On the structure of positive completion of partial matrices

by

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I. INTRODUCTION

Starting with the work of H.Dym and I.Gohberg on banded extensions [D - G], some questions regarding positive completions become of interest. Thus, in the paper [G J SW] there was solved the important problem of describing all positive patterns (partial matrices) admitting positive completions. The main result states that the associated graphs are chordal.

On the other hand, in [C] it was presented a description of the structure of positive block-matrices in terms of certain free parameters.

This result yields the description of all positive completions of positive partial matrices with proper interval graphs to be done. Subsequently it was remarked in [ACC] that even in the case of chordal graphs a completion procedure can be indicated, as a sequence of reductions to completions of proper interval graphs, and to make the above mentioned parametrization of completions available.

The main purpose of this paper is to render explicit this remark. Thus, a description of the structure of all positive completions of positive partial matrices is given in terms of certain free parameters.

2 PRELIMINARIES

1. By a positive partial matrix (or positive pattern), we denote a specification of certain entries in a (Hermitian) frame such that all specified principal matrices are positive. The graph of such a partial matrix is associated in an usual manner: the graph has vertices $\{1, 2, \ldots, n\}$ (n being the dimension of the given pattern) and an edge between i and j if the (i, j)-the entry is specified. The problem of characterizing those graphs for which any associated positive partial matrix has positive completions was solved in [GJ SW].

These are the chordal graphs, i. e. the graphs with no minimal simple circuitof four or more edges - we use [G] for the terminology and all relevent results concerning graph theory. For instance, for an undirected graph $G = (V, E)$, V is the set vertices and E is the set of edges. For $A \subset V$, we define the subgraph induced by A, to be $G_A = (A, E_A)$, where $E_A = \{xy \in E \ / \ x \in A \text{ and } y \in A\}$. A subset $A \subset V$ is called a clique if its induced subgraph is complete, i.e. every pair of distinct vertices is adjacent.

A subclass of the chordal graphs is represented by the interval graphs, i.e. graphs which are intersection graphs of a family of interval on the realline. A proper interval graphs is obtained from a family of intervals such that no interval property contains another. For these graphs, there exists an ordering of the vertices such that any associated partial matrix has a block-banded structure :

$$(2.1)\quad \begin{bmatrix} S_{11}, & S_{12}, \dots, & S_{1e} & & & \\ & S_{22}, \dots, & S_{2e} & , \dots, & S_{2e_2}, & \\ & & & & \cdot & \\ & & & & & S_{nn} \end{bmatrix}$$

For a given graph G, we will denote by M(G) the pattern associated to G, such that the graph associated to M(G) is exactly G.

2. We will need a certain structure of positive block-matrices. Such structures were develloped in [L-A-K], [L-A], [C]. Most of the notation for Hilbert space operators is taken from [Sz-N-F]. Thus, for two Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$, $\mathcal{L}(\mathcal{H},\mathcal{H}')$ is the set of the linear bounded operators from $\mathcal{H}$ into $\mathcal{H}'$. $O_{\mathcal{H}}(I_{\mathcal{H}})$ is the zero (identity) operator on the underlying space. For a contraction $T\in\mathcal{L}(\mathcal{H},\mathcal{H}')$ (i.e. $\|T\|\leq 1$), we define $D_T=(I-T^*T)^{1/2}$, $\mathcal{D}_T$ the closure of the range of D_T and the unitary operator

$$J(T):\mathcal{H}\oplus\mathcal{D}_{T^*}\longrightarrow\mathcal{H}\oplus\mathcal{D}_T$$

$$(2.2)\qquad J(T)=\begin{bmatrix} T & D_{T^*} \\ D_T & -T^* \end{bmatrix}$$

We are now concerned with the following object : for a family $\{\mathcal{H}_k\}_{1\leq k\leq n}$ we define the positive operator :

$$M:\bigoplus_{k=1}^{n}\mathcal{H}_k\longrightarrow\bigoplus_{k=1}^{n}\mathcal{H}_k$$

$$(2.3)$$

$$M=(S_{mp})\ 1\leq m,\ p\leq n.$$

where $S_{ij}\in\mathcal{L}(\mathcal{H}_j,\mathcal{H}_i)$ and $S_{ii}=I_{\mathcal{H}_i}$ (without restrictingthe generality). We follow [C] in order to state a one-to-one correspondence :

