DERIVATIONS ON ALGEBRAIC GROUPS, I :

LINEAR GROUPS

by

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§0. Introduction

Let F be an algebraically closed field of characteristic zero and let G be an irreducible affine algebraic F-group; our main object of study will be the space $\Delta(G)$ of (non necessary F-linear!) derivations of the coordinate algebra $\mathcal{P}(G)$ which take F into itself and "commute" with comultiplication, counit and antipode. We have two motivations for this study. First, the subspace $\Delta(G/F)$ of all F-linear derivations in $\Delta(G)$ can be identified with $\mathcal{L}(\mathcal{A}ut\ G)$, the Lie algebra of the automorphism functor of G. It is a fact that $\mathcal{A}ut$ G need not be representable on the category of all F-schemes, but (as shown by Borel and Serre [BS]) it is representable when restricted to the category of reduced F-schemes (by some locally algebraic group, call it Aut G). In fact there exist examples of G's (e.g. $G=G_a \times G_m$) for which the map between the corresponding Lie algebras λ: $\mathcal{L}(\text{Aut } G) \to \mathcal{L}(\mathcal{A}ut\ G)$ is not surjective (cf. [BS]) (in other words for which there exist "infinitesimal automorphisms" which do not come from an algebraic group action). The results of this paper will provide a better understanding of the above Lie algebra map λ. For instance we will prove that there exists a complex which is exact in the first two terms:

$$0 \to \mathcal{L}(\text{Aut } G) \to \mathcal{L}(\mathcal{A}ut\ G) \xrightarrow{\log} \text{Hom}(u/u\cap[g,g],\ r/u) \xrightarrow{\text{kod}} H^2(u,u)$$

where $g=\mathcal{L}(G)$ $r=\mathcal{L}(R)$, $u=\mathcal{L}(U)$, R being the radical of G and U being the unipotent radical of G. As we shall see log will be induced from the natural operation of $\Delta(G)$ on r while kod is related to the geometry of linear subspaces of the variety of all Lie algebra multiplications on u.

Our second motivation is provided by the fact (proved in [B_3]) that any irreducible linear Δ-algebraic group (in the sense of [C_1]) of finite transcendence degree (i.e. with tr.deg. $\mathcal{U}\langle\Gamma\rangle/\mathcal{U}<\infty$) has a finitely generated (in the non-differential sense) coordinate algebra $\mathcal{U}\{\Gamma\}$ hence derives from some affine algebraic $\mathcal{U}$-group G equipped with m commuting elements of $\Delta(G)$. The results of this paper will provide in particular a complete description (both from Δ-algebraic as well as analytic viewpoint) of all Δ-algebraic groups as above for which either the radical is nilpotent or the unipotent radical of $\mathcal{G}(\mathcal{U}\{\Gamma\})$ is commutative.

Note that the two kinds of applications mentioned above are related since any G with non-surjective λ provides "non-trivial" examples of Δ-algebraic groups (e.g. $G=G_a \times G_m$ leads to the Δ-algebraic group $\Gamma=\{yy''-(y')^2=0\}\subset GL_1(\mathcal{U})$ in notations of [C_1]).

Our paper is organized as follows; sections 1-4 are concerned with $\Delta(G)$ while sections 5-8 are devoted to applications to Δ-algebraic groups. Let's discuss our results in some detail. First we shall be concerned with the natural map $\log:\Delta(G)\to W(G)$ (induced by "logarithmic derivative") where W(G) is the space of all group homomorphisms from the group $X_m(G)=\mathrm{Hom}(G,G_m)$ of multiplicative characters of G to the group $X_a(G)=\mathrm{Hom}(G,G_a)$ of additive characters of G. We will prove that ker log is precisely the space $\Delta(G,\text{fin})$ of all derivations in $\Delta(G)$ which are locally finite on $\mathcal{P}(G)$; equivalently $\Delta(G,\text{fin})$ is precisely the space of all derivations in $\Delta(G)$ preserving the ideal of the unipotent radical of G. Now an element of $\mathcal{P}(G)$ is called a weight of G if it is a multiplicative character of G (i.e. a group like element) whose restriction to some (equivalently any) maximal torus T of the radical of G is a weight for the action of T on $\mathcal{P}(G)$ (by inner automorphisms). We will show that if all derivations in $\Delta(G)$ kill all the weights of G (this happens if either the radical of G is nilpotent or if the unipotent radical of G is commutative) then log has a section defined on its image (which we call exp) such that Im exp is an abelian ideal so $\Delta(G)$ is the semidirect product of $\Delta(G,\text{fin})$ by Im exp; moreover if $F=\mathbb{C}$, $\mathcal{L}(\text{Aut } G)\subset\mathcal{L}(\text{Aut } G^{an})$ where G^{an} is the underlying analytic group of G.

Now W(G) identifies with $\text{Hom}(u/u\cap[g,g],r/u)$; using this identification and a purely Lie algebraically defined map $\text{Hom}(u/[r,r],u/r)\rightarrow H^2(u,u)$ we get a map kod: $W(G)\rightarrow H^2(u,u)$. we will prove that $\ker(\text{kod}\circ\log)$ is precisely the space of all derivations in $\Delta(G)$ vanishing on the smallest algebraically closed field of definition F_G of G (the existence of F_G follows from [B_2]; we will also note that $F_G=F^{\Delta(G,\text{fin})}$). In sections 5-8 we deal with irreducible linear Δ-algebraic groups of finite transcendence degree (called here f-groups). Using our theory we will prove that such an f-group Γ is splittable (cf. [B_3] i.e. Δ-isomorphic to a Δ-algebraic group of the form $\Gamma^*\cap GL_n(\mathcal{K})$ where Γ^* is a $\mathcal{K}$-closed subgroup of $GL_n(\mathcal{U})$, in notations of [C_1], cf. also (5.1)) if and only if all group-like elements of the coordinate algebra $\mathcal{U}\{\Gamma\}$ are constants. We also prove that Γ is semi-splittable (i.e. Δ-isomorphic to $\Gamma^{*}\cap\{\delta_i y_{jk}-P_{ijk}=0\}$ with Γ^{*} as above and P_{ijk} non-differential polynomials in the y_{jk}'s) if and only if kod log $\delta_i=0$ for all i where we view δ_i as elements in $\Delta(G)$, $G=\mathcal{G}(\mathcal{U}\{\Gamma\})$ (cf. notations in [B_3], see also (5.3)). Next we will give a precise description of the set of Δ-isomorphism classes of f-groups all of whose weights are constant (the weights of Γ are by definition the weights of $\mathcal{G}(\mathcal{U}\{\Gamma\})$ viewed as elements of $\mathcal{U}\{\Gamma\}$); these groups include all Γ's for which either the radical is nilpotent or the unipotent radical of $\mathcal{G}(\mathcal{U}\{\Gamma\})$ is commutative. We will also prove that if Γ is an f-group all whose weights are constant which is defined over an algebraically closed Δ-field $\mathcal{F}$ then there exist a Picard-Vessiot extension $\mathcal{F}_1/\mathcal{F}$ and an intermediate Δ-field $\mathcal{F}_1\subset\mathcal{E}\subset\mathcal{F}_1\langle\Gamma\rangle$ such that $\mathcal{E}/\mathcal{F}_1$ is split (i.e. generated by constants) and $\mathcal{F}_1\langle\Gamma\rangle/\mathcal{E}$ is generated by exponential elements, in particular all three extensions $\mathcal{F}_1\langle\Gamma\rangle/\mathcal{E}$, $\mathcal{E}/\mathcal{F}_1$, $\mathcal{F}_1/\mathcal{F}$ have no movable singularity (NMS) in the sense of [B_1]p.5. Finally we shall define an analytic concept, namely that of Painlevé extension of Δ-fields (corresponding roughly speaking to extensions "arrising" from Painlevé foliations of the first kind in the sense of [GS]) and prove that if Γ is an f-group defined over $\mathcal{F}$ ($\mathcal{F}$ algebraically closed) all of whose weights are constant then $\mathcal{F}\langle\Gamma\rangle/\mathcal{F}$ is a Painlevé extension. Our analytic results represent our efforts to understand some very interesting remarks and conjectures made by P.Cassidy in a letter to the author [C_4]; we are endebted to P.Cassidy for her stimulating suggestions and comments on a preliminary version of this paper. We are also endebted to O.Laudal for his comments on deformations of Lie algebras.

We close our introduction by noting that most results of this paper can be extended (in a non-trivial way) to the case of non-linear algebraic groups and non-linear differential algebraic groups. This will be done in a subsequent paper.

In what follows we fix some notations and conventions.

(0.1) Terminology of affine algebraic F-groups is borrowed from [H]; however we shall also look at affine algebraic F-groups as "group schemes of finite type over F". We will often denote by the same letter an affine algebraic F-groups and its "underlying" abstract group (= its group of F-points). Recall that $\mathcal{L}(A)$, $\mathcal{L}(G)$ denote the Lie algebras associated to an associative algebra A and to an affine algebraic F-group G respectively. $\mathcal{P}(G)$ will denote the coordinate algebra of G. $\mathcal{G}(H)$ will denote the affine algebraic F-group associated to an affine Hopf algebra H. Lie algebras of algebraic groups G,R,U,... will be sometimes denoted by g,r,u,...

(0.2) In sections 1-4 terminology of differential algebra is that from $[B_1]$ while in sections 5-8 we use Kolchin's terminology $[K_1][C_1][B_3]$. So in sections 1-4 if Δ will be an arbitrary (non necessary finite) set of (non necessary commuting) derivation operators we will speak about Δ-fields, Δ-F-vector spaces, ... Recall that a Δ-F-vector space over a Δ-field F is an F-vector space together with a map $\Delta \to End_Q V$, $\delta \mapsto \delta_V$ with the property that $\delta(\lambda x) = (\delta\lambda)x + \lambda\delta x$ for all $\delta \in \Delta$, $\lambda \in F$, $x \in V$, where we have writen δv instead of $\delta_V v$ for all $v \in V$. Recall that if V,W are Δ-F-vector spaces then $V \otimes W$ and Hom(V,W) have natural structures of Δ-F-vector spaces ($\delta(x \otimes y) = \delta x \otimes y + x \otimes \delta y$ for $\delta \in \Delta$, $x \in V$, $y \in W$ and $(\delta f)(x) = \delta(f(x)) - f(\delta x)$ for $\delta \in \Delta$, $f \in Hom(V,W)$, $x \in V$). By a Δ-Lie F-algebra we understand a Δ-F-vector space h which is a Lie F-algebra such that the multiplication map $h \otimes h \to h$ is a Δ-map (this is the concept from $[B_1]$ and is different from that of Δ-F-Lie algebra in $[C_1][K_1]$), A Δ-F-vector space is called locally finite if it is a union of finite dimensional Δ-F-vector subspaces. If V and W are locally finite so is $V \otimes W$ but Hom(V,W) (and even the dual $V^{o}=$ $=$Hom(V,F)) won't be in general.

1. The space $\Delta(G)$ and the map λ.

(1.1) Let k be a field of characteristic zero, F an algebraically closed field containing k and let G be an affine algebraic F-group. Denote by $\Delta(G)$ the F-vector space of all k-derivations $\delta : \mathcal{P}(G) \longrightarrow \mathcal{P}(G)$ enjoying the following properties:

1) $\delta(F) \subset F$

2) $\mu \circ \delta = (\delta \otimes 1 + 1 \otimes \delta) \circ \mu : \mathcal{P}(G) \to \mathcal{P}(G) \otimes \mathcal{P}(G)$

3) $S \circ \delta = \delta \circ S : \mathcal{P}(G) \longrightarrow \mathcal{P}(G)$

4) $\varepsilon \circ \delta = \delta \circ \varepsilon : \mathcal{P}(G) \longrightarrow F$

where μ, S, ε are the comultiplication, antipode and counit respectively on $\mathcal{P}(G)$ [Sw]. Then $\Delta(G)$ is also a Lie k-algebra equiped with a natural map $\partial : \Delta(G) \longrightarrow \mathrm{Der}(F/k)$ and with respect to this structure it is a Lie space over F (i.e. $[\lambda\delta_1, \delta_2] = \lambda[\delta_1, \delta_2] - (\delta_2\lambda)\delta_1$ for $\lambda \in F$, $\delta_1, \delta_2 \in \Delta(G)$ where $\delta_2\lambda = \partial(\delta_2)(\lambda)$, see [NW], [$C_2$]). For any intermediate field E between k and F we may consider the Lie subspace $\Delta(G/E)$ of $\Delta(G)$ consisting of all $\delta \in \Delta(G)$ which vanish on E: then $\Delta(G/E)$ is a Lie E-algebra. The F-Lie algebra $\Delta(G/F)$ has a remarkable interpretation in terms of the automorphism functor of G.

Indeed, let $\mathcal{A}ut\, G : \{F\text{-schemes}\} \to \{\text{groups}\}$ be the functor defined by $S \mapsto \mathrm{Aut}(G \times S/S)$. This functor is not generally representable cf. [BS]; its restriction to $\{$reduced F-schemes$\}$ is however representable cf. [BS] by a locally algebraic group scheme, call it Aut G, with affine connected component of the identity $\mathrm{Aut}^0 G$. We may view Aut G as a functor $\{F\text{-schemes}\} \to \{\text{groups}\}$ by identifying it with its functor of points. Then there is an obvious homomorphism $\mathrm{Aut}\, G \to \mathcal{A}ut\, G$ inducing a homomorphism $\lambda : \mathcal{L}(\mathrm{Aut}\ G$

$\to \mathcal{L}(\mathcal{A}ut\ G)$ (here if $\mathcal{A}$ is any functor $\{$F-schemes$\}\to\{$groups$\}$, $\mathcal{L}(\mathcal{A})$ is defined to be the kernel of the map $\mathcal{A}(\mathrm{Spec}\ F_\varepsilon) \to \mathcal{A}(\mathrm{Spec}\ F)$ induced by projection of the ring $F_\varepsilon = F \oplus \varepsilon F$ of dual numbers onto F given by $\varepsilon \mapsto 0$; $\mathcal{L}(\mathcal{A})$ is apriori only a group, not a Lie algebra cf [DG]). Now the map $\theta \mapsto id + \varepsilon\theta$ clearly identifies $\Delta(G/F)$ with $\mathcal{L}(\mathcal{A}ut\ G)$ making the latter a Lie F-algebra and making the map $\mathcal{L}(\mathrm{Aut}\ G) \to \mathcal{L}(\mathcal{A}ut\ G)$ a Lie algebra map.

An important role in our paper will be played by the set $\Delta(G,\mathrm{fin})$ of all locally finite derivations in $\Delta(G)$ (here $\delta \in \Delta(G)$ is called locally finite if $\mathcal{O}(G)$ is a locally finite δ-F-vector space). Apriori this set is not even a vector subspace of $\Delta(G)$; but as we shall see below (cf. Theorem(2.1)) it is in fact a Lie subspace of $\Delta(G)$.

(1.2) PROPOSITION. The map $\lambda : \mathcal{L}(\mathrm{Aut}\ G) \to \mathcal{L}(\mathcal{A}ut G) = \Delta(G/F)$ is injective and its image equals $\Delta(G/F) \cap \Delta(G,\mathrm{fin})$ (viewed as a subset of $\Delta(G)$).

Proof. Start with a preparation. Assume W is a finite dimensional vector subspace of $\mathcal{O}(G)$ generating $\mathcal{O}(G)$ as an F-algebra and let $\mathcal{A}ut(G,W)$: $\{$affine F-schemes$\}\to\{$groups$\}$ be the subfunctor of $\mathcal{A}ut\ G$ defined by $S = \mathrm{Spec}\ B \mapsto \{f \in \mathrm{Aut}(G \times S/S);\ f^*: \mathcal{O}(G) \otimes B \to \mathcal{O}(G) \otimes B$ preserves $W \otimes B\}$. We claim that $\mathcal{A}ut(G,W)$ is representable; note that the affine group scheme $\mathrm{Aut}(G,W)$ representing it is reduced by [Sw] p.280. To prove our claim let W_1 be the intersection of all sub-F-coalgebras of $\mathcal{O}(G)$ containing $W_0 = W + SW$; by [Sw] W_1 is a finite dimensional coalgebra. Now define inductively the increasing sequence of subspaces W_i of $\mathcal{O}(G)$ by the formula $W_{i+1} = \mu(W_i \otimes W_i)$ for $i \geq 1$ and define functors $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \ldots$ from $\{$affine F-schemes$\}$ to $\{$groups$\}$ as follows. We let $\mathcal{A}_0(\mathrm{Spec}\ B)$ be the group of those B-linear automorphisms σ_o of $W_o \otimes B$ such that $\sigma_o \circ S_B = S_B \circ \sigma_o$, where $S_B = S \otimes 1_B$. For $i \geq 1$, let $\mathcal{A}_i$ (Spec B) be the group of those B-linear automorphisms σ_i of $W_i \otimes B$ such that $\sigma_i(W_{i-1} \otimes B) = W_{i-1} \otimes B$, $\sigma_i|_{W_{i-1} \otimes B} \in \mathcal{A}_{i-1}(\text{Spec } B)$ and $\sigma_i \circ \mu_B = \mu_B \circ (\sigma_{i-1} \otimes \sigma_{i-1})$ where $\mu_B = \mu \otimes 1_B$. We have canonical restriction maps $\mathcal{A}_i \to \mathcal{A}_{i-1}$ for all $i \geq 1$. Now clearly all $\mathcal{A}_i$'s are representable by affine algebraic F-groups A_i, hence we have a projective system $\dots \to A_i \to A_{i-1} \to \dots \to A_1 \to A_o$. One checks that $\mathcal{A}ut(G,W) = \varprojlim \mathcal{A}_i$. Consequently $\mathcal{A}ut(G,W)$ is represented by $\text{Spec}(\varinjlim \mathcal{O}(A_i))$ and our claim is proved.

