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POTENTIALS IN STANDARD H-CONES OF FUNCTIONS

N. Boboc and Gh. Bucur

0. Introduction. The aim of this paper is to clarify the relation between different classes of potentials which where considered in a standard H-cones of functions S on a set X.

Some of them like natural potential and fine potential are studied in the theory of harmonic spaces and they are related with the initial topology of X or with the fine topology on X. The existence of a strictly positive fine potential on X is equivalent with the fact that any universally bounded element of S is nearly continuous i.e. a sum of a series of universally cartinuous element of S). In the frame of harmonic spaces the existence of a strictly positive fine potential is equivalent with axiom D. It is proved that if p is a finite continuous potential on X or p is a finite fine potential on X then p is nearly continuous. (Other classes of potentials are related with the "harmonic carrier" or "fine harmonic carrier" and we establish the "equivalence with the preceding ones for the case of superharmonic (resp. fine superharmonic) elements of S.

A particular study is devoted to the classe of Green potentials. If X is a Green set associated with S then any fine potential is a Green potential. The converse assertion is true iff the fine topology on X is smaller then the cofine topology on X.

Other results concern the classes of pure potentials and universally potentials. It is shown that any pure potential is nearly bounded and any nearly bounded is universally potential.

Also if any universally bounded element of S is nearly continuous then any universally potential of is nearly continuous.

1. Pure potentials and nearly bounded elements in an H-cone.

In the sequel S will be an H-cone and we prove some properties concerning the pure potentials and the nearly bounded elements of S.

As in [1] we say that an element $h\in S$ is subtractible if for any $s\in S$ such that $h\leq s$ we have $h\precsim s$ where $\precsim$ is the symbol for the specific order in S.

An element $p\in S$ is termed pure potential if zero is the only subtractible minorant of p.

Theorem 1.1. Let u be a weak unit in S and let s be an element of S. Then the element

$$\bigwedge_{n\in N} R(s-nu)$$

is subtractible. Particularly if $p\in S$ ia a pure potential then

$$\bigwedge_{n\in N} R\,(p-nu) = 0$$

Proof. The last part of the statement follows from the first one sin ce the element $\bigwedge_n R(p-nu)$ is subtractible and specifically dominated by p.

We denote by f_n the positive element of S-S defined by $f_n:=s-s\wedge nu$ and by Bn the balayage on S given by

$$t\in S \Longrightarrow B_n t:=\bigvee_{n\in N} R(t\wedge nf)$$

Obviously $Bn(R(s-nu)) = R(s-nu)$ and therefore

$$Bn(R(s-mu) = R(s-mu)$$

for any $m\in N$, $m\geq n$. If we denote $s_n:=R(s-nu)$ we have $s_n\precsim s$ and therefore

$$s = s_n + s'_n\ ,\ s'_n\in S$$

Obviously the sequence $(s_n)_n$ is decreasing and the sequence $(s'_n)_n$ is increasing with respect to the natural order. For any $m\geq n$ we have

$$B_n s = B_n s_m + B_n s'_m = s_m + B_n s'_m.$$

If we put $h:=\bigwedge_n s_n$, $h':=\bigvee_n s_n$ we get

$$s = h + h', \quad B_n s = B_n h + B_n h',$$

$$B_n h' = \bigvee_m B_n s'_m$$

$$B_n s = h + \underset{m}{} B_n s'_m = h + B_n h'$$

and therefore $B_n h = h$ for any $n \in N$. On the other hand by the definition of B_n we have $B_n u \leq \frac{1}{n} s$. Hence $\bigwedge_{n \in N} B_n u = 0$ and therefore, using ([1], Theorem 1) we deduce that h is subtractible.

Definition. Suppose that S possesses a weak unit. An element $s \in S$ is called universally bounded if for any weak unit of S there exists $\alpha > 0$ such that $s \leq \alpha u$. An element $s \in S$ is termed nearly bounded if there exists a family $(s_i)_{i \in I}$ of universally bounded elements of S such that $s = \sum_{i \in I} s_i$.