(2.4) $S_{i,\ i+1} = G_{i,\ i+1}$

for $1 \leq i \leq n-1$, and otherwise,

(2.5) $S_{ij} = R_{i,\ j-1}\, U_{i+1,\ j-1}\, C_{i+1,\ j} +$

$$+ D_{G^*_{i,\ i+1}} \cdots D_{G^*_{i,\ j-1}}\, G_{ij}\, D_{G_{i+1,\ j}} \cdots D_{G_{j-1,\ j}}$$

between the set of positive operators (2.3) and the families of contractions $\mathcal{G} = \{G_{i,\ j}\}_{1 \leq i \leq j \leq n}$ such that $G_{ii} = O_{\mathcal{H}_i}$ for $1 \leq i \leq n$ and for $i < j$, $G_{ij} : \mathcal{D}_{G_{i+1,\ j}} \longrightarrow \mathcal{D}_{G^*_{i,\ j-1}}$ – we call these relation, compatibility conditions.

Let us explain the notation. For a fixed i, the family $\{G_{ik}\}_{i < k \leq j}$ defines a row contraction

(2.6) $R_{i,\ j} : \bigoplus_{k=i+1}^{j} \mathcal{D}_{G_{i+1,\ k}} \longrightarrow \mathcal{H}_i$

$$R_{i,j} = (G_{i,i+1},\ D_{G^*_{i,i+1}}\, G_{i,i+1}, \ldots, D_{G^*_{i,i+1}} \cdots D_{G^*_{i,j-1}}\, G_{ij})$$

By an obvious duality there are defined the column contractions C_{ij}.

When necessary, the parameters on which is R_{ij} or C_{ij} constructed, are explicitely written, $R_{ij} = R_{ij}(G_{i,\ i+1}, \ldots, G_{ij})$, and $C_{ij} = C_{ij}(G_{j-1,\ j},\ G_{j-2,\ j}, \ldots,\ G_{ij})$.

The unitary operators U_{ij} are given $U_{ii} = I_{\mathcal{H}_i}$, $1 \leq i \leq n$ and for $j > i$,

$$U_{ij} : \bigoplus_{k=-j}^{-i} \mathcal{D}_{G^*_{-k,\ j}} \longrightarrow \bigoplus_{k=i}^{j} \mathcal{D}_{G_{ik}}$$

(2.7)

$$U_{ij} = J_j(G_{i,i+1})\, J_j(G_{i,i+2}) \cdots J_j(G_{ij})\,(U_{i+1,j} \oplus I_{\mathcal{D}_{G^*_{ij}}}).$$

As a consequence of the algorithm (2.5), we get the following formula for the determinant of M in case all underlying Hilbert spaces are of finite dimension. That is, (2.8) $$\det M = \prod_{1 \leq i < j \leq n} \det D^2_{G_{ij}} .$$

From (2.8) we deduce a variant of Fisher-Hadamard inequality : for $A, B \subset \{1,\ldots,n\}$ two sets of induces,

(2.9) $$\det M(A \cup B) = \frac{\det M(A)\ \det M(B)}{\det M(A\ \ B)} \prod_{(i,j) \notin (A\times A) \atop (B\times B)} \det D^2_{G_{ij}}$$

where $M(A)$ is the principal submatrix of M subordinate to the index set A.

3. Consider $F_{ii} = I_{\mathcal{L}_i}$, $1 \leq i \leq n$ and for $j > i$

$$F_{ij} : \bigoplus_{k=i}^{j} \mathcal{H}_k \longrightarrow \bigoplus_{k=i}^{j} \mathcal{D}_{G_{ik}}$$

(2.10) $$F_{ij} = \begin{bmatrix} F_{i,j-1}, & U_{i,j-1} & C_{ij} \\ O & , D_{G_{ij}} \ldots & D_{G_{j-1,j}} \end{bmatrix}$$

The following relation were proved in [C] :

(2.11) $$M_{ij} = F^*_{ij} F_{ij}$$

and

(2.12) $$(S_{i,i+k},\ldots,S_{i+k-1,i+k})^t = F^*_{i,i+k-1} U_{i,i+k-1} C_{i,i+k}.$$

Based on (2.12), we define

(2.13) $$H_{ij} = F^*_{ij} U_{ij}$$

and we have

(2.14) $$H_{ij} H^*_{ij} = F^*_{ij} U_{ij} U^*_{ij} F_{ij} = M_{ij} .$$

Moreover, it is quite easy to see that H_{ij} are lower triangular.