Let's prove that $\text{Im}\,\lambda = \Delta(G/F) \cap \Delta(G,\text{fin})$. The inclusion "$\subset$" is clear. Conversely if $\delta \in \Delta(G/F) \cap \Delta(G,\text{fin})$ we may choose W above such that $\delta W \subset W$. Then $id + \varepsilon\delta \in \mathcal{A}ut(G,W)(\text{Spec } F_\varepsilon)$, hence we get a morphism $f:\text{Spec } F_\varepsilon \to \text{Aut}(G,W)$ such that $id + \varepsilon\delta = f^* \varphi_{G,W}$ where $\text{Aut}(G,W)$ is the affine group scheme representing $\mathcal{A}ut(G,W)$ and $\varphi_{G,W}$ is the universal $\text{Aut}(G,W)$-automorphism of $G \times \text{Aut}(G,W)$. Now $\text{Aut}(G,W)$ being reduced there exists a morphism $h:\text{Aut}(G,W) \to \text{Aut } G$ such that $\varphi_{G,W} = h^* \varphi_G$ where φ_G is the universal Aut G-automorphism of $G \times \text{Aut } G$. Consequently $id + \varepsilon\delta = (h \circ f)^* \varphi_G$ hence $\delta \in \text{Im}\,\lambda$.

Finally, let's prove that λ is injective. Let $\text{Aut}^o G = \text{Spec } R$; we may choose a finitely dimensional subspace W of $\mathcal{O}(G)$ generating $\mathcal{O}(G)$ as an F-algebra such that $\varphi_G^*(W \otimes R) = W \otimes R$ (here $\varphi_G^*: \mathcal{O}(G) \otimes R \to \mathcal{O}(G) \otimes R$ is induced by φ_G). Exactly as above, there exists a morphism $h:\text{Aut}(G,W) \to \text{Aut } G$ such that $h^* \varphi_G = \varphi_{G,W}$. There is also a natural morphism $c:\text{Aut}^o G \to \text{Aut}(G,W)$ defined at the level of S-points by $(\text{Aut}^o G)(S) \to \text{Aut}(G,W)(S)$, $s \mapsto \tilde{s}$ where $s^* \varphi_G = \tilde{s}^* \varphi_{G,W}$. Consider the affine group scheme $A = \text{Aut}(G,W) \times_{\text{Aut } G} \text{Aut}^o G$; note that the projection $p_1: A \to Aut(G,W)$ is a closed embedding. Now the map $A \xrightarrow{p_2} Aut^o G \xrightarrow{c} Aut(G,W)$ equals the map $p_1: A \to Aut(G,W)$ for if $(f,f') \in A(S)$ is an S-point of A we have $h \circ f = i \circ f'$ (where $i: Aut^o G \to Aut\, G$ is the inclusion) so the image of (f,f') via $(c \circ p_2)(S)$ is a map $s \in Aut(G,W)(S)$ such that $s^* \varphi_{G,W} = f'^* i^* \varphi_G = f^* h^* \varphi_G = f^* \varphi_{G,W}$; consequently $s=f$ by the universality of $Aut(G,W)$. We get that p_2 is a closed embedding, so A is an affine algebraic group. Since the map $p_2: A \to Aut^o G$ induces a bijection at the level of F-points this map is an isomorphism. Consequently c is a closed embedding hence $Aut^o G$ is a subfunctor of $Aut(G,W)$ hence of $\mathcal{A}ut\, G$ and injectivity of λ follows.

(1.3) We end this section by reviewing some results from [B_3] which will be used in what follows. Note that in [B_3] we worked in the setting of partial Δ-fields; but there is no difficulty (using for instance [B_1]) to extend these results to the setting of Δ-fields (Δ arbitrary). Let G be an irreducible affine algebraic F-group. Then $g = \mathcal{L}(G)$ has a natural structure of $\Delta(G)$-Lie F-algebra defined as follows. $\mathcal{P}(G)$ is a $\Delta(G)$-F-vector space hence so is its dual $\mathcal{P}(G)^o$. Then with respect to convolution, $\mathcal{P}(G)^o$ becomes a $\Delta(G)$-F-algebra, so $\mathcal{L}(P(G)^o)$ becomes a $\Delta(G)$-Lie F-algebra. Finally one checks that $\mathcal{L}(G)$ (viewed as a subspace of $\mathcal{L}(\mathcal{P}(G)^o)$ cf [H] p.36 is $\Delta(G)$-stable. Now we proved in [B_3] that the Lie algebra r of the radical R of G is a $\Delta(G)$-subalgebra of g. Arguments similar to those in [B_3] also show that if H is an irreducible algebraic subgroup of G whose Lie algebra is a Δ-subalgebra of g (Δ being some subset of $\Delta(G)$) then the defining ideal of H in G is a Δ-ideal in $\mathcal{P}(G)$. In particular the defining ideal of R in G is a $\Delta(G)$-ideal of $\mathcal{O}(G)$. But we should note (and this will be a crucial point here) that the Lie algebra u of the unipotent radical U of G is not in general a $\Delta(G)$-subalgebra of g; nor is the defining ideal of U in G a $\Delta(G)$-ideal in $\mathcal{O}(G)$! The above discussion implies that we have a natural "restriction map" $\Delta(G)\to\Delta(R)$. We will also need the following fact: if $i:G\to G'$ is an isogeny (i.e. a homomorphism of irreducible affine algebraic groups with finite kernel) then there is a natural "lifting map" $i^*: \Delta(G')\to\Delta(G)$ defined as follows. Note that $G\to G'$ is an étale map, hence any k-derivation δ' on $\mathcal{O}(G')$ lifts uniquely to a k-derivation δ on $\mathcal{O}(G)$ (see [B_1]p.13). But if $\delta'\in\Delta(G')$ then δ must belong to $\Delta(G)$ because the map $\mu\circ\delta-(\delta\otimes 1+1\otimes\delta)\circ\mu$ (respectively $S\circ\delta-\delta\circ S$, $\varepsilon\circ\delta-\delta\circ\varepsilon$) is a μ-F-derivation $\mathcal{O}(G)\to\mathcal{O}(G)\otimes\mathcal{O}(G)$ (respectively an S-F-derivation, an ε-derivation) vanishing on $\mathcal{O}(G')$; such a derivation must vanish on the whole of $\mathcal{O}(G)$.

Finally we have:

(1.4) THEOREM [B_3] Assume the radical of G is unipotent. Then $\Delta(G)=\Delta(G,fin)$ and $F^{\Delta(G)}$ is a field of definition for G.

Since this will play a key role in our approach let's say a few words about the proof. We prove in fact that $\mathcal{O}(G)$ is locally finite as a $\Delta(G)$-F-vector space; this will imply the statement about the field of definition (use the method in [B_1], chapter II).

As for local finitness one proceeds as follows: using a version of Kolchin's theorem on the surjectivity of the logarithmic derivative one can replace F by a $\Delta(G)$-field extension of it such that $g=\mathcal{L}(G)$ splits (i.e. it is spanned over F by constants) so

$g = g^{\Delta} \otimes F$, $\Delta = \Delta(G)$. Let $g^{\Delta} = r_o + s_o$ be a decomposition of g^{Δ} with r_o its radical and s_o a complementary semisimple Lie algebra. Then by [H] p.112 $s = s_o \otimes F$ is an algebraic Lie subalgebra of g, $s = \mathcal{L}(S)$, $S \subset G$. Both ideals defining R(=radical of G) and S are Δ-ideals in $\mathcal{P}(G)$; therefore the multiplication map $R \times S \to G$ is a Δ-map and we are reduced to prove that both $\mathcal{P}(R)$ and $\mathcal{P}(S)$ are locally finite. This follows for instance by inspecting their embeddings into the corresponding continuous duals of the universal envelloping algebras of their Lie algebras: for $\mathcal{P}(R)$ the image of this embedding lies in the algebra of R-nilpotent representative functions which is locally finite while for $\mathcal{P}(S)$ the whole continuous dual is locally finite (for bockground see [H]).

2. The maps log and exp

Let G be an irreducible affine algebraic F-group. Put $X_m(G)=\mathrm{Hom}(G,G_m)$, $X_a(G)=\mathrm{Hom}(G,G_a)$; note that $X_m(G)$ and $X_a(G)$ viewed as contained in $\mathcal{P}(G)$ are precisely the group of group-like elements of $\mathcal{P}(G)$ respectively the group of primitive elements of $\mathcal{P}(G)$. Put $W(G)=\mathrm{Hom}(X_m(G), X_a(G))$; since $X_a(G)$ is an F-vector space so is $W(G)$. We define a remarkable subspace $W_o(G)$ of $W(G)$ as follows. Let T be radical of some maximal reductive subgroup of G (equvalently, T=a maximal torus of the radical of G) and let $\Phi(T,\mathcal{P}(G))\subset X_m(T)$ be the set of weights of the action of T on $\mathcal{P}(G)$ by inner automorphisms (i.e. $\Phi(T,\mathcal{P}(G))=\{\chi\in X_m(T);\mathcal{P}(G)_\chi\neq 0\}$). Moreover let $\Phi(G)$ be the subset of $X_m(G)$ consisting of all characters of G whose restriction to T belongs to $\Phi(T,\mathcal{P}(G))$. It can be easily shown that $\Phi(G)$ does not depend on the choice of the maximal reductive subgroup of G. For con-convenience the elements of $\Phi(G)$ will be called simply the weight of G(or of $\mathcal{P}(G)$). Now let $W_o(G)$ be the space of morphisms in $W(G)$ vanishing on $\Phi(G)$. Clearly $W_o(G)=W(G)$ if the radical of G is nilpotent.

Our main result is:

(2.1) THEOREM. There exist F-linear maps $\log:\Delta(G)\to W(G)$ and $\exp:W_o(G)\to\Delta(G/F)$ with the following properties:

1) $\ker\log=\Delta(G,\mathrm{fin})$. Moreover $\Delta(G,\mathrm{fin})$ is a Lie subspace of $\Delta(G)$ and $\mathcal{P}(G)$ is locally finite as a $\Delta(G,\mathrm{fin})$-F-vector space.

2) $\log\circ\exp$ is the natural inclusion $W_o(G)\to W(G)$ and $\mathrm{Im}\exp$ is an abelian subalgebra of $\Delta(G/F)$ such that $[\mathrm{Im}\lambda, \mathrm{Im}\exp]\subset\mathrm{Im}\exp$.

3) For any $\sigma \in \mathrm{Aut}\, G$ and $a \in W_o(G)$ we have $\sigma^{-1}(\exp a)\sigma = \exp \sigma a$ where we also denoted by σ the induced automorphisms of $\mathcal{P}(G)$ and $W_o(G)$.

4) For any field of definition E of G contained in F and for any $\delta \in \mathrm{Der}(F/E)$ upon letting $G \simeq G_E \otimes_E F$ (G_E an E-group) and letting δ^* be the trivial lifting of δ to $\mathcal{P}(G) = \mathcal{P}(G_E) \otimes F$ and to $W_o(G) = W_o(G_E) \otimes F$ we have $[\delta^*, \exp a] = \exp \delta^* a$ for all $a \in W_o(G)$.

(2.2) Let's define the map log. The map exp will be defined later. Note first that if $\chi \in X_m(G)$ and $\delta \in \Delta(G)$ then $\chi^{-1}\delta\chi \in X_a(G)$; indeed $\mu\delta\chi = (\delta \otimes 1 + 1 \otimes \delta)\mu\chi = (\delta \otimes 1 + 1 \otimes \delta)(\chi \otimes \chi) = \delta\chi \otimes \chi + \chi \otimes \delta\chi$ and multiplying this equality by the equality $\mu(\chi^{-1}) = \chi^{-1} \otimes \chi^{-1}$ we get $\mu(\chi^{-1}\delta\chi) = \chi^{-1}\delta\chi \otimes 1 + 1 \otimes \chi^{-1}\delta\chi$ showing that $\chi^{-1}\delta\chi$ is primitive as required. Then define $(\log\delta)(\chi) = \chi^{-1}\delta\chi$.

A similar computation shows that $X_a(G)$ is a $\Delta(G)$-F-vector subspace of $\mathcal{P}(G)$. Hence W(G) has a natural structure of $\Delta(G)$-F-vector space given by $(\delta a)(\chi) = \delta(a(\chi))$ for all $\delta \in \Delta(G)$, $a \in W(G)$, $\chi \in X_m(G)$. In fact W(G) is even a $\Delta(G)$-F-module (i.e. the map $\Delta(G) \to \mathrm{End}_k(W(G))$ is a map of Lie k-algebras). Then one immediately checks that log is a cocycle of $\Delta(G)$ in W(G) in the sense that

$$\log[\delta_1, \delta_2] = \delta_1 \log \delta_2 - \delta_2 \log \delta_1$$

for all $\delta_1, \delta_2 \in \Delta(G)$. In particular this shows that ker log is a Lie subspace.

In what follows we shall repeatedly use the following:

(2.3) Remarks. 1) Let $G = G_1 \rtimes G_2$ be a semidirect product of irreducible affine algebraic groups and identify $\mathcal{P}(G)$ with $\mathcal{P}(G_1) \otimes \mathcal{P}(G_2)$ by the multiplication map $G_1 \times G_2 \to G$. If $X_m(G_1)=1$ then $X_m(G)$ identifies with $X_m(G_2)$ via the identification of $1 \otimes \mathcal{P}(G_2)$ with $\mathcal{P}(G_2)$. On the other hand, if $X_a(G_2)=0$ then $X_a(G)$ identifies with $(X_a(G_1))^{G_2}$ (= G_2-invatiant space of $X_a(G_1)$ with G_2 acting by inner automorphisms on G_1) via the identification of $\mathcal{P}(G_1) \otimes 1$ with $\mathcal{P}(G_1)$.

2) Assume $G_2 \to G_1$ is an isogeny. Then the map $X_m(G_1) \longrightarrow X_m(G_2)$ is injective with finite cokernel and the map $X_a(G_1) \to X_o(G_2)$ is an isomorphism. In particular there is an induced isomorphism $W(G_1) \xrightarrow{\sim} W(G_2)$.

(2.4) To prove Theorem (2.1) we fix some notations: let U be the unipotent radical of G, H a maximal reductive subgroup of G and T the radical of H; put $S=[H,H]$, $M=U \rtimes S$ and $\tilde{G}=M \rtimes T=U \rtimes (S \times T)$. The isogeny $S \times T \to H$ induces an isogeny $\tilde{G} \to G=U \rtimes H$. We write $X=X_m(T)$. Note that the map $X_m(G) \to X$ induced by restriction is injective and has finite kernel; indeed this map is easily seen to identify with the map $X_m(G) \xrightarrow{\sim} X_m(H) \longrightarrow X_m(S \times T) \xrightarrow{\sim} X_m(T)$ cf. (2.3) above and we are done also by (2.3).

Let $\rho : T \times M \to T \times M$ be defined by $\rho(t,m)=(t,tmt^{-1})$. Then the multiplication on G is defined (after identifying G with $M \times T$ as an algebraic variety) by:

$$M \times T \times M \times T \xrightarrow{1 \times \rho \times 1} M \times T \times M \times T \xrightarrow{1 \times \tau \times 1} M \times M \times T \times T \xrightarrow{\mu_M \times \mu_T} M \times T,$$

where $\tau : T \times M \to M \times T$ is the twist map. For $\chi \in X$ we denote by $\mathcal{P}(M)_\chi$, $\mathcal{L}(M)_\chi$ the eigenspaces corresponding to χ (with T acting via ρ); note that we shall always view here $\mathcal{L}(M)$ as the

Lie algebra of left invariant derivations on $\mathcal{P}(M)$. We will still denote by μ the comultiplications on $\mathcal{P}(M)$, $\mathcal{P}(\tilde{G})$ and by $\rho: \mathcal{P}(T) \otimes \mathcal{P}(M) \to \mathcal{P}(T) \otimes \mathcal{P}(M)$, $\tau: \mathcal{P}(T) \otimes \mathcal{P}(M) \to \mathcal{P}(M) \otimes \mathcal{P}(T)$ the maps induced by ρ, τ.