Theorem 1.2. If $s \in S$ is nearly bounded then

$$\bigwedge_{n \in N} R(s - nu) = 0$$

for any weak unit $u \in S$.

Conversselly, if there exists a weak unit v of S which is nearly bounded then any $s \in S$ for which $\bigwedge_n R(s-nu) = 0$ is nearly bounded.

Particularly, in this case, any pure potential is nearly bounded.

Proof. Suppose that $s \in S$ is nearly bounded and let $(s_i)_{i \in I}$ be a family of universally bounded elements of S such that $s = \sum_{i \in I} s_i$. Then for any weak unit u of S and any finite subset $J \subset I$ there exists $n_0 \in N$ such that

$\sum_{y \in J} s_y \leq n_0 u$ and therefore

$$R(s - n_0 u) \leq \sum_{i \notin J} s_i,$$

$$\bigwedge_n R(s-nu) \leq \bigwedge_{J \subset I} \sum_{L \notin J} s_i = 0$$

Let now v be a weak unit of S which is nearly bounded and $s \in S$ be such that

$$\bigwedge_n R(s-nv) = 0$$

We denote

$$s_n := R(S-nv)$$

Since $s_n \precsim s$ and $s-s_n \leqslant nv$ we get that $s-s_n$ is nearly bounded. Since $\bigwedge_n s_n = 0$ we deduce that $\curlyvee_n (s-s_n) = s$ and therefore s is nearly bounded.

<u>Corollary.</u> 1-3 If S is a standard H-cone then $p \in S$ is nearly bounded iff for any weak unit u of S we have

$$\mathop{}_{n} R(p-nu) = 0$$

Particularly any pure potential is nearly bounded.

<u>Proof.</u> The assertion follows using the fact that there exists in S a weak unit which is nearly continuous and therefore a nearly bounded element of S.

<u>Theorem 1-4.</u> Suppose that the dual S^* of S is reprezented as a stan - dard H-cone of functions on a semisaturated set Y. Then an element $s \in S$ is nearly bounded iff s is an H-measure on Y which does not charge any polar subset of Y.

<u>Proof.</u> Suppose that s is an H-measure on X which does not charge any polar subset of X and let s' be a specific minorant of s such that $s' \curlywedge q = 0$ for any nearly bounded element q.

We want to shaw that s' = 0. If s'(A) = 0 for any semipolar subset of Y then using ([9], Proposition 5.4.2) it follows that s' is nearly continuous and therefore s=0.

Suppose that there exists a semipolar subset A of Y with s'(A) = 0. Obviously we may suppose that A is a Borel subset of Y. As in ([10], Theorem 1.10 we may construct a nearly bounded element $\mu \in S$ which is an H-measure carried by A and such that a Borel subset B of A is polar iff $\mu(B) = 0$. From the above considerations we deduce that the measure λ on A given by

$$\lambda(M) = s'(M)$$

is absolutely continuos with respect to μ and therefore there exists a positive Borel function $f:A \longrightarrow [0,\infty)$ such that $\lambda = f\cdot\mu$. Since μ is nearly bounded then is also nearly bounded. Since s' we get =0,

$$s'(A) = \lambda(A) = 0$$

wich contradicts the assumption $s'(A) \neq 0$.

Suppose now s nearly bounded. Then s is an H-measure on the saturated set Y_1 of Y. Since $s = \sum_n s_n$ where s_n is universally bounded for any $n \in N$ then s does not charge any polar subset of Y_1.

Indeed in the contrary case there exists $n_o \in N$ and a Borel polar subset A of Y_1 such that $s_{n_o}(A) > 0$.

Let $t \in \mathcal{S}^*$ be such that $t = \infty$ on A. We have the contradictory relations

$$ks_{n_o}(A) \leq s_{n_o}(t) < \infty \qquad (\forall)\quad k \in N.$$

2. Potentials in a standard H-cone of functions on a topological space.

Let S be a standard H-cone of functions on a set X and let $\mathcal{T}$ be a topology on X which is smaller than the fine topology $\mathcal{T}_1$ on X. We denote also by $\mathcal{T}_o$ the natural topology on X.