3. COMPLETION SEQUENCES

The connection between Gaussian elimination and Schur reduction is illustrated from the very begining by the following remark : performing a Gaussian elimination in $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ means to compute

$$\begin{bmatrix} I & O \\ -BA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \begin{bmatrix} I & -A^{-1}B \\ O & I \end{bmatrix} = \begin{bmatrix} A & O \\ O & C-BA^{-1}B \end{bmatrix}$$

and, as $\begin{bmatrix} I, & -A^{-1}B \\ O & I \end{bmatrix} = \begin{bmatrix} I & A^{-1}B \\ O & I \end{bmatrix}^{-1}$, we get Frobenius-Schur identity

$$\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} = \begin{bmatrix} I & O \\ B^*A^{-1} & I \end{bmatrix} \begin{bmatrix} A & O \\ O & C-B^*A^{-1}B \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ O & I \end{bmatrix} \tag{3.1}$$

This is the basis for the structure of positive block-matrices as described by (2. 5). Moreover, from (2. 5) we get the structure of all positive completions of a partial positive matrix associated to a proper interval graph. Indeed, in this case, we can reorder the vertices of the graph in such a way to put the associated matrix in a block- banded form :

$$\begin{bmatrix}
I & S_{12} \; \ldots \; S_{1,e_1-1} & X_{1e_1} \; \ldots\ldots & & & X_{1n} \\
S_{12}^*, & I \; \ldots & & & & \\
\cdot \quad \cdot & S_{q_1,e_1-1} & X_{q_1,e_1} \; \ldots\ldots & & & \cdot \\
\cdot \quad \cdot & & S_{q_1+1,e_1} \ldots S_{q_1+1,e_2-1}, & X_{q_1+1,e_2} & & \cdot \\
\cdot \quad \cdot & & & X_{q_2e_2}, \ldots & & \cdot \\
\cdot \quad \cdot & & & & X_{q_pe_p} \ldots & X_{q_p,n} \\
 & & & & & S_{n-1,n} \\
\cdot \quad \cdot & & & & & \\
X_{1n}^* \quad \cdot & \cdot & \cdot & \cdot \quad \cdot & & I
\end{bmatrix}$$

where $1 \leq e_1 \leq \ldots \leq e_p \leq n$, $1 \leq q_1 < q_2 \ldots < q_p \leq n$ are positive integers, the entries marked by S are those imposed and the entries marked by X must be determined. It is convenable to use (2.1) in this form in order to obtain that any positive completion is given by a set of contractions :

$$
\begin{array}{lllll}
G_{1e_1}, \ldots\ldots & & & & G_{1n} \\
\vdots & & & & \vdots \\
G_{q_1 e_1} & & & & G_{q_1 n} \\
& & G_{q_1+1,\, e_2} & \cdots\cdots & G_{q_1+1,n} \\
& & G_{q_2\, e_2} & \cdots\cdots & G_{q_2,n} \\
& & & & G_{q_p,n}
\end{array} \tag{3.3}
$$

satisfying obvious compatibility relations. Moreover,

$$X_{mk} = A_{mk} + L_{mk}\, G_{mk}\, R_{mk} \tag{3.4}$$

with all elements determined by (2.5) - for details see [C] and [ACC].

Now, we take into account a chordal graph $G = (V, E)$ and let M_0 be a positive partial block-matrix associated to it.

The following notion of completion sequence is introduced.

A sequence of positive integers :

$$1 < r_1 < r_2 \quad \ldots < r_s = n$$

is called a completion sequence for the chordal graph G if it satisfies the properties :

(a) For r_1 there exists an ordering $\sigma(r_1)$ of r_1 vertices in V (we denote by $V(r_1)$ the set of these r_1 vertices) such that the partial matrix $M(r_1)$ associated to $G_{V(r_1)}$ is block- banded.