For $\chi \in X$ let $p_\chi : \mathcal{P}(M) \to \mathcal{P}(M)_\chi \subset \mathcal{P}(M)$ be the projection onto the χ-component; then we have the formula $\rho(\chi \otimes a) = \sum_{\chi'} \chi\chi' \otimes p_{\chi'}(a)$ for all $a \in \mathcal{P}(M)$. Moreover, note that $\mathcal{L}(M)_\chi$ is the subspace of $\mathcal{L}(M)$ consisting of all $\theta \in \mathcal{L}(M)$ such that $\theta \circ p_{\chi'} = p_{\chi\chi'} \circ \theta$ for all $\chi' \in X$ (as linear endomorphisms of $\mathcal{P}(M)$).

Finally, for all $\chi \in X$ and for any k-derivation δ of $\mathcal{P}(M) \otimes \mathcal{P}(T)$ such that $\delta F \subset F$ we define k-endomorphisms δ_χ of $\mathcal{P}(M)$ by the formula $\delta_\chi(a) = (1 \otimes q_{\chi^{-1}})\delta(a \otimes 1)$ for all $a \in \mathcal{P}(M)$ where $q_\chi : \mathcal{P}(T) = \bigoplus_{\chi'} F\chi' \to F\chi$ is the canonical projection. In other words we have the formula $\delta(a \otimes 1) = \sum (\delta_\chi a) \otimes \chi^{-1}$. Clearly, δ_1 will be a k-linear derivation of $\mathcal{P}(M)$ while for $\chi \neq 1$, δ_χ are F-linear derivations of $\mathcal{P}(M)$. We put $\delta^0 = \sum_\chi \delta_\chi$.

(2.5) LEMMA. Let $\delta \in \mathrm{Der}(\mathcal{P}(M) \otimes \mathcal{P}(T)/k)$, $\delta F \subset F$ be as in (1.4) above. Then $\delta \in \Delta(G)$ if and only if the following conditions are satisfied:

1) For any $\chi \in X$ we have $\delta(1 \otimes \chi) = a_\chi \otimes \chi$ for some $a_\chi \in X_a(M)^T$.

2) For any $\chi, \chi' \in X$, $\chi \neq 1$ we have

$$(1 \otimes p_{\chi\chi'}) \circ \mu_M \circ \delta_\chi = (1 \otimes \delta_\chi \circ p_{\chi'}) \circ \mu_M$$

3) For any $\chi \in X$ we have

$$(1 \otimes p_\chi) \circ \mu_M \circ \delta_1 = (1 \otimes \delta_1 \circ p_\chi + a_\chi \otimes p_\chi + \sum_{\chi'} \delta_{\chi'} \otimes p_{\chi'\chi}) \circ \mu_M$$

4) $\delta^0 \circ \varepsilon = \varepsilon \circ \delta^0$

5) For all $\chi \in X$ we have

$$\sum_{\chi'} p_{\chi'} \circ \delta_{\chi'\chi} = \sum_{\chi'} S_M \circ \delta_{(\chi')^{-1}\chi^{-1}} \circ S_M \circ p_{\chi'} + a_\chi p_\chi$$

Proof. 1) is equivalent to the fact that $\mu \circ \delta$ and $(\delta \otimes 1 + 1 \otimes \delta) \circ \mu$ agree on X. We claim that 2)+3) are equivalent to the fact that the two maps above agree on $\mathcal{P}(M) \otimes 1$. Indeed for $a \in \mathcal{P}(M)$ we have

$$\mu\delta(a \otimes 1) = \mu(\sum_\chi (\delta_\chi a) \otimes \chi^{-1}) = (1 \otimes \rho \otimes 1)(\sum_\chi (\delta_\chi a)_{(1)} \otimes \chi^{-1} \otimes (\delta_\chi a)_{(2)} \otimes \chi^{-1}) =$$

$$= \sum_{\chi,\chi'} (\delta_\chi a)_{(1)} \otimes \chi^{-1}\chi' \otimes p_{\chi'}((\delta_\chi a)_{(2)}) \otimes \chi^{-1}$$

$$(\delta \otimes 1 + 1 \otimes \delta)\mu(a \otimes 1) = (\delta \otimes 1 + 1 \otimes \delta)(\sum a_{(1)} \otimes \chi' \otimes p_{\chi'}(a_{(2)}) \otimes 1) =$$

$$= \sum_{\chi''} \delta(a_{(1)}) \otimes (\chi'')^{-1}\chi' \otimes p_{\chi'}(a_{(2)}) \otimes 1 +$$

$$+ \sum a_{(1)} a_{\chi'} \otimes \chi' \otimes p_{\chi'}(a_{(2)}) \otimes 1 +$$

$$+ \sum a_{(1)} \otimes \chi' \otimes \delta_\chi(p_{\chi'}(a_{(2)})) \otimes \chi^{-1}$$

Now using the fact that $\mathcal{P}(M) \otimes \mathcal{P}(T) \otimes \mathcal{P}(M) \otimes \mathcal{P}(T)$ is a free $\mathcal{P}(M) \otimes \mathcal{P}(M)$ - module with basis $1 \otimes \chi' \otimes 1 \otimes \chi$ and identifying coefficients we get our claim.

Condition 3) is equivalent to $\delta \circ \varepsilon = \varepsilon \circ \delta$. Finally, using the fact that the antipode on $\mathcal{P}(M) \otimes \mathcal{P}(T)$ is given by $(1 \otimes S_T) \circ \rho^\tau \circ (S_M \otimes 1)$ where $\rho^\tau : \mathcal{P}(M) \otimes \mathcal{P}(T) \to \mathcal{P}(M) \otimes \mathcal{P}(T)$ is obtained by composing ρ with the twists we see using a similar computation that condition 4) is equivalent to $\delta \circ S = S \circ \delta$. Our Lemma is proved.

(2.6) Assume δ satisfies the conditions 1)-5) from (2.5). Then taking in 2) the sum over all χ' we get

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$$(2.6.1)\qquad \mu_M \circ \delta_\chi = (1 \otimes \delta_\chi) \circ \mu_M \qquad \text{for all } \chi \neq 1$$

which says that $\delta_\chi \in \mathcal{L}(M)$. Combining the above formula with 2) in (2.5) we get

$$(1 \otimes p_{\chi\chi'} \circ \delta_\chi) \circ \mu_M = (1 \otimes \delta_\chi \circ p_{\chi'}) \circ \mu_M$$

applying $\varepsilon_M \otimes 1$ to this equality we get that $p_{\chi\chi'} \circ \delta_\chi = \delta_\chi \circ p_{\chi'}$ which says that in fact

$$(2.6.2)\qquad \delta_\chi \in \mathcal{L}(M)_\chi \qquad \text{for all } \chi \neq 1$$

Taking in 3) from (2.5) the sum over all χ we get

$$(2.6.3)\qquad \mu_M \circ \delta_1 = (1 \otimes \delta_1 + \delta^0 \otimes 1 + \sum_\chi a_\chi \otimes p_\chi) \circ \mu_M$$

Summing up (2.6.3) with (2.6.1) for all $\chi \neq 1$ we get

$$(2.6.4)\qquad \mu_M \circ \delta^0 = (1 \otimes \delta^0 + \delta^0 \otimes 1 + \sum_\chi a_\chi \otimes p_\chi) \circ \mu_M$$

Finally taking in 5) from (2.5) the sum over all χ we get

$$(2.6.5)\qquad \delta^0 = S_M \circ \delta^0 \circ S_M + \sum_\chi a_\chi p_\chi$$

In particular we see that if $a_\chi = 0$ for all χ then $\delta^0 \in \Delta(M)$.

(2.7) Let us prove that ker log $= \Delta(G, \text{fin})$. To prove that ker log is contained $\Delta(G, \text{fin})$ consider the lifting map $\Delta(G) \to \Delta(\tilde{G})$ (1.3) and since the maps $\Delta(G) \to \Delta(\tilde{G}) \to W(\tilde{G})$ and $\Delta(G) \to W(G) \simeq W(\tilde{G})$ are equal it is sufficient to prove the corresponding assertion for G.

Let $\delta\in\Delta(\tilde{G})$ with $\log\delta=0$, hence in the notations of (2.6) we have $a_\chi=0$ for all $\chi\in X$. Since $\mathcal{L}(M)_\chi=(\mathcal{L}(U)\oplus\mathcal{L}(S))_\chi=$ $=\mathcal{L}(U)_\chi$ for $\chi\neq 1$ we have by (2.6) above that $\delta_\chi\in\mathcal{L}(U)$ for $\chi\neq 1$ and $\delta_1\in\Delta(M)+\mathcal{L}(U)$ (the latter sum being taken inside $\mathrm{Der}(\mathcal{P}(M)/k)$).

<u>Claim 1.</u> There exists a finite dimensional F-vector subspace V of $\mathcal{P}(M)$ generating $\mathcal{P}(M)$ as an F-algebra and which is preserved by both $\Delta(M)$ and $\mathcal{L}(M)$.

Indeed, since the radical of M is unipotent, by (1.4), M is defined over $F_o=F^{\Delta(M)}$, hence $M=M_o\otimes_{F_o}F$ for some F_o-group M_o (and clearly $U=U_o\otimes_{F_o}F$, U_o the radical of M_o); moreover by (1.2) and (1.4) any $\delta\in\Delta(M)$ can be written as $\delta=\delta^*+\theta$ where δ^* is the trivial lefting of $\delta/_F$ to $M_o\otimes_{F_o}F$ and $\theta\in\mathcal{L}(\mathrm{Aut}^oM)=$ $=\mathcal{L}(\mathrm{Aut}^oM_o)\otimes_{F_o}F$. Since M_o Aut^oM_o (with Aut^oM_o acting naturally on M_o) acts rationally on $\mathcal{P}(M_o)$ (with M_o acting via left translations) there is a finite dimensional F_o-subspace V_o of $\mathcal{P}(M_o)$ stable under the actions of both M_o and Aut^oM_o. Our claim follows by putting $V=V_o\otimes F$.

<u>Claim 2.</u> $[\Delta(M),\mathcal{L}(U)]\subset\mathcal{L}(U)$.

Since $[\delta^*,\mathcal{L}(U)]\subset\mathcal{L}(U)$ for all $\delta\in\Delta(M)$ it is sufficient to check that $[\mathcal{L}(\mathrm{Aut}\ M),\mathcal{L}(U)]\subset\mathcal{L}(U)$ (in $\mathrm{End}_F(\mathcal{P}(M))$). It is sufficient to check the some relation viewed in $\mathrm{End}_F(V)$. But this follows from the fact that $[\mathrm{Aut}^oM,U]\subset U$ in the group $M\rtimes\mathrm{Aut}^oM$.

<u>Claim 3.</u> Let $N=\dim V$ and $\theta_1,\dots,\theta_n\in\Delta(M)+\mathcal{L}(U)$. If card $\{i;\ \theta_i\in\mathcal{L}(U)\}\geq N$ then $\theta_1\theta_2\dots\theta_n(V)=0$.

Indeed if $\theta_1,\dots,\theta_N \in \mathcal{L}(U)$ then use the fact that with respect to some basis of V, the restrictions of θ_i to V are upper triangular with zero on the diagonal for $1 \leq i \leq N$. The general case follows using Claim 2 and an induction on the number $(i_1-1)+ +(i_2-2)+\dots+(i_N-N)$ where $i_1,\dots,i_N$ are numbers chosen such that $\theta_{i_j} \in \mathcal{L}(U)$ for $1 \leq j \leq N$ and $\theta_p \notin \mathcal{L}(U)$ for all $p < i_N$ for which $p \notin \{i_1,\dots,i_N\}$.

Now let $\Phi^* = \Phi^*(T, \mathcal{L}(M))$ be the set of non-trivial weights of the T-action on $\mathcal{L}(M)$ i.e. $\Phi^* = \{\chi \in X; \chi \neq 1, \mathcal{L}(M)_\chi \neq 0\}$; it is a finite set. Denote by $\Phi^{(N)}$ the F-span of all products of the form $\chi_1\chi_2\dots\chi_N$ with $\chi_i \in \Phi \cup \{1\}$.

<u>Claim 4.</u> For all $a \in V$ we have that $a \otimes 1$ is a Picard-Vessiot element for $\Delta = \ker \log$. Recall that an element x in a Δ-field extension $\mathcal{E}$ of a Δ-field $\mathcal{F}$ is called Picard-Vessiot if it is contained in a finite dimensional Δ-$\mathcal{F}$-vector subspace of $\mathcal{E}$; in our case $\mathcal{F}=(F,\Delta)$ and $\mathcal{E}=(\Omega(\mathcal{O}(\tilde{G})),\Delta))$. Assuming for a moment that this holds let's see that Claim 4 implies that $\ker \log \subset \Delta(\tilde{G}, \text{fin})$. Indeed, since $1 \otimes \mathcal{O}(T)$ still consists of Picard-Vessiot elements and since $\mathcal{O}(\tilde{G})$ is generated as an F-algebra by $V \otimes 1$ and $1 \otimes \mathcal{O}(T)$ it follows [BB] that any element of $\mathcal{O}(\tilde{G})$ is a Picard-Vessiot element. Now to check claim 4 note simply that

$$\delta^n(a \otimes 1) = \Sigma\, \delta_{\chi_n}\delta_{\chi_{n-1}}\dots\delta_{\chi_1}a \otimes \chi_1^{-1}\chi_2^{-1}\dots\chi_n^{-1}$$

the sum being taken for all χ_i's belonging to $\Phi^* \cup \{1\}$. By Claim 3 each product $\delta_{\chi_n}\dots\delta_{\chi_1}$ vanishes unless card $\{i;\ \chi_i \in \Phi^*\} \leq N$. Consequently $\delta^n(a \otimes 1) \in V \otimes \Phi^{(N)}$ and we are done.

To conclude the proof of assertion 1) in theorem (2.1) we are left to prove that $\ker \log \supset \Delta(G, \text{fin})$. Assume the contrary and take $\delta \in \Delta(G,\text{fin})$ with $(\log\delta)(\chi) = a \neq 0$ for some $X_m(G)$. By (2.2) $X_a(G)$ is stable under $\Delta(G)$. By [H] p.88, the symmetric algebra $S'(X_a(G))$ embeds into $\mathcal{O}(G)$ and then of course each homogeneous component $S^n(X_a(G))$ is stable under $\Delta(G)$. By induction we get that

$$\delta^n\chi = (a^n + P_n)\chi \quad \text{for } n \geq 0$$

with $P_n \in \bigoplus_{p \leq n-1} S^p(X_a(G))$. By our assumption the family $(\delta^n\chi)_n$ is F-linearly independent; this implies that $(a^n+P_n)_n$ is F-linearly independent. But this is impossible since $a^n \in S^n(X_a(G))$. Assertion 1) in Theorem (2.1) is proved.

(2.8) Now let's define $\exp: W_o(G) \to \Delta(G)$. The action of T on H by left (or, which is the same, right) translations gives an X-gradation $\mathcal{P}(H) = \bigoplus_\chi \mathcal{P}(H)^\chi$; let $p^\chi: \mathcal{P}(H) \to \mathcal{P}(H)^\chi \subset \mathcal{P}(H)$ be the corresponding projections. Identify $Hom(X_m(G), X_a(G))$ with $Hom(X, X_a(U)^H)$. Then any $a \in W_o(G)$ can be viewed as a homomorphism $X \to X_a(U)^H$ vanishing on $\Phi(T, \mathcal{P}(G))$. Moreover identify $\mathcal{P}(G)$ with $\mathcal{P}(U) \otimes \mathcal{P}(H)$ via the multiplication map $U \times H \to G$ and define the F-linear endomorphism exp a of $\mathcal{P}(G)$ by the formula

$$(\exp a)(x \otimes y) = \sum_\chi xa(\chi) \otimes p^\chi(y)$$

for $x \in \mathcal{P}(U)$, $y \in \mathcal{P}(H)$. Clearly exp a is a $\mathcal{P}(U)$-derivation. To check that $\exp a \in \Delta(G)$ it is sufficient to check that its unique lifting δ to $\mathcal{P}(\tilde{G}) = \mathcal{P}(U) \otimes \mathcal{P}(S \times T)$ belongs to $\Delta(\tilde{G})$. But we have

$$\delta(x \otimes z) = \sum_\chi xa(\chi) \otimes \tilde{p}^\chi(z)$$

for all $x \in \mathcal{P}(U)$, $z \in \mathcal{P}(S \times T)$ where $\tilde{p}^\chi: \mathcal{P}(S \times T) \to \mathcal{P}(S \times T)^\chi \subset \mathcal{P}(S \times T)$ is the corresponding projection and $\mathcal{P}(S \times T)^\chi = \mathcal{P}(S) \otimes F\chi$.
In particular, using notations from (2.5) for our δ we see that $\delta_\chi = 0$ for all $\chi \in X$ and $\delta(1 \otimes \chi) = a(\chi) \otimes \chi$. Since a vanishes also on

$\Phi(T,\rho(M))$ we see that for any $\chi\in X$ either $a_\chi=a(\chi)=0$ or $p_\chi=0$ so δ satisfies the conditions 1)-5) in (2.5) hence by (2.5) $\delta\in\Delta(\tilde{G})$. We conclude that exp is well-defined. Clearly log exp is the natural inclusion and Im exp is an abelian Lie subspace of (G/F).