Definition. An element $p \in S$ is termed $\mathcal{T}$-potential if for any increasing sequence $(Gn)_n$ of $\mathcal{T}$ such that $\overline{G_n}^{\mathcal{T}} \subset G_{n+1}$ for any $n \in N$ and such that $\bigcup_n Gn = X$ we have $\bigwedge_n B^{X \setminus Gn} p = 0$.

If $\mathcal{T} = \mathcal{T}_o$ (resp. $\mathcal{T} = \mathcal{T}_1$) then a $\mathcal{T}$- potential will be called potential (resp. fine potential) on X.

Obviously the set of all $\mathcal{T}$-potentials on X (denoted by $P_{\mathcal{T}}(X)$ is a convex subcone of S which is solid with respect to the natural order and for any sequence $(p_n)_n$ from $P_{\mathcal{T}}(X)$ such that $\sum_n p_n \in S$ we have $\sum_n p_n \in P_{\mathcal{T}}(X)$.

We rememeber ([4], [6]) the following definition. An element $p \in S$, is called <u>quasicontinuous</u> if there exists a decreasing sequence $(G_n)_n$ of open subset of X such that

$$\bigwedge_n B^{G_n} 1 = 0$$

and the restriction of f to any subset $X \setminus G_n$ is finite continuous.

<u>Theorem 2-1.</u> If $p \in S$ is a potential on X wich is bounded and quasi-continuous then p is nearly continuous.

<u>Proof.</u> First we remark that for any $p' \in S$, $p' \neq 0$, $p' \lneqq p$ we have $\operatorname{carr}_{\bar{X}} p' = 0$. Indeed, p' is obviously a bounded potential on X and therefore $\operatorname{carr}_{\bar{X}} p' \neq \emptyset$. If $\operatorname{carr}_X p' = 0$ we deduce, using the relation $\operatorname{carr}_X p' = X \cap \operatorname{carr} p'$, that $\operatorname{carr}_{\bar{X}} p' \subset \bar{X} \setminus X$. Let $(G_n)_n$ be a decreasing sequence of open subset of $\bar{X}$ such that $G_n \supset \bar{G}_{n+1}$ for any $n \in N$ and $\bigcap_n G_n = \operatorname{carr}_{\bar{X}} p'$. Hence $p' = B^{G_n \cap X} p' = B^{\bar{G}_n \cap X} p'$, for any $n \in N$. On the other hand the sequence $(D_n)_n$ of open subset of X defined by $D_n = X \setminus \bar{G}_n$ is increasing to X and therefore, p' being a potential on X, we have the contradictory relation

$$0 = \bigwedge_n B^{X \setminus D_n} p' = \bigwedge_n B^{\bar{G}_n \setminus X} p' = p'.$$

Hence $\operatorname{carr}_X p' \neq \emptyset$.

Further for any Borel subset A of X we denote by p_A the element of S defined by

$$p_A = \bigvee \{ p_K \mid K \text{ closed}, \ K \subset A \}$$

(see [9], § 3-4). Since p is quasicontinuous there exists an increasing sequence $(F_n)_n$ od closed subset of X such that the restriction of p to any F_n is continuous and

$$\bigwedge_n B^{X \setminus F_n} 1 = 0$$

From the last relation we deduce that $p_{\bigcap_n (X \setminus F_n)} = 0$. Indeed for any closed subset F of $\bigcap_n (X \setminus F_n)$ we have

$$p_F = B^{X \setminus F_n} p_F \leq \alpha B^{X \setminus F_n} 1 \text{ where } p \leq \alpha \text{ on } X \text{ and therefore}$$

$p_F = 0$, $p_{\bigcap_n (X \setminus F_n)} = 0$.

Hence $p = \bigvee_n p_{F_n}$. To finish the proof it will be sufficient to show that p_{F_n} is nearly continuous for any $n \in N$. For this we prove that p_{F_n} satisfies the domination principle, i.e for any specific minorant q of p_{F_n}, and any $s \in S$ such that $s \geq q$ on carr q we have $s \geq q$ on X ([7], § 2).