(b) For any $1 \leq k \leq s$, there exists an ordering $\sigma(r_k)$ of r_k vertices including $V(r_{k-1})$ (and we denote by $V(r_k)$ the set of all these vertices) such that the partial matrix $M(r_k)$ associated to $G_{V(r_k)}$ has the following three properties :

(i) the first r_{k-1} are still those in $V(r_{k-1})$, although it is possible they appear in other order.

(ii) $M(r_k)$ is "almost banded", i.e. it is banded in the sense of (3.2) with $e_1 = r_{k-1}$, excepting the fact that the principal $r_{k-1} \times r_{k-1}$ matrix is viewed as a "whole" (and it may be not banded in the sense of (3.2)).

(iii) let $h_k < r_{k-1} + 1$ be the **least** integer with position $(h_k, r_{k-1} + 1)$ imposed (i.e. marked with an S). Then all entries in the principal matrix given by the set of indices

$(i, j)_{h_k \leq i,\, j \leq r_{k-1}}$ are also imposed.

3.1 <u>PROPOSITION</u> If $G = (V, E)$ is a chordal graph, than there exists at least one completion sequence of G.

<u>PROOF</u> By a result in the theory of chordal graphs (Theorem 4.1 in [G]), G has a perfect elimination scheme, that is, there exists an ordering $\sigma = [v_1, \ldots, v_n]$ of the vertices of G such that each v_i is a simplicial vertex of the graph $G\{v_i, \ldots, v_n\}$ .Recall that a vertex v of G is simplicial if Adj(v), the set of all adjacent vertices of v, is a clique.

Now, take σ such a perfect elimination scheme of G and define $A_k = \{v_k, \ldots, v_n\}$. Let e be the **least** integer for which A_e is a clique. Then, A_{e-1} is partitioned as :

(3.5) $A_{e-1} = \{v_{e-1}\} \cup (A_e \cap \mathrm{Adj}(v_{e-1})) \cup B_{e-1}$

where B_{e-1} is the complement of $\{v_{e-1}\} \cup (A_e \cap Adj(v_{e-1}))$ in A_{e-1}.

Define $r_1 = n - e + 2$, $V(r_1) = A_{e-1}$ and

$$(3.6) \qquad \sigma(r_1) = \left[B_{e-1} \; , \; [A_e \cap Adj(v_{e-1})] \; , \; v_{e-1} \right]$$

where B_{e-1} and $A_e \cap Adj(v_{e-1})$ denote orderings (arbitrarely choosen of the sets B_{e-1} and $A_e \cap Adj(v_{e-1})$.

It is obvious that $M(r_1)$ is block-banded in the sense of (3.2).

Further on, for $k = 2, \dots, e-1$, we define

$$r_k = r_{k-1} + 1, \; V(r_k) = A_{e-k}$$

and A_{e-k} is partitioned as

$$(3.7) \qquad A_{e-k} = \{v_{e-k}\} \cup (A_{e-k+1} \cap Adj(v_{e-k})) \cup B_{e-k}$$

Findly, we define

$$(3.8) \qquad \sigma(r_k) = \left[[B_{e-k}] \; , \; [A_{e-k+1} \cap Adj(v_{e-k})] \; , \; v_{e-k} \right] .$$

$M(r_k)$ obviously satisfies (b)(i) and (ii) in the definition of completion sequences, and, as σ was chosen as a perfect elimination scheme, $A_{e-k+1} \cap Adj(v_{e-k})$ is a clique, i.e. (b)(iii) also holds. ▢

Based on Proposition 3.1 and on the existence of positive completions of positive partial matrices of the block-banded form (3.2), we already obtained another proof of Theorem 7 in [G J SW]. Moreover, we can get a description of all positive completions of a positive partial matrix associated to a chordal graph.

Fix a chordal graph $G = (V, E)$, M_o a positive partial block-matrix associated to it and $1 < r_1 < r_2 \dots < r_s = n$ a completion sequence of G (for a certain simplicity, we can suppose that this completion sequence is produced as in the Proof of Proposition 3.1, i.e. $r_k = r_{k-1} + 1$).

Denote by σ_k, $k = 1,\ldots,s$ the permutation of $\{1,\ldots,n\}$ associated to the orderings $\sigma(r_k)$. Moreover, we define the unitary operators $U_k = (\delta_{\sigma_k(i),\sigma_{k-1}(j)})_{1\leq i,\, j\leq r_{k-1}}$, $2\leq k\leq s$. The set of the specified entries in M_o is denoted by $\mathcal{S}$ andc the set of unknown positions is denoted by $\mathcal{X}$.