(2.9) To prove the remaining claims of assertions 2) and 3) in Theorem (2.1), note that Aut G is generated by Int G and the group Aut(G,H) of all automorphisms of G preserwing H. Consequently by (12) Im $\lambda=\mathcal{L}$(Aut G) is generated by $\mathcal{L}$(Int G) and $\mathcal{L}$(Aut(G,H)). So it is sufficient to check that for all $a\in W_o(G)$ we have:

(2.9.1) $\sigma^{-1}(\exp a)\sigma=\exp a$ for all $\sigma\in \mathrm{Im}(M\to \mathrm{Int}(G))$

(2.9.2) $[d,\exp a]=0$ for all $d\in \mathrm{Im}(\mathcal{L}(M)\to\mathcal{L}(\mathrm{Int}(G)))$

(2.9.3) $\sigma^{-1}(\exp a)\sigma=\exp \sigma a$ for all $\sigma\in \mathrm{Aut}(G,H)$

(2.9.4) $[d, \exp a]\in \mathrm{Im}\exp$ for all $d\in\mathcal{L}(\mathrm{Aut}(G,H))$.

To prove the assertions above we may assume $G=\tilde{G}$. Start with (2.9.1) and (2.9.2). The action of M on $G=\tilde{G}$ by interior automorphisms is given by the composition of maps:

$$M\times M\times T\xrightarrow{a}M\times M\times T\xrightarrow{b}M\times M\times T\xrightarrow{c}M\times T$$

$$(m,x,t)\mapsto(mx,m,t)\mapsto(mx,tm^{-1}t^{-1},t)\mapsto(mxtm^{-1}t^{-1},t)$$

where a and c are induced by multiplications while b is induced by taking first the antipode on the middle factor and then applying the action by interior automorphisms $M\times T\to M$ of T on M. This immediately implies that for any $z\in\mathcal{P}(M)$ we have that the image of $z\otimes 1$ via the map $a^*b^*c^*:\mathcal{P}(M)\otimes\mathcal{P}(T)\to\mathcal{P}(M)\otimes\mathcal{P}(M)\otimes\mathcal{P}(T)$ belongs to $\mathcal{P}(M)\otimes\mathcal{P}(M)\otimes\langle\Phi(G)\rangle$ where $\langle\Phi(G)\rangle\subset\mathcal{P}(T)$ is the F-linear span of $\Phi(G)=\Phi(T,\mathcal{P}(G))$. This shows that if σ and d are as in (2.9.1) and (2.9.2) we have $\sigma(z\otimes 1)$, $d(z\otimes 1)\in\mathcal{P}(M)\otimes\otimes\langle\Phi(G)\rangle$. This immediately implies that the derivations $\sigma^{-1}(\exp a)\sigma-\exp a$ and $[d,\exp a]$ vanish on $\mathcal{P}(M)\otimes 1$. On the other hand these derivations vanish on $1\otimes\mathcal{P}(T)$ (use the fact that σ is the identity on $1\otimes\mathcal{P}(T)$ and on $X_a(G)$ and d is zero on these spaces) hence they vanish on $\mathcal{P}(M)\otimes\mathcal{P}(T)$ and (2.9.1), (2.9.2) are proved. To prove (2.9.3) let σ still denote the restriction of σ to M and T. Then for $x\in\mathcal{P}(M)$, $\chi\in X_m(T)\subset\mathcal{P}(T)$ we have

$$\sigma^{-1}(\exp a)\sigma(x\otimes\chi)=\sigma^{-1}(\exp a)(\sigma x\otimes\sigma\chi)=\sigma^{-1}(a(\sigma\chi)\sigma x\otimes\sigma\chi)=$$
$$=\sigma^{-1}(a(\sigma\chi))x\otimes\chi=(\exp\sigma a)(x\otimes\chi).$$

To prove (2.9.4) let d still denote the derivation induced on $\mathcal{P}(M)$; noting that d kills all elements of $X_m(T)$ we have for x and χ as above

$$[d,\exp a](x\otimes\chi)=xd(a(\chi))\otimes\chi=(\exp da)(x\otimes\chi)$$

So assertions 2) and 3) in (2.1) are proved. A computation similar to the latter proves assertion 4) and our Theorem (2.1) is now completely proved.

In what follows we discuss a remarkable property of exp in the analytic case i.e. if $F=\mathbb{C}$.

Indeed, let G be an irreducible affine algebraic $\mathbb{C}$-group. Then there is a commutative diagram

$$\begin{array}{ccc} \mathcal{L}(\mathrm{Aut}\ G) & \xrightarrow{\ \lambda\ } & \Delta(G/\mathbb{C}) \\ \downarrow & & \downarrow j \\ \mathcal{L}(\mathrm{Aut}\ G^{an}) & \xrightarrow[\ \lambda^{an}\ ]{} & \Delta(G^{an}) \end{array}$$

where G^{an} is the underlying analytic Lie group of G, $\mathrm{Aut}\ G^{an}$ is the Lie group of analytic isomorphisms of G^{an} (whese identity component is algebraisable cf. [HM]) and $\Delta(G^{an})$ is the Lie algebra of analytic vector fields on G^{an} vanishing at $1\in G^{an}$ and for which the multiplication and the inverse map are equivariant. We have the following result (which will play a key role in Section 8):

(2.10) PROPOSITION. In notations above $j(\mathrm{Im}\ \exp)\subset \mathrm{Im}\ \lambda^{an}$.

Proof. Let $a\in W_o(G)=\mathrm{Hom}(X_m(T)/\langle\Phi(T,P(G))\rangle,\ X_a(U)^H)$ where T,U and H are as in (2.4). Let $\chi_1,\dots,\chi_N$ be a basis of $X_m(T)\simeq\mathbb{Z}^N$ and put $a_i=a(\chi_i)$. We shall identify in what follows T with $(\mathbb{C}^*)^N$ by means of $(\chi_1,\dots,\chi_N):T\to(\mathbb{C}^*)^N$. Let us construct a 1-parameter subgroup of analytic automorphisms of G^{an} whose derivative in zero is $j(\exp a)$. Let σ: $:(\mathbb{C}^*)^N\times H\to H$ be the action induced by multiplication and define for each t a map $\varphi_t:U^{an}\rtimes H^{an}\to U^{an}\rtimes H^{an}$ by the formula

(*) $\quad\varphi_t(x,y)=(x,\sigma(e^{ta(x)},y)),\quad x\in U^{an},\ y\in H^{an}$

where we write $a(x)$ for the row vector $(a_j(x))_j$ in $\mathbb{C}^N$ and $e^{ta(x)}$ is the corresponding row vector in $(\mathbb{C}^*)^N$. Since a vanishes on $\Phi(G)$ and since $\chi(e^{ta(x)})=e^{ta(\chi)(x)}$ for all $\chi\in X$, $e^{ta(x)}$ is central in G hence $\varphi_t\in\mathrm{Aut}(G^{an})$; clearly $\varphi_{t_1+t_2}=\varphi_{t_1}\circ\varphi_{t_2}$ so we got a 1-parameter subgroup.

Assume now that in (*) x (resp. y) is a collection of local coordinate on U (resp. on H) while $\lambda=(\lambda_1,\ldots,\lambda_N)$ will be the standard coordinates on $(\mathbb{C}^*)^N$. By $\frac{\partial}{\partial x}$ we hsall mean the column vector $(\frac{\partial}{\partial x_j})_j$ (similarly for $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial \lambda}$). Then consider the vector field on $U^{an} \times H^{an}$ defined by

$$v= (\frac{d\varphi_t}{dt}\Big|_{t=0}) \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = a(x)\, \frac{\partial \sigma}{\partial \lambda}(1,y)\, \frac{\partial}{\partial y}$$

To check that v=j(exp a) take any character $\chi \in X_m(T)$, $\chi(\lambda)=\lambda^{\alpha}$ (α a column of integers) and $b \in \mathcal{P}(H)$ such that $b(\sigma(\lambda,y))=\chi(\lambda)b(y)$ for all λ and y. Applying $\frac{\partial}{\partial \lambda}$ to the latter equality and then putting $\lambda=1$ we get

$$\frac{\partial \sigma}{\partial \lambda}(1,y)\, \frac{\partial b}{\partial y}(y)=\alpha b(y)$$

Multiplying this equality on the left by the row a(x) we get that $(vb)(x,y)=a(x)\alpha b(y)=((\exp a)b)(x,y)$ and we are done.

§3. The map kod

Throughout this section we fix an irreducible affine algebraic F-group G with radical R and unipotent radical U, we pick a maximal torus T of R and put $r=\mathcal{L}(R)$, $u=\mathcal{L}(U)$, $t=\mathcal{L}(T)$.

(3.1) We start with the remark that W(G) naturally identifies with $\mathrm{Hom}(u/[g,g]\cap u, r/u)$. Indeed identify both $\mathcal{L}(G_m)$ and $\mathcal{L}(G_a)$ with the field F via the identification $G_a=\mathrm{Spec}\, F[t]$, $G_m=\mathrm{Spec}\, F[t,t^{-1}]$. Moreover write $G/[G,G]=G^{(a)}\times G^{(s)}$ with $G^{(a)}$ a vector group and $G^{(s)}$ a torus; clearly the natural map $U/U\cap[G,G]\to G^{(a)}$ is an isomorphism while the map $T\to G^{(m)}$ is an isogeny (2.4). We have identifications $\mathcal{L}(G^{(a)})^{\circ}\simeq \mathrm{Hom}(\mathcal{L}(G^{(a)}),\mathcal{L}(G_a))\simeq \mathrm{Hom}(G^{(a)}, G_a)=X_a(G)$ and $\mathcal{L}(G^{(m)})^{\circ}\simeq \mathrm{Hom}(\mathcal{L}(G^{(m)}),\mathcal{L}(G_m))\simeq X_m(G)\otimes$ $\otimes F$, hence we have an identification

$$W(G)=\mathrm{Hom}(X_m(G)\otimes F,\ X_a(G))\simeq \mathrm{Hom}(\mathcal{L}(G^{(m)})^{\circ},\mathcal{L}(G^{(a)})^{\circ})\simeq$$
$$\simeq \mathrm{Hom}(\mathcal{L}(G^{(a)}),\mathcal{L}(G^{(m)}))\simeq \mathrm{Hom}(u/u\cap[g,g],r/u)$$

(3.2) Next let's note that for $\delta\in\Delta(G)$, the image of $\log\delta$ in $\mathrm{Hom}(u/[g,g]\cap u,r/u)$ (still denoted so) has the following particularly simple description: upon letting δ still denote the k-endomorphism induced by δ on r (cf.(1.3)) and by $\pi:r\to r/u$ the natural projection we claim that $(\log\delta)(\hat{x})=\pi(\delta x)$ for all $x\in u$ (where $\hat{x}$ denotes the image of x in $u/[g,g]\cap u$). In particular $\log\delta=0$ if and only if $\delta u\subset u$. To check our claim note that under the identification in (3.1) we have the following formula: $(d\chi)\circ\log\delta=d(\chi^{-1}\delta x)$ for all character $\chi:R/U\to G_m$ (where

$d\chi: = \mathcal{L}(R/U) \to \mathcal{L}(G_m) \simeq F$ is the tangent map of χ, $\chi^{-1}\delta\chi \in X_a(G)$ is viewed as an additive character $U/[G,G] \cap U \to G_a$ and similarily $d(\chi^{-1}\delta\chi)$ is its tangent map).

Identifying now $\mathcal{L}(U/U \cap [G,G])$ (resp. $\mathcal{L}(R/U)$) with a subspace of $P(U/U \cap [G,G])^0$ (resp. $P(R/U)^0$) the above formula reads $(\log\delta)(\hat{x})(\chi) = \hat{x}(\chi^{-1}\delta\chi)$ for all $x \in \mathcal{L}(U)$. But $\hat{x}(\chi^{-1}\delta\chi)$ coincides with the image of χ via the map

$$P(R/U) \to P(R) = P(U) \otimes P(T) \xrightarrow{\delta} P(U) \otimes P(T) \xrightarrow{1 \otimes \varepsilon_T} P(U) \xrightarrow{x} F$$

and our claim is proved.

Next we have the following Lie algebra theoretic construction.

(3.3) LEMMA. Let r be any Lie F-algebra, u an ideal in r containing $[r,r]$ and $s: r/u \to r$ a Lie algebra section of the projection $r \to r/u$. Define the linear map

$$b: \mathrm{Hom}(u/[r,r], r/u)) \to \mathrm{Alt}^2(u,u)$$

($\mathrm{Alt}^2(u,u)$ = space of alternating bilinear maps $u \times u \to u$) by the formula

$$b(\varphi)(x,y) = [s\varphi\hat{x}, y] - [s\varphi\hat{y}, x]$$

for all $\varphi \in \mathrm{Hom}(u/[r,r], r/u)$, $x,y \in u$, where $\hat{x},\hat{y}$ are the images of x,y in $u/[r,r]$. Then the following properties hold for any φ:

1) $b(\varphi) \in Z^2(u,u)$=space of 2-cocycles of u in u.

2) $b(\varphi)$ is a Lie algebra multiplication on u and $(u, b(\varphi))$ is a metaabelian Lie algebra (i.e. solvable in 2 steps).

3) for every $z \in r/u$, ad $s(z)$ induces a derivation of the Lie algebra $(u, b(\varphi))$.

Proof. A computation involving only definitions.

In particular the above Lemma shows that in our specific situation when $u=\mathcal{L}(U)$, $r=\mathcal{L}(R)$, taking s to correspond to our choice of a maximal torus T of R we are provided with a map b as above and hence with a map

$$\beta : \mathrm{Hom}(u/[r,r], r/u) \to H^2(u,u)$$

Noting that $\mathrm{Hom}(u/u\cap[g,g], r/u)$ is a subspace of $\mathrm{Hom}(u/[r,r], r/u)$ we get a map kod: $W(G) \to H^2(u,u)$ by composing β above with the identification isomorphism $W(G) \simeq \mathrm{Hom}(u/u\cap[g,g], r/u)$.

(3.4) PROPOSITION. The smallest algebraically closed field of definition F_G of G contained in F and containing k(cf [B$_2$]) coincides with $F^{\Delta(G,fin)}$. Moreover $\ker(\mathrm{kod}\circ\log) = \Delta(G/F_G)$.

Proof. The equality $F_G = F^{\Delta(G,fin)}$ follows easily from the fact that $\mathcal{P}(G)$ is a locally finite $\Delta(G,fin)$-F-vector space (cf. Theorem (2.1) by using the "splitting arguments" from [B$_3$]. To check the equality $\ker(\mathrm{kod}\circ\log) = \Delta(G/F_G)$ it is sufficient to show that for a derivation $\delta \in \Delta(G)$ we have $\mathrm{kod}\log\delta = 0$ if and only if F^{δ} is a field of definition for G. By [B$_2$] the latter happens if and only if F^{δ} is a field of definition for u. Now assume $\mathrm{kod}\log\delta = 0$ hence that there is an F-linear map $\theta : u \to u$ such that

(*) $b(\log\delta)(x,y)=\theta[x,y]-[\theta x,y]-[x,\theta y]$ for all $x,y\in u$

Letting $e_1:r\to r$ and $e_2:r\to r$ be the projections onto u and t respectively we have by (3.2) and (3.3) that

(**) $b(\log\delta)(x,y)=[e_2\delta x,y]-[e_2\delta y,x]$ for all $x,y\in u$

Projecting the equality $\delta[x,y]=[\delta x,y]+[x,\delta y]$ on u and using (**) we get

(***) $e_1\delta[x,y]=[e_1\delta x,y]+[x,e_1\delta y]+b(\log\delta)(x,y)$

for all $x,y\in u$. From (*) and (***) we get that $\tilde{\delta}:=e_1\circ\delta-\theta$ is a k-derivation on u and one checks immediately that $\tilde{\delta}(\lambda x)=(\delta\lambda)x+\lambda\delta x$ for all $\lambda\in F$, $x\in u$. By [B_1] p.86, F^{δ} is a field of definition for u.

Conversely, if the latter holds, then writing $G=G_1\otimes_{F_1}F$ ($F_1=F^{\delta}$, G_1 an affine algebraic F_1-group) we may consider $\partial(\delta)\in Der(F/F_1)$ and lift it to an F_1-derivation $\delta^*:=1\otimes\partial(\delta)$ on $\mathcal{P}(G)$. Clearly δ^* preserves U, hence preserves u. Then view $\theta=$ $=e_1\circ\delta-\delta^*$ as a map from u to u; clearly, it is F-linear. Substracting the equality (***) from the equality $\delta^*[x,y]=[\delta^* x,y]+$ $+[x,\delta^* y]$ we get that formula (*) holds for our θ just defined, hence $b(\log\delta)$ is a coboundary and we are done.

(3.5) COROLLARY. If the radical of G is nilpotent, $F^{\Delta(G)}$ is a field of definition for G (in fact it equals F_G).

Proof. Since in this case $[u,t]=0$ we have b=0 hence kod=

$=0$ hence $F_G=F^{\Delta(G)}$.

