\xi_n|\xi_n) - (x\xi|\xi)|$$

$$\leq |(x\xi_n|\xi_n) - (x\xi|\xi_n)| + |(x\xi|\xi_n) - (x\xi|\xi)|$$

$$\leq \|x\| \cdot \|\xi_n - \xi\| \cdot \|\xi\| + \|x\| \cdot \|\xi\| \cdot \|\xi_n - \xi\| \leq \varepsilon \|x\|$$

so $\|\omega_{\xi_n} - \omega_\xi\| \xrightarrow[n \to \infty]{} 0.$

Then, for every $x \in M \otimes M$ fixed

$$\|(i \otimes \omega_{\xi_n})(x) - (i \otimes \omega_\xi)(x)\| \xrightarrow[n \to \infty]{} 0$$

so, if we pass to the limit in relation (14) we obtain

$$m((i \otimes \omega_\xi)^{\Gamma}(x)) = m(x)\omega_\xi(1), \text{ for every } x \in M,$$

that is K is amenable.

3.6. COROLLARY. Let K be a Katz algebra of discrete type, $A \subset \mathcal{B}(H)$ a von Neumann algebra and $\alpha : A \to A \otimes M$ an action of K on A. Then K is amenable if and only if α is weakly amenable and there exists a state m of A such that

$$m((i \otimes \omega)\alpha(x)) = m(x)\omega(1), \text{ for all } x \in A \text{ and } \omega \in M_*$$

PROOF. For every Katz algebra $K = (M, \Gamma, k, \phi)$ of discrete type, M is atomic (cf. 2.7), so we can apply Theorem 3.5 b).

3.7. COROLLARY. A Katz algebra K is amenable if and only if there exists a von Neumann algebra A such that the trivial action of K on A : $x \to x \otimes 1_M$ is amenable.

3.8. PROPOSITION. Let α be an amenable action of the Katz algebra $K = (M, \Gamma, k, \phi)$ on a von Neumann algebra A and B a von Neumann subalgebra of A such that $\alpha(B) \subset B \otimes M$. We suppose that there exists a conditional expectation P from A to B such that $(i \otimes \omega) \circ P = P \circ (i \otimes \omega)$ for all $\omega \in M_*$. Then the action of K on B obtained by restriction of α is amenable.

PROOF. Because α is amenable, there exists a conditional expectation $E : A \otimes M \to A$ such that

$$\omega(1)[(i \otimes \omega)\alpha] \circ E = E \circ [((i \otimes \omega)\alpha) \otimes ((i \otimes \omega)\Gamma)], \text{ for all } \omega \in M_*$$

Then if we denote by E' the restriction of $P \circ E$ at $B \otimes M$, E' is a conditional expectation from $B \otimes M$ onto B; using the above equality we obtain

$$\omega(1)[(i \otimes \omega)(\alpha|_B)] \circ E' = E' \circ [((i \otimes \omega)(\alpha|_B)) \otimes ((i \otimes \omega)\Gamma)], \text{ for all } \omega \in M_* ,$$

so $\alpha|_B$ is amenable.

3.9. PROPOSITION. Let A,B be two von Neumann algebras and α an amenable action of the Katz algebra K on A. Then

(i) $i_B \otimes \alpha$ is an amenable action of K on $B \otimes A$

(ii) $(i \times c)(\alpha \otimes i_B)$ is an amenable action of K on $A \otimes B$, where $c : M \otimes B \to B \otimes M$ is defined by $c(x \otimes b) = b \otimes x$, for all $x \in M$ and $b \in B$.

The proof is straightforward.

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