Now, we can describe a completion procedure along the fixed completion sequence of G.

Every positive completion of $M(r_1)$ is given by a frame of parameters :

$$
(3.9)\qquad
\begin{array}{ccc}
F^{(1)}_{\sigma_1(1)\sigma_1(2)} \cdots\cdots & F^{(1)}_{\sigma_1(1)\sigma_1(r_1-1)} & G^{(1)}_{\sigma_1(1)\sigma_1(r_1)} \\
 & & \vdots \\
 & & G^{(1)}_{\sigma_1(h_1-1)\sigma_1(r_1)} \\
 & & F^{(1)}_{\sigma_1(h_1)\sigma_1(r_1)} \\
 & & \vdots \\
 & & F^{(1)}_{\sigma_1(r_1-1)\sigma_1(r_1)}
\end{array}
$$

where the parameters denoted by $F^{(1)}$ are actually uniquely determined by M_0 and those denoted by $G^{(1)}$ are the real parameters of the completion. The correspondence between $\{G^{(1)}_{ij}\}_{\substack{(i,j)\in\mathcal{X}\\ i\leq r_1\\ j\leq r_1}}$ and the positive completions of $M(r_1)$ is one-to-one.

Fix such an extension M_1.

Further on, every positive completion $M(r_2)$ having the positions in $M(r_1)$ fixed by M_1 is given by a frame of parameters :

$$F^{(2)}_{\sigma_2(1)\sigma_2(2)} \cdots F^{(2)}_{\sigma_1(1)\sigma_2(r_1)} \qquad G^{(1)}_{\sigma_2(1)\sigma_2(r_2)}$$

$$\vdots$$

$$G^{(1)}_{\sigma_2(h_2-1)\sigma_2(r_2)}$$

(3.10)

$$F^{(1)}_{\sigma_2(h_2)\sigma_2(r_2)}$$

$$\vdots$$

$$F^{(1)}_{\sigma_2(r_2-1)\sigma_2(r_2)}$$

where the parameters $F^{(2)}$ are associated to $U_2 M_1 U_2^*$, the parameters $F^{(1)}_{\sigma_2(r_2-1)\sigma_2(r_2)}, \ldots, F^{(1)}_{\sigma_2(h_2)\sigma_2(r_2)}$ are derived in a positive matrix containing only elements from M_0 and $G^{(1)}_{\sigma_2(h_2-1)\sigma_2(r_2)}, \ldots, G^{(1)}_{\sigma_2(1)\sigma_2(r_2)}$ are the real parameters of the positive completion of $M(\mathcal{R}_2)$. We are faced with two problems. First, the parameters $G^{(1)}_{\sigma_2(h_2-1)\sigma_2(r_2)}, \ldots, G^{(1)}_{\sigma_2(1)\sigma_2(r_2)}$ satisfy obvious compatibility relations, but not in terms of the parameters with upper index 1.

Second, it is desirable to obtain formulas for $X_{\sigma_2(1)\sigma_2(r_2)}, \ldots, X_{\sigma_2(h_2-1)\sigma_2(r_2)}$ in terms of the parameters with upper index 1.

A possible variant, goes as follows by (2.14),

$$H_{\sigma_2(1),\sigma_2(r_2-1)} H^*_{\sigma_2(1),\sigma_2(r_2-1)} = U_2 M_1 U_2^* =$$

$$= U_2 H_{\sigma_1(1),\sigma_1(r_1-1)} H^*_{\sigma_1(1),\sigma_1(r_1-1)} U_2.$$

and it results that, there exists a uniquely determined unitary operator :

$$\Omega_2 : \mathcal{H} \oplus \mathcal{D}^{*}_{F^{(2)}_{\sigma_2(r_1-1),\sigma_2(r_1)}} \oplus \dots \oplus \mathcal{D}^{*}_{F^{(2)}_{\sigma_2(1),\sigma_2(r_1)}} \longrightarrow$$

$$\longrightarrow \mathcal{H} \oplus \mathcal{D}^{*}_{F^{(1)}_{\sigma_1(r_1-1)\sigma_1(r_1)}} \oplus \dots \oplus \mathcal{D}^{*}_{G^{(1)}_{\sigma_1(1)\sigma_1(r_1)}}$$

such that

$$H_{\sigma_2(1)\sigma_2(r_1-1)} = U_2 \, H_{\sigma_1(1)\sigma_1(r_1-1)} \Omega_2$$