In fact a more precise statement holds:

(3.6) PROPOSITION. Assume Δ is a subset of $\Delta(G)$ such that $\log\Delta\subset W_o(G)$. Then F^Δ is a field of definition for G and $W_o(G)\cap$ $\cap\,\mathrm{Im}\log\subset\ker\mathrm{kod}$.

Proof. By (2.1) and $\delta\in\Delta$ can be written as $\delta=d+\exp a$ for some $d\in\Delta(G,\mathrm{fin})$, $a\in W_o(G)$. Since exp a is F-linear δ and d have the same restriction to F. Now apply (3.4) to conclude.

§4. Consequences

Start with some immediate consequences of the preceeding theory. As above G is an irreducible affine algebraic F-group, R its radical, U its unipotent radical.

(4.1) COROLLARY. The following hold:

1) There is a complex exact in the first two terms:

$$0\to\mathcal{L}(\mathrm{Aut}\ G)\to\mathcal{L}(\mathcal{A}ut\ G)\xrightarrow{\log}\mathrm{Hom}(u/u\cap[g,g],r/u)\xrightarrow{\mathrm{kod}}H^2(u,u)$$

2) A derivation in $\Delta(G)$ belongs to $\Delta(G,\mathrm{fin})$ if and only if it preserves the ideal of U in G or, equivalently, if and only if it preserves u.

3) The natural restriction map $\Delta(G)/\Delta(G,\mathrm{fin})\to\Delta(R)/\Delta(R,\mathrm{fin})$ is injective.

4) For any isogeny $i:G\to G'$ the lifting map $i^*:\Delta(G')\to\Delta(G)$ has the property that $(i^*)^{-1}(\Delta(G,\mathrm{fin}))=\Delta(G',\mathrm{fin})$.

5) $\Delta(G)=\Delta(G,\mathrm{fin})$ is any of the following cases: a) the radical of G is unipotent, b) G is reductive, c) $G/[G,G]$ is unipotent (i.e. a vector group, d) $G/[G,G]$ is reductive (i.e. a torus)

6) $\dim(\Delta(G)/\Delta(G/F)) \leq \mathrm{tr.deg.}(F/F_G) + \dim(\mathrm{Im}\, b/\mathrm{Im}\, b \cap B^2(u,u))$.

In the last assertion, we used notations from §3; morally it says that $\dim(\Delta(G)/\Delta(G/F))$ is restricted by the geometry of the linear subspaces contained in the variety V of all Lie algebra multiplications on the underlying vector space of u (since Im b is such a linear subspace of V). Some information on this geometry is provided by work of O. Laudal.

Proof. 1) follows from (1.2), (2.1) and (3.4).

2) follows from (2.1), (3.2) and (1.3).

3) follows from (2.1) and the fact that the map $W(G) \longrightarrow W(R)$ is injective (see (2.4) or (3.1)).

4) follows from (2.1) and the fact that the map $W(G') \rightarrow W(G)$ is an isomorphism (see (2.3))

5) can be checked as follows: a) is (1.4); b) follows from representability of Aut G(cf. [D]) but can be derived in a more elementary way from (2.1) since $X_a(G)=0$; c) follows since in this case the radical of G must be unipotent; d) follows from (2.1). Finally to prove e), start with a preparation. Assume V is an N-dimensional Δ-F-vector space (Δ an arbitrary set). Then the coordinate algebra $\mathcal{P}(GL(V))$ of GL(V) has a natural structure of Δ-F-algebra defined by identifying $\mathcal{P}(GL(V))$ with $S^{\cdot}(\mathfrak{gl}(V)^{o})[1/d]$ where $S^{\cdot}$="symmetric algebra" and $d \in S^N(\mathfrak{gl}(V)^{o})$ is "the determinant". We claim that $\mathcal{P}(GL(V))$ is locally finite; indeed $S(\mathfrak{gl}(V)^{o})$ clearly is so and we are done by noting that d is a Δ-constant (to check this replace (cf. $[B_3]$) F by some Δ-field extension of it such that V splits (i.e. has a Δ-constant F-basis). Associated to this basis there is a Δ-constant basis X_{ij} of $\mathfrak{gl}(V)^{o}$; now d is a polynomial in

the X_{ij}'s with Q-coefficients hence is Δ-constant). Coming back to our group G, let Ad:G→GL(g) be its adjoint representation. Using the description of Ad in [H] p.51 one cheks that Ad^*: :$\mathcal{P}$(GL(g))→ $\mathcal{P}$(G) is a Δ(G)-algebra map. Consequently if Z(G) is the center of G, $\mathcal{P}$(G/Z(G)) is a locally finite Δ(G)-F-vector space (being identified with $\mathcal{P}$(GL(g))/ker Ad^*). By assertion 4) $\mathcal{P}$(G) must be locally finite as a δ-F-vector space for all $\delta \in \Delta(G)$ and we are done.

Finally to check 6) one uses (3.4) and the obvious exact sequence $0 \to \Delta(G/F) \to \Delta(G/F_G) \to Der(F/F_G) \longrightarrow 0$.

From the discussion of assertion 5) above we get the following useful:

(4.2) Remark. The ideal defining the center of G is a Δ(G)-ideal of $\mathcal{P}$(G).

Proof. Indeed one checks that the ideal m defining 1_g in GL(g) is a Δ(G)-ideal of $\mathcal{P}$(GL(g)) (use once again a splitting of g). Then the ideal defining the center of G is $(Ad^*m)\mathcal{P}(G)$ hence it is a Δ(G)-ideal of $\mathcal{P}$(G).

In what follows we adress the question of describing the Lie space structure of Δ(G). We succed to do this for a remarkable subspace of it namely for $\Delta_o(G)=\log^{-1}W_o(G)$. Indeed we have

(4.3) COROLLARY.

1) $\Delta_o(G)=\Delta(G,fin) \oplus$ Im exp.

2) Im exp is an abelian ideal in $\Delta_o(G)$

3) $\Delta(G,fin) \simeq Der(F/F_G) \oplus \mathcal{L}(Aut\ G)$ (where $Der(F/F_G)$ embeds into Δ(G,fin) by $\delta \mapsto \delta^*$ = trivial lifting of δ to G(cf.(3.4)) and

$[\delta^*, \exp a] = \exp \delta^* a$ for all $\delta \in \mathrm{Der}(F/F_G)$ and $a \in W_o(G)$.

5) If in addition $F = \mathbb{C}$, upon letting $\Delta_o(G/\mathbb{C}) = \Delta_o(G) \cap \Delta(G/\mathbb{C})$ we have $j(\Delta_o(G/\mathbb{C})) \subset \mathrm{Im}\,\chi^{an}$ (notations as in (2.10)).

Proof. Just put together (1.2), (2.1), (2.10), (3.6).

(4.4) PROPOSITION. Assume one of the following holds:

1) The radical of G is nilpotent

2) The unipotent radical of G is commutative.

Then $\Delta_o(G) = \Delta(G)$; in other words any weight of G is $\Delta(G)$-constant.

Proof. We must prove that $\mathrm{Im}\log \subset W_o(G)$. This is clear in case 1). In case 2) upon letting R be the radical of G, the map $W(G) \to W(R)$ is injective and viewing it as an inclusion we have $W(G) \cap W_o(R) = W_o(G)$. So we may assume G solvable. Borowing notations from (2.4)-(2.6), we have $G = \tilde{G}$, $M = U$, $H = T$. If $\delta \in \Delta(G)$ then we have by (2.6.4) that $\mu_U \circ \delta^o = (1 \otimes \delta^o + \delta^o \otimes 1 + \sum a(\chi) \otimes p_\chi) \circ \mu_U$ where $a = \log \delta$. Applying $\tau : P(U) \otimes P(U) \to P(U) \otimes P(U)$ (twist map) to the above formula and using commutativity of μ_U (i.e. $\tau \circ \mu_U = \mu_U$) we get $_U$ ^{o}= ^{o} x 1 + 1 x ^{o} + p x a( )) $_U$; consequently we have

$$(*) \quad \left(\sum_\chi a(\chi) \otimes p_\chi\right) \circ \mu_U = \left(\sum_\chi p_\chi \otimes a(\chi)\right) \circ \mu_U$$

Let $x \in X_a(U)_\chi \subset P(U)$ be a primitive element of weight χ; applying (*) to x we get

$$(**) \qquad a(\chi) \otimes x = x \otimes a(\chi)$$

This implies of course that either $a(\chi)=0$ or $x=0$ or

$0 \neq x = \lambda a(\gamma)$ for some $\lambda \in F^*$. The latter equality is impossible because $a(\gamma) \in X_a(U)_1$. So a vanishes on all weights of $X_a(U)$. But $\rho(U)$ is the symmetric algebra on $X_a(U)$ hence all weights of $\rho(U)$ are products of weights of $X_a(U)$ and hence a vanishes on all weights $\rho(U)$. The proposition is proved.

5. Split and semisplit groups

In this section and the following we place ourselves in the setting of $[K_1],[C_1],[B_3]$. So Δ will be a fixed finite set ($m = \text{card}\,\Delta$) and all our Δ-objects will be "partial differential" i.e. derivations by which Δ acts are always pairwise commuting. $\mathcal{U}$ will denote a universal Δ-field with field of constants $\mathcal{K}$; $\mathcal{F}$ will generally denote an algebraically closed Δ-subfield of $\mathcal{U}$ over which $\mathcal{U}$ is universal and $\mathcal{C}$ will denote the field of constants of $\mathcal{F}$. We will also freely use terminology from $[B_1]$ involving Δ-schemes, Δ-varieties , Δ- function fields with no movable singularity a.s.o.

(5.1) Definitions

1) By an f-group we understand an irreducible linear Δ-algebraic group Γ such that tr.deg. $\mathcal{U}\langle\Gamma\rangle/\mathcal{U}<\infty$ (see also $[B_3]$).

2) An f-group is called split if it has the form $\Gamma^* \cap GL_n(\mathcal{K})$ where Γ^* is a $\mathcal{K}$-closed subgroup of $GL_n(\mathcal{U})$. It is called splittable if it is Δ-isomorphic to a split f-group (cf. B_3).

3) An f-group is called semisplet if it has the form $\Gamma^* \cap X$ where Γ^* is a $\mathcal{K}$-closed subgroup of $GL_n(\mathcal{U})$ and X is a Δ-closed subset of $GL_n(\mathcal{U})$ defined by equations of the form $0 = \delta_i y_{jk} - P_{ijk}$ with $P_{ijk} \in \mathcal{U}[(y_{jk})]$, $1 \le i \le m$, $1 \le j,k \le n$. Clearly "split" implies "semisplit" (Take $P_{ijk}=0$). Γ is called semisplit table if it is Δ-isomorphic to a semisplit f-group.

(5.2) In spite of Cassidy's deep results in [C_1][C_2] [C_3] a satisfactory picture of f-groups is still missing. What we intend to do here is initiate a study of f-groups based on the concepts introduced in (5.1) (cf. also [B_3]) and using our theory developed in the preceeding sections.

By [B_3] $\Gamma=\{yy''-(y')^2=0\}\subset GL_1(\mathcal{U})=\mathcal{U}^*$ is an example of a semisplittable f-group which is not splittable.

The relation between f-groups and the theory from sections (1-4) is given by the following.

(5.3) THEOREM. [B_3]. Let Γ be an f-group. Then the Δ-coordinate algebra $\mathcal{U}\{\Gamma\}$ is finitely generated as a non-differential $\mathcal{U}$-algebra.

Thus to any f-group Γ one can associate an affine algebraic $\mathcal{U}$-group $G=\mathcal{G}(\mathcal{U}\{\Gamma\})$ together with m commuting derivations $\delta_1,\dots,\delta_m\in\Delta(G)$. By Theorem (2.1) we have well defined elements $\log\delta_i\in W(G)$ with $\delta_i\log\delta_j=\delta_j\log\delta_i$ for all i, j. Moreover by (3.3) we have well defined cohomology classes kod $\log\delta_i\in H^2(u,u)$ (u=Lie algebra of the unipotent radical of G). Finally the weights of $\mathcal{G}(\mathcal{U}\{\Gamma\})$ (or of $\mathcal{U}\{\Gamma\}$, cf. the discussion preceeding (2.1)) will be sometimes simply called the weights of Γ.

Since rational maps between algebraic groups commuting with multiplications must be everywhere defined one gets that any surjective homomorphism $\Gamma\to\Gamma'$ of f-groups induces a natural surjective homomorphism between the corresponding affine algebraic groups $G\to G'$: clearly if $\Gamma\to\Gamma'$ is a Δ- isomorphism (respectively a Δ-isogeny) then $G\to G'$ is an isomorphism (respectively an isogeny). Here $\Gamma\to\Gamma'$ is called isogeny if (it is surjective and) $[\mathcal{U}\langle\Gamma\rangle : \mathcal{U}\langle\Gamma'\rangle] < \infty$.

A Δ-algebraic group will be called nilpotent if it is so as an abstract group; it will be called unipotent [C_3] if it consists of unipotent matrices. Note that an f-group Γ is nilpotent (respectively unipotent) if and only if $\mathcal{G}(\mathcal{U}\{\Gamma\})$ is so.

Recall that the radical $R(\Gamma)$ of a linear Δ-algebraic group Γ is the unique maximal element in the set of all Δ-closed normal irreducible solvable subgroups of G.

Assertion 1) of the following Lemma was proved in [B_3] (while its assertion 2) can be proved similarly by using (4.2)).

(5.4) LEMMA. Let Γ be an f-group, let $R(\Gamma)$ be its radical and $Z^o(\Gamma)$ be the connected component of the center of Γ. Moreover, let $G=\mathcal{G}(\mathcal{U}\{\Gamma\})$, let $R(G)$ be the radical of G and $Z^o(G)$ the connected component of the center of G. Then

1) $R(G)=\mathcal{G}(\mathcal{U}\{R(\Gamma)\})$ as subgroups of G.

2) $Z^o(G)=\mathcal{G}(\mathcal{U}\{Z^o(\Gamma)\})$ as subgroups of G.

Finally recall from [B_3]:

(5.5) LEMMA. An f-group Γ is splittable if and only if $\mathcal{U}\{\Gamma\}$ is locally finite as a Δ-$\mathcal{U}$-vector space. Moreover if this is the case and if Γ is defined over $\mathcal{F}$ ($\mathcal{F}$ algebraically closed) then one can find a Picard-Vessiot extension $\mathcal{F}_1/\mathcal{F}$ and a Δ-$\mathcal{F}_1$-isomorphism of Γ with an f-group of the form $\Gamma^* \cap GL_n(\mathcal{K})$, Γ^* a $\mathcal{C}$-closed subgroup of $GL_n(\mathcal{U})$.

(5.6) COROLLARY. An f-group Γ is splittable if and only if all group-like elements of $\mathcal{U}\{\Gamma\}$ are constants.

Furthermore we have

(5.7) COROLLARY. An f-group Γ is semisplittable if and only if $\mathcal{K}$ is a field of definition for $\mathcal{G}(\mathcal{U}\}\Gamma\})$ or, equivalently, if and only if kod log $\delta_i=0$ for all i. Moreover if all weights of Γ are constant then Γ is semisplittable. In particular if the radical of Γ is nilpotent, Γ is semisplittable.

Proof. The assertion that Γ is semisplettable if and only if $\mathcal{K}$ is a field of definition for $\mathcal{G}(\mathcal{U}\}\Gamma\})$ is an easy exercise. The rest of the assertions follow from (3.4) and (3.6).

Now putting together (5.4), (5.5) (5.7) and (4.1) we get

(5.8) COROLLARY. For an f-group Γ the following hold:

1) Γ is splittable (respectively semisplittable) if and only if its radical is so.

2) If there is an isogeny $\Gamma\to\Gamma'$ then Γ is splittable (respectively semisplittable) if and only if Γ' is so.

3) Γ is splittable in each of the following cases: a) the radical of Γ is unipotent (cf [B_3]), b) the center of Γ is finite, c) Γ has no non-trivial umipotent commutative queotient.

6. The classification problem

Let G be an irreducible affine algebraic $\mathcal{U}$-group. We propose ourselves to describe the set $\Gamma(G)$ of Δ-isomorphism classes of f-groups Γ for which $\mathcal{G}(\mathcal{U}\}\Gamma\})$ is $\mathcal{U}$-isomorphic with G. We restrict ourselves to the case when $\mathcal{K}$ is a field of definition for G (hence $G\simeq G_{\mathcal{K}}\otimes\mathcal{U}$ for some $\mathcal{K}$-group $G_{\mathcal{K}}$); by (5.7) this restriction is not effective if we are interested only in semisplittable Γ's.

To get an idea of $\Gamma(G)$ note that $\Gamma(G)$ has precisely one element in each of the following cases: a) the radical of G is unifinite center, cf. (4.1) and (5.5).

In general it is clear that $\Gamma(G) \simeq \Delta(G)^{\text{int}}/\text{Aut } G$ where $\Delta(G)^{\text{int}}$ is the set of m-uples $(\delta_1, \dots, \delta_m)$ of pairwise commuting elements of $\Delta(G)$ and Aut G acts on $\Delta(G)^{\text{int}}$ by the formula (lifting the derivations of $\mathcal{U}$)

$$(\sigma, (\delta_1, \dots, \delta_m)) \mapsto (\sigma^{-1}\delta_1\sigma, \dots, \sigma^{-1}\delta_m\sigma), \quad \sigma \in \text{Aut } G$$

where we also denoted by σ the corresponding automorphism of $\mathcal{P}(G)$. But this description of $\Gamma(G)$ is quite unsatisfactory.

Let δ_i^* be the trivial lifting of δ_i from $\mathcal{U}$ to $\mathcal{P}(G) = \mathcal{P}(G_{\mathcal{K}}) \otimes \mathcal{U}$. Then the map $\Delta(G)^m \to \Delta(G/\mathcal{U})^m$ defined by $(\delta_1, \dots, \delta_m) \mapsto (\delta_1 - \delta_1^*, \dots, \delta_m - \delta_m^*)$ induces a bijection

$$\Gamma(G) \simeq \Delta(G/\mathcal{U})^{\text{int}}/\text{Aut } G$$

where

1) $\Delta(G/\mathcal{U})$ is viewed as a Δ-$\mathcal{U}$-vector space by letting $\delta_i^{\#}\theta = [\delta_i^*, \theta]$ for all $\theta \in \Delta(G/\mathcal{U})$ (bracket taken in the Lie space $\Delta(G)$).

2) $\Delta(G/\mathcal{U})^{\text{int}}$ is the $\mathcal{K}$-space of m-uples $(\theta_1, \dots, \theta_m)$ of elements $\theta_i \in \Delta(G/\mathcal{U})$ such that $\delta_i^{\#}\theta_j - \delta_j^{\#}\theta_i + [\theta_i, \theta_j]$ for all i,j.

3) Aut G acts by the "Loewy-type" [C1] action:

$$(\sigma, (\theta_1, \dots, \theta_m)) \mapsto (\sigma^{-1}\theta_1\sigma + (\sigma^{-1}\delta_1^*\sigma - \delta_1^*), \dots \quad)$$

This description is as unsatisfactory as the first one because everything is hidden by the Δ-algebraic action of AutG; what one should be looking for is description involving an algebraic (rather than a Δ-algebraic) action of Aut $G_{\mathcal{K}}$. Such a description will be given [illegible]

But more generally we will give a description in the general case of the subset $\Gamma_o(G)$ of $\Gamma(G)$ consisting of all Δ-isomorphism classes of f-groups Γ in $\Gamma(G)$ all of whose weights are constant. As above we have

$$\Gamma_o(G) \simeq \Delta_o(G)^{int}/\mathrm{Aut}\ G$$

where $\Delta_o(G)^{int} = \Delta(G)^{int} \cap (\Delta_o(G) x \ldots x \Delta_o(G))$. Moreover, since $\S_j^*$ kill all weights of G we have

$$\Gamma_o(G) \simeq \Delta_o(G/\mathcal{U})^{int}/\mathrm{Aut}\ G$$

where $\Delta_o(G/\mathcal{U})^{int} = \Delta(G/\mathcal{U})^{int} \cap (\Delta_o(G) x \ldots x \Delta_o(G))$.

Note that $\Gamma_o(G) = \Gamma(G)$ provided G is in one of the two situations from (4.4).

Our main result here is:

(6.1) THEOREM. We have

$$\Gamma_o(G) \simeq W_o(G)^{int}/\mathrm{Aut}\ G_{\mathcal{K}}$$

where:

1) $W_o(G)=W_o(G_{\mathcal{K}}) \otimes \mathcal{U}$ is viewed as a Δ-$\mathcal{U}$-vector space by letting Δ act trivially on $W_o(G_{\mathcal{K}})$.

2) $W_o(G)^{int}$ is the space of m-uples $(a_1,\dots,a_m)$ of elements $a_i \in W_o(G)$ such that $\delta_i^* a_j = \delta_j^* a_i$ for all i,j.

3) Aut $G_{\mathcal{K}}$ acts on $W_o(G)$ via its natural action on $X_m(G)$ and $X_a(G)$.

Proof. We will show that the map $\exp: W_o(G) \to \Delta_o(G/\mathcal{U})$ induces a bijection $e: W_o(G)^{int}/\mathrm{Aut}\ G_{\mathcal{K}} \xrightarrow{\sim} \Delta_o(G/\mathcal{U})^{int}/\mathrm{Aut}\ G$. That exp induces a map e as above follows from assertions 3) and 4) in (4.3). To check that e is injective, assume $(a_1,\dots,a_m),(a_1',\dots,a_m') \in W_o(G)^{int}$ are such that there exists $\sigma \in \mathrm{Aut}\ G$ with

$$\exp a_i' = \sigma^{-1}(\exp a_i)\sigma + (\sigma^{-1}\delta_i^*\sigma - \delta_i^*)$$

Since $\sigma^{-1}\delta_i^*\sigma - \delta_i^* \in \Delta(G,\mathrm{fin})$ and since $\mathrm{Im}\ \exp \cap \Delta(G,\mathrm{fin})=0$ we get $a_i' = \sigma a_i$ and $\sigma^{-1}\delta_i^*\sigma = \delta_i^*$; the latter equality implies that $\sigma \in \mathrm{Aut}\ G_{\mathcal{K}}$ and injectivity follows. To check surjectivity of e, take any $(\theta_1,\dots,\theta_m) \in \Delta_o(G/\mathcal{U})^{int}$. By (4.3) we may write $\theta_i = d_i + \exp a_i$ for some $a_i \in W_o(G)$ and $d_i \in \Delta(G/\mathcal{U}) \cap \Delta(G,\mathrm{fin})$. We claim that $(d_1,\dots,d_m) \in \Delta_o(G/\mathcal{U})^{int}$. Indeed:

$$0 = \left[\delta_i^* + \theta_i, \delta_j^* + \theta_j\right] = \left[\delta_i^* + d_i, \delta_j^* + d_j\right] +$$

$$+ \left[\delta_i^* + d_i, \exp a_j\right] - \left[\delta_j^* + d_j, \exp a_i\right]$$

Since by (4.3) Im exp is an ideal in $\Delta_o(G)$ the last two terms of the right hand side expression belong to Im exp while the first belongs to $\Delta(G,\mathrm{fin})$, because by (2.1) $\Delta(G,\mathrm{fin})$ is a Lie subspace of $\Delta(G)$. Consequently the first term vanishes and our claim is proved. Now the f-group obtained by considering $\mathcal{P}(G)$ as a Δ-ring with derivations $\delta_1^* + d_1, \ldots, \delta_m^* + d_m$ is splittable by (5.5), hence there exists $\sigma \in \mathrm{Aut}\, G$ such that $d_i = \sigma^{-1}\delta_i^*\sigma - \delta_i^*$ for all i. Applying formula 3) from (4.3) we get that $(\theta_1, \ldots, \theta_m)$ is Aut G - conjugate to $(\exp \sigma^{-1}a_1, \ldots, \exp \sigma^{-1}a_m)$ and surjectivity follows. Our theorem is proved.

7. Link with NMS extensions

Next, we are dealing with the question [C_4] of how Δ-field extensions arrising from f-groups are related to Δ-function fields with no movable singularity (NMS) (in the sense of our book [B_1], p.5). P. Cassidy has shown [C_4] that if Γ is the Δ-algebraic group $\{yy''' - (y')^2 = 0\} \subset \mathcal{U}^*$ and $\mathcal{F}$ is any Δ-field then the extension $\mathcal{F}\langle\Gamma\rangle / \mathcal{F}$ does not split (in the sense of [B_1], i.e. is not generated by constants). In particular such an extension cannot have NMS [B_1]. On the other hand we will prove that extensions of this type are "not too far" from having NMS:

(7.1) THEOREM. Let Γ be an f-group all of whose weights are constant and $\mathcal{F}$ an algebraically closed Δ-field of definition for Γ. Then there exists a Picard-Vessiot extension $\mathcal{F}_1/\mathcal{F}$

and an intermediate Δ-field $\mathcal{F}_1 \subset \mathcal{E} \subset \mathcal{F}_1\langle\Gamma\rangle$ such that the extension $\mathcal{E}/\mathcal{F}_1$ is a split Δ-function field and $\mathcal{F}_1\langle\Gamma\rangle/\mathcal{E}$ is a Δ-function field generated by exponential elements (i.e. elements whose logarithmic derivative belong to $\mathcal{E}$).

We will also prove

(7.2) PROPOSITION. Any Δ- function field generated by Picard-Vessiot elements has NMS.

Consequently we get:

(7.3) COROLLARY. In notations of (7.1) all three extensions $\mathcal{F}_1\langle\Gamma\rangle/\mathcal{E}$, $\mathcal{E}/\mathcal{F}_1$, $\mathcal{F}_1/\mathcal{F}$ have NMS.

(7.4) Proof of (7.1). Let $\mathcal{C}$ be the constant field of $\mathcal{F}$. Since by (5.7) $\mathcal{G}(\mathcal{U}\}\Gamma\})$ is defined over $\mathcal{K}$ it follows from [B_2] that $G_{\mathcal{F}} = \mathcal{G}(\mathcal{F}\}\Gamma\})$ is defined over $\mathcal{C} = \mathcal{K} \cap \mathcal{F}$ so $G_{\mathcal{F}} \simeq G_{\mathcal{C}} \otimes \mathcal{F}$, $G_{\mathcal{C}}$ a $\mathcal{C}$-- group.

In particular the derivations δ_i on $\mathcal{P}(G_{\mathcal{F}})$ have the form $\delta_i = \delta_i^* + \theta_i$ where δ_i^* is the trivial lifting of δ_i from $\mathcal{F}$ to $\mathcal{P}(G_{\mathcal{F}})$ and $\theta_i \in \Delta(G_{\mathcal{F}}/\mathcal{F})$. Now look into the proof of Theorem (6.1) where we checked the surjectivity of the map e, borrow the notations from there and let's be careful about "rationality problems". Indeed in writing $\theta_i = d_i + \exp a_i$ we have $a_i \in W_0(G_{\mathcal{F}})$ and $d_i \in \Delta(G_{\mathcal{F}}/\mathcal{F}) \cap \Delta(G_{\mathcal{F}}, \text{fin})$. Next by (5.5) we may choose $\sigma \in \operatorname{Aut} G_{\mathcal{F}_1}$ for some Picard-Vessiot extension $\mathcal{F}_1/\mathcal{F}$. Now the extension $\mathcal{F}_1\langle\Gamma\rangle/\mathcal{F}_1$ is isomorphic to $Q(\mathcal{P}(G_{\mathcal{F}_1}))/\mathcal{F}_1$ with derivations $\delta_i^* + \exp \sigma^{-1} a_i$, $1 \le i \le m$. Let U and H be as in (2.4) and put $\mathcal{E} = Q(\mathcal{P}(U_{\mathcal{F}_1}))$; then clearly $\Delta\mathcal{E} \subset \mathcal{E}$ and $\mathcal{E}/\mathcal{F}_1$ is split. Moreover $\mathcal{F}_1\langle\Gamma\rangle/\mathcal{E}$ is generated by elements of the form $1 \otimes y$ with $y \in \bigcup \mathcal{P}(H)^{\chi}$ (notation from (2.8)) and any such $1\otimes y$ is an exponential element for the extension $\mathcal{F}_1\langle\Gamma\rangle/\mathcal{E}$. Our Theorem is proved.

(7.5) Proof of (7.2). Let $\mathcal{E}/\mathcal{F}$ be a Δ-function field generated by Picard-Vessiot elements. Then clearly a finite set of them suffices so $\mathcal{E}=Q(A)$ where A is a sub Δ-$\mathcal{F}$- algebra of $\mathcal{E}$, finitely generated as an $\mathcal{F}$-algebra and locally finite as a Δ-$\mathcal{F}$--vector space. Let X=Spec A be the corresponding Δ-variety and $V\subset A$ be a finite dimensional Δ-$\mathcal{F}$-vector space generating A as an $\mathcal{F}$-algebra. Then the symmetric algebra $S'V$ has a natural structure of Δ-$\mathcal{F}$-algebra and the closed embedding $X\to\mathbb{A}=\mathrm{Spec}\,(S'V)$ is a Δ- map. Let the polynomial algebra $(S'V)[T]$ be a Δ-algebra extension by putting $\Delta T=0$; since Δ preserves the gradation on $(S'V)[T]$ it follows that $\mathbb{P}=\mathrm{Proj}((S'V)[T])$ is a Δ-variety and the open embedding $\mathbb{A}\to\mathbb{P}$ is a Δ-map. Let $\bar{X}$ be the Zariski closure of X in $\mathbb{P}$. We claim that X is a projective Δ-model of its function field (and this will close the proof). Indeed, let Spec B be an open subset of $\mathbb{P}$ not contained in $\mathbb{A}$, let $P\subset B$ be the prime ideal corresponding to $\mathbb{P}\setminus\mathbb{A}$ and Q the prime ideal corresponding to X. Now $\Delta QB_Q\subset QB_Q$ because $Q\in X$, hence $\Delta Q\subset QB_Q\cap B=Q$ this showing that $\bar{X}\cap\mathrm{Spec}\,B$ is a Δ-subscheme of Spec B and we are done.

Remark. Theorem (7.1) applies of course to any Γ whose radical is nilpotent or to any Γ for which $\mathcal{G}(\mathcal{U}\{\Gamma\})$ has a commutative unipotent radical.

§8. Painlevé extensions

In this section we introduce the notion of Painlevé extension of Δ-fields, discuss its connection with "logarithmic derivatives" of analytic actions on algebraic varieties and give an application to f-groups whose radical is nilpotent. Start with an analytic preparation.

(8.1) Let $f:X\to Y$ be a map between analytic (complex) manifolds and suppose we are given analytic commuting vector fields $\delta_1,\dots,\delta_m$ on Y lifting to some analytic commuting vector fields (still denoted by) $\delta_1,\dots,\delta_m$ on X. We say that f has the Painlevé property (with respect to $\delta_1,\dots,\delta_m$) if, upon letting $\tilde{Y}\to Y$ be the universal covering of Y, there is an analytic $\tilde{Y}$-isomorphism $X\times_Y\tilde{Y}\simeq Z\times\tilde{Y}$ $(Z=f^{-1}(y_o),\ y_o\in Y)$ sending $\tilde{\delta}_i$ into δ_i^* where $\tilde{\delta}_i$ is the unique lifting of δ_i from X to $X\times_Y\tilde{Y}$ and δ_i^* is the trivial lifting of δ_i from $\tilde{Y}$ to $Z\times\tilde{Y}$.

Note that if dim Y=m and if we throw away from Y the locus where $\delta_1,\dots,\delta_m$ do not generate the tangent space and from X the preimage of this locus then we get a foliation on X, transverse to f and being a "feuilletage de Painlevé de 1^{re} epèce" in the sens of [GS], i.e. having the property that any path on Y starting from $y_o\in Y$ can be lifted in the leaf passing through any point of $f^{-1}(y_o)$.

(8.2) From now on $\mathcal{F}$ will be assumed to contain $\mathbb{C}$, to be algebraically closed and to be such that Δ provides an $\mathcal{F}$-basis of $Der(\mathcal{F}/\mathbb{C})$; in particular $tr.deg\,\mathcal{F}/\mathbb{C}=card\,\Delta=m$.

Let V be a Δ-variety over $\mathcal{F}$ (in the sense of [B_1] p.4). We say that V is a Painlevé variety if there is a cartesian diagram of

Δ - schemes

$$\begin{array}{ccc} U & \longrightarrow & V \\ f \downarrow & & \downarrow \\ S & \longrightarrow & \mathrm{Spec}\ \mathcal{F} \end{array}$$

with U and S smooth Δ-varieties over $\mathbb{C}$, dim S=m, such that $f^{an}: U^{an} \to S^{an}$ has the Painlevé property (here f^{an} denotes as usual the analytic map underlying f).

Finally let's say that a Δ-function field $\mathcal{E}/\mathcal{F}$ is a Painlevé extension if it has a Painlevé Δ-model.

Remarks. 1) Any smooth projective Δ-variety over $\mathcal{F}$ is a Painlevé Δ-variety; in particular, any extension with no movable singularity (NMS) in the sense of [B_1] p.5 is a Painlevé extension. This is a trivial consequence of "Ehresmann's theorem" [J].

2) A consequence of the above remark and of our result in [B_1], p.103 is that any strongly normal extension of $\mathcal{F}$ is a Painlevé extension (as it has NMS).

3) Painlevé extensions need not have NMS, for they may have "movable singularities hidden at infinity". For instance K.Okamoto's work (in Sém. F.Norguet, Février 1977) shows that the famous Painlevé second order equations lead to Painlevé extensions; these are not necessarily NMS extensions (cf. work of Nishioka and also cf. [C_4]).

Our main result will be:

(8.3) THEOREM. Let Γ be an f-group over $\mathcal{F}$ such that all weights of Γ are constant. Then $\mathcal{F}\langle\Gamma\rangle/\mathcal{F}$ is a Painlevé extension.