Consequently, by (2.12),

$$\begin{bmatrix} X_{\sigma_2(1)\sigma_2(r_2)} \\ \\ S_{\sigma_2(r_2-1)\sigma_2(r_2)} \end{bmatrix} = U_2 \, H_{\sigma_1(1)\sigma_1(r_1-1)} \Omega_2 \, C(F^{(1)}_{\sigma_2(r_2-1),\sigma_2(r_2)}, \dots, G^{(1)}_{\sigma_2(1)\sigma_2(r_2)}$$

Of course, taking

$$(3.12) \qquad C_2 = \Omega_2 \, C(F^{(1)}_{\sigma_2(r_2-1),\sigma_2(r_2)}, \dots, G^{(1)}_{\sigma_2(1),\sigma_2(r_2)})$$

C_2 is a column contraction given by certain parameters. These parameters satisfy now good compatibility conditions, $X_{\sigma_2(1),\sigma_2(r_2)}, \dots, X_{\sigma_2(h_1-1),\sigma_2(r_2)}$ are expressed in terms of the parameters with upper index 1 and these new parameters, but now, we can not disscern that there are free and imposed parameters.

Now, this procedure can be continued in an obvious way.

3.2 <u>COROLLARY</u> If the underlying Hilbert spaces are finite dimensional and M is a positive completion of M_0, then

$$\det M = \prod_{(i,j)\in\mathcal{G}} \det D^2_{F^{(1)}_{ij}} \prod_{(i,j)\in\mathcal{X}} \det D^2_{G^{(1)}_{ij}}.$$

<u>PROOF</u> By (2.8),

$$\det M_1 = \prod_{\substack{(i,j)\in\mathcal{G} \\ i,j\le r_1}} \det D^2_{F^{(1)}_{ij}} \prod_{\substack{j=r_1 \\ i>h_1}} \det D^2_{G^{(1)}_{ij}} .$$

Further on, if M_2 is the principal matrix of M given by $V(r_2)$, then again by (2.8),

$$\det M_2 = \prod_{\substack{(i,j)\in\mathcal{G} \\ i,j\leq r_2}} \det D^2_{F^{(1)}_{ij}} \prod_{\substack{j=r_2 \\ i>h_2}} \det D^2_{G^{(1)}_{ij}} \cdot$$

$$\cdot \prod_{1\leq i,j\leq r_1} \det D^2_{F_{ij}} =$$

$$= \prod_{\substack{(i,j)\in\mathcal{G} \\ i,j\leq r_2}} \det D^2_{F^{(1)}_{ij}} \prod_{\substack{j=r_2 \\ i>h_2}} \det D^2_{G^{(1)}_{ij}} \det M_1.$$

The formula now follows by induction. ∎

As a consequence we obtain the interpretation of the maximum determinant principle in [D - G] and [G JSW] .

3.3 <u>COROLLARY</u> If G is a chordal graph, among the positive completions of M_0, there exists a unique such a completion with maximum determinant. This completion is given by the parameters $G^{(1)}_{ij} = 0$, $(i,j)\in\mathcal{X}$. ∎

Inheritance (or permanence) principles were discussed in [EGL] for proper interval graphs and in [JR] for chordal graphs. It was shown in [JR] that this principles are connected to the notion of increasing chordal sequence of a chordal graph G, which is a sequence of chordal graphs $G_0 = G, G_1, \ldots, G_t$ such that G_t is the complete graph and each G_j is obtained from G_{j-1} by adding exactly one new edge in a way that G_j contains exactly one maximal clique which is not a clique of G_{j-1}. It turns out that the existence of such an increasing chordal sequence of G is equivalent with the chordality of G (see [G JSW]).

As a completion sequence generates in an obvious way an increasing chordal sequence of G, as consequence of Proposition 3.1 and Corollary 3.2, we obtain a proof for Theorem 3.3 in [JR].

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