(8.4) Let us introduce some introduce some notational conventions which will simplify our exposition in what follows. For any analytic manifolds X,Y we denote by Map(X,Y) the set of all maps from X to Y. For $f \in$ Map(X,Y) we denote by f_* the tangen map viewed as a map between the manifolds T_X, T_Y (=tangent bund les) so $f_* \in$ Map(T_X,T_Y). Analytic vector fields on X will be vie- wed as sections of the canonical projection $T_X \longrightarrow X$ hence their space $H^0(X,T_X)$ lies in Map(X,T_X). So if $f \in$ Map(X,Y) and $v \in H^0(X,T_X)$ then $f_* v \in$ Map(X,T_Y). Moreover, if Λ is a finite dimen- sional linear subspace of $H^0(X,T_X)$, there is a natural injective map Map(Y,Λ) $\longrightarrow H^0(X \times Y, T_{X \times Y})$; we shall usually identify elements of Map(Y,Λ) with their image under this map. If W is a finite dimensional complex vector space and Y is an analytic manifold then for any $h \in$ Map(Y,W) and any vector field $v \in H^0(Y,T_Y)$ there is a well defined map $vh \in$ Map(Y,W) (take a basis $(e_i)_i$ of W, write $h = \sum h_i e_i$ with $h_i \in \mathcal{O}(Y)$ and let $vh = \sum (vh_i) e_i$; this definition does not depend on the choice of the basis). Moreover if $A: W \rightarrow W'$ is a linear map of vector spaces and v,h are as above, then $v(A \circ h) = A \circ vh$.

Now if G is a complex Lie group and $f \in$ Map(Y,G), define $L_f \in$ Map(GxY,GxY) by the formula $L_f(g,y) = (f(y)g,y)$.

Here is an analytic analogue of Kolchin's surjectivity theorem for the logarithmic derivative.

(8.5) PROPOSITION. Assume G is an algebraisable Lie group (i.e. admits a structure of affine algebraic group) and Y is a simply connected analytic manifold with trivial tangent bundle generated at each point by commuting fields $\delta_1,\ldots,\delta_m \in H^o(Y,T_Y)$, $m=\dim Y$. Then for any $h_j \in \mathrm{Map}(Y,\mathcal{L}(G))$, $1\le j\le m$, such that $\delta_j h_i + \delta_i h_j + [h_j,h_i] = 0$ holds for all $1\le i,j\le n$, there exists $f\in \mathrm{Map}(Y,G)$ such that

$$L_{f_*}\delta_j^* - \delta_j^* = h_j \quad \text{for all } j$$

where $\delta_j^* \in H^o(GxY,T_{GxY})$ is the trivial lifting of δ_j from Y to GxY.

Proof. First let's prove the proposition for $G=(GL_n)^{an}$. In this case if we identify any $f\in \mathrm{Map}(Y,G)$ with a nxn matrix with entries belonging to the ring $\mathcal{O}(Y)$ of global analytic functions on Y then an easy computation shows that for such an f we have $L_{f_*}\delta_j^* - \delta_j^* = f^{-1}\delta_j f$. So given h as in the proposition and identifying it with an nxn matrix with entries in $\mathcal{O}(Y)$ (so $h\in gl_n(\mathcal{O}(Y))$) we are looking for an $f\in GL_n(\mathcal{O}(Y))$ such that $\delta_j f = fh_j$ for all j. Since the δ_j's form a basis of the tangent space of Y at each point, Y is simply connected and the h_j's satisfy the "integrability conditions" from the statement of the Proposition it is classically known (see for instance [D]) that an f as above exists and the case $G=(GL_n)^{an}$ is done.

Now let G be an arbitrary algebraisable Lie group and view it as a Zariski closed algebraic subgroup of GL_n; in particular $\mathcal{L}(G)\subset gl_n$ so the h_j's define maps from Y to gl_n still satisfying the integrability condtions. By the discussion above there exists $f\in \mathrm{Map}(Y,(GL_n)^{an}) = GL_n(\mathcal{O}(Y))$ such that $L_{f_*}\delta_j^* - \delta_j^* = h_j$ for all j. On the other hand by Kolchin's surjectivity theorem for the logarithmic derivative (cf. its version in $[B_1]$p.51) there is a Picard-Vessiot extension $\mathcal{E}$ of the quotient field of $\mathcal{O}(Y)$ and $f_1 \in G(\mathcal{E})$ such that $\ell\delta_j f_1 = h_j$ for all j ($\ell\delta_j$=Kolchin's logarithmic derivative). By $[B_1]$ p.25 we see that $\ell\delta_j f = L_{f_*}\delta_j^* - \delta_j^* = h_j$ which immediately implies that $\ell\delta_j(f_1 f^{-1}) = 0$ for all j hence that $f_1 f^{-1} \in GL_n(\mathcal{E}$ $= GL_n(\mathbb{C}) \subset GL_n(\mathcal{O}(Y))$. Since $f^{-1} \in GL_n(\mathcal{O}(Y))$ we get $f_1 \in GL_n(\mathcal{O}(Y)) \cap$ $\cap G(\mathcal{E}) = G(\mathcal{O}(Y))$ and we are done.

(8.6) Let X be an analytic manifold. A Lie subalgebra of $H^0(X,T_X)$ will be called analytically bounded if there exists an algebraisable Lie group G and a faithful analytic action $G \times X \to X$ such that Λ is contained in the image of the natural map $\mathcal{L}(G) \to H^0(X,T_X)$.

(8.7) PROPOSITION. Let X be an analytic manifold, Λ an analytically bounded Lie subalgebra of $H^0(X,T_X)$, Y a simply connected manifold with trivial tangent bundle generated at each point by commuting vector fields $\delta_1,\dots,\delta_m \in H^0(Y,T_Y)$, m=dim Y and $\gamma_j \in \mathrm{Map}(Y,\Lambda)$, $1 \le j \le m$ such that $\delta_j\gamma_i + \delta_i\gamma_j + [\gamma_j,\gamma_i] = 0$ for all $i,j=1,\dots,m$. Then there exists an analytic Y-isomorphism $\varphi: X\times Y \to X\times Y$ such that $\varphi_*\delta_j^* - \delta_j^* = \gamma_j$ for all j where δ_j^* is the trivial lifting of δ_j to $X \times Y$.

Proof. If $\Lambda \subset \mathrm{Im}(\mathcal{L}(G) \to H^0(X,T_X))$ where $G \times X \to X$ is an action as in (8.6), then $\gamma_j: Y \to \Lambda$ induce maps $h_j: Y \to \mathcal{L}(G)$; since the natural map $\Lambda \to \mathcal{L}(G)$ is a Lie algebra map, h_j will still satisfy the integrability conditions. By (8.5) there exists

$f \in Map(Y,G)$ such that

$$(8.7.1) \qquad L_{f_*}\hat{\delta}_j - \hat{\delta}_j = h_j \qquad \text{for all } j$$

where $\hat{\delta}_j$ is the trivial lifting of δ_j to $G\text{x}Y$. Now consider the Y-isomorphism $\varphi : X\text{x}Y \to X\text{x}Y$, $\varphi(x,y) = (f(y)x, y)$. To check that

$$(8.7.2) \qquad \varphi_* \delta^*_j - \delta^*_j = \gamma_j \qquad \text{for all } j$$

consider the diagram with commutative squares:

$$\begin{array}{ccccc}
G \text{ x } Y & \xleftarrow{\;p\;} & G \text{ x } Y \text{ x } X & \xrightarrow{\;q\;} & X \text{ x } Y \\
\downarrow{\scriptstyle L_f} & & \downarrow{\scriptstyle L_f \text{xid}} & & \downarrow{\scriptstyle \varphi} \\
G \text{ x } Y & \xleftarrow[p]{} & G \text{ x } Y \text{ x } X & \xrightarrow[q]{} & X \text{ x } Y
\end{array}$$

with $q(g,y,x) = (gx,y)$ and $p(g,y,x) = (g,y)$ for all $g \in G$, $x \in X$, $y \in Y$. Now (8.7.1) implies that

$$(8.7.3) \qquad (L_f \text{ x id})_* \tilde{\delta}_j - \tilde{\delta}_j = \tilde{h}_j \qquad \text{for all } j$$

where $\tilde{h}_j : Y \to H^0(G\text{x}X, T_{G\text{x}X})$ is the trivial lifting of h_j and $\tilde{\delta}_j$ is the trivial lifting of δ_j to $G\text{x}Y\text{x}X$. Applying q_* to (8.7.3) we get

$$q_* \tilde{h}_j = q_*(L_f \text{xid})_* \tilde{\delta}_j - q_* \tilde{\delta}_j = \varphi_* q_* \tilde{\delta}_j - q_* \tilde{\delta}_j$$

But $q_* \tilde{\delta}_j = \delta^*_j q$ and $q_* \tilde{h}_j = \gamma_j q$ so (8.7.2) follows because q is surjective and we are done.

(8.8) COROLLARY. Lev V be a Δ-variety over $\mathcal{F}$. Assume

$V=Z\otimes\mathcal{F}$ for some smooth algebraic $\mathbb{C}$-variety Z and assume there exists a finite dimensional Lie subalgebra Λ of $H^o(Z,T_Z)$ whose image in $H^o(Z^{an},T_{Z^{an}})$ is analytically bounded such that

$$\delta_j - \delta_j^* \in \Lambda\otimes\mathcal{F} \quad \text{for all } j$$

where δ_j above is viewed as a $\mathbb{C}$-derivation of $\mathcal{O}_{Z\otimes\mathcal{F}}$ while δ_j^* is the trivial lifting of δ_j from $\mathcal{F}$ to $\mathcal{O}_{Z\otimes\mathcal{F}}$. Then V is a Painlevé Δ-variety.

Proof. Since tr deg $\mathcal{F}/\mathbb{C}<\infty$ any finite subset of $\mathcal{F}$ is contained in a Δ- subfield of $\mathcal{F}$ which is finitely generated as a field extension of $\mathbb{C}$. So we may choose a Δ-variety S over $\mathbb{C}$ with $Q(S)\subset\mathcal{F}$, dim S=m such that $\delta_j-\delta_j^*\in\Lambda\otimes H^o(S,\mathcal{O}_S)$ and T_S is trivial, generated at each point by $\delta_1,\dots,\delta_m$. Let $Y\to S^{an}$ be the universal covering of S^{an}. Then the unique lifting of $\delta_j-\delta_j^*$ to Z x Y (still denoted so) gives rise to a map $\sigma_j\in \mathrm{Map}(Y,\Lambda)$. By (8.7) there exists an Y-isomorphism φ of Z x Y such that $\varphi_*\delta_j^*=\delta_j^*+\sigma_j=\delta_j$ hence the projection $Z^{an}\times S^{an}\to S^{an}$ has the Painlevé property and we are done.

To prove Theorem (8.3) we need:

(8.9) LEMMA. For any affine algebraic $\mathbb{C}$-group G, we have $\Delta(G\otimes\mathcal{F}/\mathcal{F})\simeq\Delta(G/\mathbb{C})\otimes\mathcal{F}$.

Proof. We have a natural injective map $\Delta(G/\mathbb{C})\otimes\mathcal{F}\to\Delta(G\otimes\mathcal{F}/\mathcal{F})$ so we have to check surjectivity. Let $\delta\in\Delta(G\otimes\mathcal{F}/\mathcal{F})$ and choose an affine Δ-variety S such that δ defines a regular (i.e. everywhere defined rational) vector field on G x S, $(Q(S)\subset\mathcal{F})$. Let A and

B be the coordinate rings of G and S respectively. It is clear that for any maximal ideal M of B, the derivation (δ mod M): $:A \to A$ induced by $\delta: A \otimes B \to A \otimes B$ belongs to $\Delta(G/\mathbb{C})$.

Let V be a finite dimensional $\mathbb{C}$-subspace of A generating A as a $\mathbb{C}$-algebra and let W be a finite dimensional $\mathbb{C}$-subspace of A such that $\delta(V \otimes B) \subset W \otimes B$ and $\Delta(G/\mathbb{C})(V \otimes B) \subset W \otimes B$ (this is possible because $\dim_{\mathbb{C}} \Delta(G/\mathbb{C}) < \infty$ of (2.1) and this is the point in the proof of our Lemma). Now let E be the B-submodule of $H = \mathrm{Hom}_B(V \otimes B, W \otimes B)$ generated by the image of $\Delta(G/\mathbb{C})$ and E' be the B-submodule of H generated by E and the image of δ. We have that for any maximal ideal M of B the images of E and E' in $H/MH \doteq \mathrm{Hom}_C(V, W)$ coincide. Replacing B by B_f for some $f \in B, f \neq 0$ we may assume that H/E' is a free B-module, hence that $MH \cap E' = ME'$. Nakayama immediately implies that E=E'. Since two derivations on $A \otimes B$ which agree on $V \otimes B$ must agree everywhere we get that $\delta \in \Delta(G/\mathbb{C}) \otimes B$ and our Lemma is proved.

(8.10) Proof of Theorem (8.3). As in (7.4), $\mathcal{G}(\mathcal{F}\rangle\Gamma\}) = G_{\mathbb{C}} \otimes \mathcal{F}$ for some algebraic $\mathbb{C}$-group G. Let as usual δ_j^* denote the trivial lifting of δ_j from $\mathcal{F}$ to $G_{\mathbb{C}} \otimes \mathcal{F}$ and δ_j the derivation corresponding to the operator δ_j on $\mathcal{P}(G_{\mathbb{C}} \otimes \mathcal{F})$. Then by (8.9)
$\delta_j - \delta_j^* \in \Delta_o(G_{\mathbb{C}} \otimes \mathcal{F}) \cap \Delta(G_{\mathbb{C}} \otimes \mathcal{F}/\mathcal{F}) = \Delta_o(G_{\mathbb{C}} \otimes \mathcal{F}) \cap (\Delta(G_{\mathbb{C}}/\mathbb{C}) \otimes \mathcal{F}) = \Delta_o(G_{\mathbb{C}}/\mathbb{C}) \otimes \mathcal{F}$. By (4.3) $\Delta_o(G_{\mathbb{C}}/\mathbb{C})$ is analytically bounded hence by (8.8) $\mathcal{G}(\mathcal{F}\rangle\Gamma\})$ is a Painlevé variety and Theorem (8.3) is proved.

Appendix . Derivations of general Hopf algebras

Let A be a Hopf algebra [Sw] over a field F of arbitrary characteristic and k a subfield of F. Then define $\Delta(A)$ to be the Lie F-space of all k-derivations of A satisfying conditions 1)-4) from (1.1) (with A instead of $\mathcal{O}(G)$). The aim of this Appendix is to study $\Delta(A)$ in this general context. The "algebraic group version" of the result we are going to obtain can be easily deduced from our Theorem (2.1) plus standard structure theory of algebraic groups; since these tools are not available in the "general Hopf algebra" case we found it interesting to show how one can develop some purely Hopf algebraic techniques in order to deal with "infinitesimal automorphisms". Apart from the fact that (non-necessary commutative or finitely generated) Hopf algebras have their own right and beauty we are also motivated by the fact that linear Δ-algebraic groups Γ of infinite transcendence degree lead to (commutative) Hopf algebras $\mathcal{U}\{\Gamma\}$ which are not finitely generated as $\mathcal{U}$-algebras.

We start with a preparation on cocenters and centers of Hopf algebras. Then our main result will describe the structure of $\Delta(A \otimes B)$ with B co-semisimple. Our background here is [Sw].

(A.1) Let F be a field and C a F-coalgebra. A coideal I of C is called cocentral if $\overline{\tau} \circ (1 \otimes p) \circ \mu = (p \otimes 1) \circ \mu : C \to C/I \otimes C$ where μ is the comultiplication, $p: C \to C/I$ is the canonical surjection and $\overline{\tau}: C \otimes C/I \to C/I \otimes C$ is the twist map (equivalently, if $\mu x - \tau\mu x \in I \otimes C$ for all $x \in C$, where $\tau: C \times C \to C \times C$ is the twist map). If $\dim C < \infty$, I is cocentral iff the subalgebra $(C/I)^{0}$ of

C^o is central. One easily sees that if dim $C<\infty$ then, there is a minimum cocentral coideal I_C of C; indeed let A be the center of C^o and put $I_C=\ker(C\simeq C^{oo}\to A^o)$. Now we claim that any coalgebra C (possibly of infinite dimension) has a minimum cocentral coideal I_C. Indded, take any family $(C_i)_i$ of subcoalgebras such that $C=\sum C_i$ and dim $C_i<\infty$; then $I_C=\sum I_{C_i}$ is easily seen to be the minimum cocentral coideal of C.

(A.2). A coalgebra C will be called cocentral if $I_C=$ $=\ker\varepsilon$ ($\varepsilon=$ the counit). If dim $C<\infty$ then C is cocentral iff C^o is a central algebra. Hence, if F is algebraically closed and C is simple, then C is automatically cocentral.

(A.3) Recall from [Sw] p.161 the following basic property of simple coalgebras: if C is a coalgebra, $C=\sum C_i$, C_i subcoalgebras, then any simple subcoalgebra of C lies in one of the C_i's. Recall that a coalgebra is called co-semisimple if it is the sum of its simple subcoalgebras. The above property implies that if C is co-semisimple then any subcoalgebra of C has a complementary coalgebra and is the sum of its simple subcoalgebras.

(A.4) Let B be a Hopf F-algebra. A Hopf ideal J is called cocentral if it is so as a coideal. Any Hopf algebra B has a minimum cocentral Hopf ideal J_B; indeed put $J_B=B(\sum S^n I_B)B$ where S=antipode and I_B is the minimum cocentral coideal in B. The quotient $B^c=B/J_B$ is called the cocenter of B.

(A.5) Assume in (A.4) above that B^c is a group algebra and let $B=\oplus B_g$ be the $G(B^c)$-gradation corresponding to the coaction of B^c of B on the right $\rho: B\to B\otimes B^c$ (here G(A) means the group of group like elements of A); since J_B is cocentral, this gradation coincides with the gradation corresponding to the left coaction $\lambda: B \to B^c \otimes B$. We claim that all B_g's are subcoalgebras of B. Indeed $\mu: B \to B \otimes B$ is equivariant with respect to the right coactions of B^c on B and $B \otimes B$ (for the latter take the coaction ρ, on the second factor); in particular $\mu(B_g) \subset B \otimes B_g$. Similarly, μ is equivariant with respect to the left coaction of B^c on B and $B \otimes B$ (for the latter take the coaction λ on the first factor) and get $\mu(B_g) \subset B_g \otimes B$; consequently $\mu(B_g) \subset B_g \otimes B_g$.

Note also that $B_g \neq 0$ for all $g \in G(B^c)$; indeed, if $b \in B$ lies above $g \in G(B^c)$ then b_g, the g-homogeneous piece of b also lies above g, hence $b_g \neq 0$!

(A.6) A Hopf algebra is called co-semisimple [Sw] if it is so as a coalgebra. In [Sw] p.294 several characterisations of co-semisimple Hopf algebras are given; in particular if H is an affine algebraic F-group then $\mathcal{O}(H)$ is co-semisimple if and only if H is linearly reductive. Note that if B is co-semisimple anf F is algebraically closed then the cocenter B^c is a group algebra. Indeed, write $B = \bigoplus C_i$, C_i simple (cocentral) subcoalgebras. Then $B/I_B = \bigoplus (C_i/I_{C_i}) = \bigoplus (C_i/\ker \varepsilon_{C_i})$ is generated by group-like elements, hence so will be B^c, being a quotient of B/I_B.

(A.7) Let's discuss the notion of center of a Hopf algebra. A subset of a Hopf F-algebra A will be called central if each element of it commutes will all elements of A. Then A contains a maximum central sub Hopf algebra ^cA (we let ^cA be the maximum element of the family of all central, S-stable subcoalgebras of A). We call ^cA the center of A; it is contained (but apriori not equal to) the center of the underlying algebra of A. Note that if P(A) (resp. $P({}^cA)$) denotes the space of primitive elements of A (resp. ${}^cA$) then $P({}^cA)$ consists precisely of the central elements of P(A).

(A.8) Let A and B be two Hopf F-algebras. We denote simply by $\Delta(A) \oplus_F \Delta(B)$ the fibred product $\Delta(A) \times_{\mathrm{Der}\, F} \Delta(B)$. Start our investigation of $\Delta(A \otimes B)$ by noting that there is a natural F-linear projection $\Delta(A \otimes B) \to \Delta(A) \oplus_F \Delta(B)$, $\delta \mapsto (\delta_A, \delta_B)$ where $\delta_A = (1_A \otimes \varepsilon_B) \circ \delta \circ i_A$ where $i_A : A \to A \otimes B$ is the natural map $i_A(a) = a \otimes 1$ (and δ_B is defined similarily). Our projection admits a section (which is a Lie space map) $\Delta(A) \oplus_F \Delta(B) \to \Delta(A \otimes B)$ defined by $(\delta_1, \delta_2) \mapsto \delta_1 \otimes 1 + 1 \otimes \delta_2$. We shall identify from now on $\Delta(A) \underset{F}{\oplus} \Delta(B)$ with its image in $\Delta(A \otimes B)$.

(A.9) Next assume in (A.8) above that the cocenter B^c is a group algebra; by (A.6) this is the case for instance when B is co-semisimple. Then we shall define a remarkable Lie subalgebra $\Delta(B:A)$ of $\Delta(A \otimes B)$ lying in the kernel of the projection $\Delta(A \otimes B) \to \Delta(A) \underset{F}{\oplus} \Delta(B)$. Indeed let $P({}^cA)$ be the group of primitive elements of ${}^cA$, consider the $G(B^c)$-gradation $B = \bigoplus B_g$ on B defined by the coaction of B^c on B (cf.(A.5)) and for any group homomorphism $a \in \mathrm{Hom}(G(B^c), P({}^cA))$ define the F-linear map $\delta_a : A \otimes B \longrightarrow A \otimes B$ by the formula

$$\delta_a(x \otimes b) = \sum_g xa(g) \otimes b_g, \quad x \in A,\ b \in B,$$

where $b = \sum b_g$ is the decomposition of $b \in B$ into homogeneous pieces. Then clearly δ_a is an A-derivation and using the fact that all B_g's are subcoalgebras of B one checks that in fact $\delta_a \in \Delta(A \otimes B)$.
We defined a linear map

$$\mathrm{Hom}(G(B^c),\ P(^cA)) \to \Delta(A \otimes B)$$

whose image will be denoted by $\Delta(B:A)$.

Since $B_g \neq 0$ for all g (cf.(A.5)) it follows that $\mathrm{Hom}(G(B^c)$ $P(^cA)) \simeq \Delta(B:A)$ is an abelian Lie subspace of $\Delta(A \otimes B)$ (it is of course the analog of Im exp from (2.1)).

Our main result is the following:

(A.10) Theorem. Let A and B be Hopf F-algebras with B co-semisimple and F algebraically closed. Then

$$\Delta(A \otimes B) = (\Delta(A) \underset{F}{\oplus} \Delta(B)) \oplus \Delta(B:A)$$

Proof. We must show that every $\delta \in \ker(\Delta(A \otimes B) \to \Delta(A) \underset{F}{\oplus} \Delta(B))$ lies in $\Delta(B:A)$. Clearly δ is F-linear.

Step 1. We show that δ is an A-derivation. It is sufficient to check that $\delta(A \otimes 1) \subset A \otimes 1$. By (A.3) $B = F \oplus E$ for some coalgebra E. Consequently, for any $a \in A$ we may write $\delta(a \otimes 1) \in a_o \otimes 1 + A \otimes E$ with $a_o \in A$. We get that

$$\mu(\delta(a \otimes 1)) \in \mu(a_o \otimes 1) + A \otimes E \otimes A \otimes E,\ \mu(a_o \otimes 1) \in A \otimes 1 \otimes A \otimes 1$$

On the other hand

$$\mu(\delta(a \otimes 1)) = (\delta \otimes 1 + 1 \otimes \delta)\mu(a \otimes 1) \in (\delta \otimes 1 + 1 \otimes \delta)(A \otimes 1 \otimes A \otimes 1) \subset$$
$$\subset A \otimes 1 \otimes A \otimes 1 + A \otimes E \otimes A \otimes 1 + A \otimes 1 \otimes A \otimes E$$

We get that $\mu(\delta(a \otimes 1)) = \mu(a_o \otimes 1)$ hence $\delta(a_o \otimes 1) = a_o \otimes 1$ and our claim is proved.

Step 2. We show that for any subcoalgebra C of B, we have $\delta(1\otimes C)\subset A\otimes C$. Indeed by (A.3) C has a complementary coalgebra C'. For each $b\in C$ we have $\delta(1\otimes b)\in x_b + A\otimes C'$ with $x_b\in A\otimes C$.

We get

$$\mu(\delta(1\otimes b))\in\mu(x_b)+A\otimes C'\otimes A\otimes C',\ \mu(x_b)\in A\otimes C\otimes A\otimes C$$

On the other hand

$$\mu(\delta(1\otimes b))=(\delta\otimes 1+1\otimes\delta)\mu(1\otimes b)\in(\delta\otimes 1+1\otimes\delta)\ (1\otimes C\otimes 1\otimes C)\subset$$
$$\subset A\otimes B\otimes 1\otimes C+1\otimes C\otimes A\otimes B$$

We get that $\mu(\delta(1\otimes b))=\mu(x_b)$ hence $\delta(1\otimes b)=x_b$ and we are done.

Step 3. We show that for any simple subcoalgebra C of B there exists $a_C\in P(^CA)$ such that for all $b\in C$ we have $\delta(1\otimes b)=$ $=a_C\otimes b$.

Start with the following general remark: if C is any simple F--coalgebra and $L:C\to C$ is an F-linear map such that $\mu\circ L=$ $=(L\otimes 1_C)\circ\mu=(1_C\otimes L)\circ\mu:C\to C\otimes C$ then L is a scalar multiple of the identity. Indeed choose $t\in F$ such that $L-t1_C$ is not invertible. The equalities $\mu\circ(L-t1_C)=((L-t1_C)\otimes 1_C)\circ\mu=(1_C\otimes(L-t1_C))\circ\mu$ show that the image V of $L-t1_C$ is a subcoalgebra of C; since V is not thw whole of C, it must be zero, hence $L=t1_C$ and our remark is proved.

Now let $a^*\in A^o$ and

$$L_{a^*}=(a^*\otimes 1_C)\circ\delta\circ i_C: C\to A\otimes C\to A\otimes C\quad C$$

where $i_C:C\quad A\times C$, $i_C(x)=1\times x$. We claim that L_{a^*} is a secular multiple of 1_C .

To see this note first that $a^* = (a^* \otimes \varepsilon_A)\circ\mu : A \to F$ which implies that $\mu_C \circ (a^* \otimes 1_C) = (a^* \otimes 1_C \otimes \varepsilon_A \otimes 1_C)\circ\mu_{A\otimes C} : A \otimes C \longrightarrow$ $\to C \otimes C$. Now if $x \in C$ we get

$$\mu_C(L_{a^*}x) = \mu_C((a^* \otimes 1)\delta(1 \otimes x)) =$$
$$= (a^* \otimes 1_C \otimes \varepsilon_A \otimes 1_C)\mu_{A\otimes C}\delta(1 \otimes x) =$$
$$= (a^* \otimes 1_C \otimes \varepsilon_A \otimes 1_C)(\delta \otimes 1_{A\otimes C} + 1_{A\otimes C} \otimes \delta)\mu_{A\otimes C}(1 \otimes x) =$$
$$= (((a^* \otimes 1_C)\circ\delta) \otimes \varepsilon_A \otimes 1_C)\mu_{A\otimes C}(1 \otimes x) +$$
$$+ (a^* \otimes 1_C) \otimes ((\varepsilon_A \otimes 1_C)\circ\delta)\mu_{A\otimes C}(1 \otimes x)$$

The second term of the latter sum vanishes because δ projects into $0 \in \Delta(A) \oplus_F \Delta(B)$. Using the usual sigma notation $\mu_C x = \sum x_{(1)} \otimes$ $\otimes x_{(2)}$ we get

$$\mu_C(L_{a^*}x) = \sum(((a^* \otimes 1_C)\circ\delta) \otimes \varepsilon_A \otimes 1_C)(1 \otimes x_{(1)} \otimes 1 \otimes x_{(2)}) =$$
$$= \sum(L_{a^*}x_{(1)}) \otimes x_{(2)} = (L_{a^*} \otimes 1_C)(\mu_C x)$$

in other words $\mu_C \circ L_{a^*} = (L_{a^*} \otimes 1_C)\circ\mu$. Similarily, starting with the equality $a^* = (\varepsilon_A \otimes a^*)\circ\mu$ we get $\mu_C \circ L_{a^*} = (1_C \otimes L_{a^*})\circ\mu$. By a previous remark, L_{a^*} must be a scalar multiple of the identity and our claim is proved.

Our claim implies that there exists $a \in A$ such that $\delta(1 \otimes b) =$ $= a \otimes b$ for all $b \in C$. All we have to check now is that $a \in P({}^cA)$. Writing

$$0 = \mu_{A\otimes B}\delta(1 \otimes b) - (\delta \otimes 1_{A\otimes B} + 1_{A\otimes B} \otimes \delta)\mu_{A\otimes B}b =$$
$$= (1_A \otimes \overline{T} \otimes 1_B)((\mu_A a - a \otimes 1 - 1 \otimes a) \otimes \mu_B b)$$

where $\tau: A\otimes B\to B\otimes A$ is the twist map we get that $a\in P(A)$. Writing $xa\otimes b=\delta(x\otimes b)=\delta((1\otimes b)(x\otimes 1))=ax\otimes b$ for $x\in A$ we ge get $a\in P({}^cA)$.

<u>Step 4</u>. We show that whenever C and D are two simple subcoalgebras of B contained in the same B_g, we have $a_C=a_D$.

Indeed, by (A.1) and (A.4) the ideal $J_B=\ker(B\to B^c)$ is generated (as a 2-sided ideal) by elements of the form S^nx where $n\geq 0$ and x belongs to the union of all simple subalgebras of B. This together with Step 3 shows that the ideal $A\otimes J_B$ is stable under δ; in particular δ induces a derivation $\bar{\delta}\in(A\otimes B^c)$ and we have a commutative diagram:

$$\begin{array}{ccccc}
A\otimes B & \xrightarrow{\mu} & A\otimes B\otimes A\otimes B & \xrightarrow{1\otimes 1\otimes 1\otimes p} & A\otimes B\otimes A\otimes B^c \\
\downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta\otimes 1+1\otimes\delta} & & \downarrow{\scriptstyle\delta\otimes 1+1\otimes\bar{\delta}} \\
A\otimes B & \xrightarrow[\mu]{} & A\otimes B\otimes A\otimes B & \xrightarrow[1\otimes 1\otimes 1\otimes p]{} & A\otimes B\otimes A\otimes B^c
\end{array}$$

Now clearly the projection of $\bar{\delta}$ in $\Delta(A)\oplus_F\Delta(B^c)$ is zero, hence by the Step 3 applied to kg and B^c instead of C and B we get that $\bar{\delta}(1\otimes g)=a_g\otimes g$ for some $a_g\in A$. We have for any $b\in C$:

$$(\delta\otimes 1+1\otimes\bar{\delta})\circ(1\otimes 1\otimes 1\otimes p)\circ\mu(1\otimes b)=(\delta\otimes 1+1\otimes\bar{\delta})(1\otimes b\otimes 1\otimes g)=$$
$$=a_C\otimes b\otimes 1\otimes g+1\otimes b\otimes a_g\otimes g$$

and on the other hand

$(1\otimes 1\otimes 1\otimes p)\circ\mu\circ\delta(1\otimes b)=(1\otimes 1\otimes 1\otimes p)\mu(a_C\otimes b)=$

$$=a_C\otimes b\otimes 1\otimes g + 1\otimes b\otimes a_C\otimes g$$

from which we get $a_C=a_g$. Similarily $a_D=a_g$ and we are done.

<u>Step 5:</u> conclusion of the proof. By (A.3) each B_g is a sum of simple subcoalgebras, hence by Step 4 we get a function $G(B^c)\to P(A)$, $g\mapsto a(g)$ such that $\delta(1\otimes b)=a(g)\otimes b$ for all $b\in B_g$. To check that a is a group homomorphism, compute $(1\otimes b_1b_2)$ for $b_1\in B_{g_1}, b_2\in B_{g_2}$ with $b_1b_2\neq 0$ (note that $B_{g_1}B_{g_2}\neq 0$ for all g_1, g_2 because by (A.5) $0\neq B_{g_2}\subset B_{g_1^{-1}}B_{g_1}B_{g_2}$).

(A.11) Remark. The proof of Step 2 in (A.10) shows that for any co-semisimple Hopf algebra B, any subcoalgebra C of B and any $\delta\in\Delta(B)$ we have $\delta C\subset C$; in particular B is locally finite as a $\Delta(B)$-F-vector space.

This, in its turn implies the following statement: let Γ be a linear Δ-algebraic group such that the Hopf algebra $\mathcal{U}\{\Gamma\}$ is co-semisimple; then $\mathcal{U}\{\Gamma\}$ is finitely generated as a (non-differential) $\mathcal{U}$-algebra (i.e. Γ is automatically an f-group!). Indeed, using arguments similar to those in $[B_3]$ one sees that $\mathcal{U}\{\Gamma\}=(\mathcal{U}\{\Gamma\}^{\Delta})\otimes_{\mathcal{K}}\mathcal{U}$. But the right hand side of the latter equality cannot be finitely generated as a Δ-$\mathcal{U}$-algebra unless $\mathcal{U}\{\Gamma\}^{\Delta}$ is finitely generated as a $\mathcal{K}$-algebra and our asswertion follows. Our Theorem (A.10) can be used to produce some classification statements for Δ-algebraic groups (in the same way as (4.3) was used in section 6); since the results obtainable are far less complete than in the case of f-groups we shall not give them here

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