Let $q \in S$ be a specific minorant of p_{F_n} and let $s \in S$ be such that $s \geq q$ on carr q. Obviously carr $q \subset$ carr $p_{F_n} \subset F_m$ for any $m \subset N$, $m \geq n$. Since the restriction of p to F_m is continuous we deduce that the restriction of q to F_m is also continuous and therefore there exists an open subset G_m of X such that

$$F_m \cap G_m = [s > q] \cap F_m$$

If $\alpha \in R_+$ is such that $p \leq \alpha$ on X and $t \in S$ is such that $t > 1$ on $X \setminus F_m$ then $s + \alpha t > q$ on the open set $G_m \cup (X \setminus F_m)$ and therefore (see [9], Proposition 3.4.3) $s + \alpha t \geq q$. The element t being arbitrary we get $s + \alpha B^{X \setminus F_m} 1 \geq q$ for any $m \geq n$ and therefore $s \geq q$.

Since p_{F_n} satisfies the domination principle it follows ([7], Theorem 2-6) that p_{F_n} is nearly continuous.

<u>Remark.</u> In the previous theorem we can'not replay the condition "bounded" by the condition "finite". Indeed in the case of harmonic space asociated on R^2 to the heat operator, the function s equal to $\frac{1}{\sqrt{t}} e^{-\frac{x^2}{t}}$ if $t > 0$ and equal zero on $t \leq 0$ is a finite potential which is also quasicontinuous and on the other hand it is not nearly continuous.

<u>Corollary 2.2.</u> Suppose that there exists a strictly positive potential on X. Then the following assertions are equivalent

1) S satisfies the axiom of nearly continuity

2) Any element of S is quasicontinuous

3) Any universally bounded element of S is quasicontinuous.

Proof. The assertion 1) $\Rightarrow$ 2) follows from [6], Theorem 1.7. The assertion 2) $\Rightarrow$ 3) is obvious and the relation 3) $\Rightarrow$ 1) follows from the previous theorem.

Theorem 2.3. Let p be an element of S any let Y be the set $[p>0]$. Then the convex cone S_Y of all restrictions to Y of the elements of S is a standard H-cone of functions on Y such that:

a) an element $s \in S$ such that $s = 0$ on $X \setminus Y$ is nearly continuous (resp. nearly bounded) iff the function $s/_Y$ is a nearly continuous (resp. nearly bounded) element of the H-cone S_Y.

b) an element $s \in S$, such that $s=0$ on $X \setminus Y$ and such that s is continous, is a potential in S iff $s/_Y$ is a potential in S_Y.

Proof. The fact that S_Y is a standard H-cone of functions on Y follows from the fact that S_Y is isomorphic with the solid subcone in S of all element $s \in S$ equal zero on $X \setminus Y$. The statement a) follows immediately.

b) We suppose now that q is a potential on X such that q is continuous and $q/_{X \setminus Y} = 0$. We show that the element $q' := q/_Y$ is a potential on Y.

Let $(D_n)_n$ be an increasing sequence of open subset of X such that $\bigcup_n D_n = Y$. For any $\varepsilon > 0$ let $(Gn)_n$ be the sequence of open subset of X given by $G_n :=$ $:= D_n \cup [q < \varepsilon]$. Obviously $(Gn)_n$ is increasing and $\bigcup_n Gn = X$. Hence we have

$$\bigwedge_n B^{X \setminus Gn} q \leq \bigwedge_n B^{X \setminus Gn} p = 0$$

$${}^Y B^{Y \setminus D_n} q' = (B^{Y \setminus D_n} q)|_Y \leq (B^{X \setminus G_n} q + \varepsilon)|_Y ,$$

$$\bigwedge_n {}^Y B^{Y \setminus D_n} q' \leq \varepsilon .$$

The number ε being arbitrary we get

$$\bigwedge_n {}^{Y}B^{Y\setminus D_n} q' = 0$$

<u>Theorem. 2.4.</u> Let p be a potential on X. Then the following assertions are equivalent:

1) any universally bounded element of S which is dominated by p is nearly continuous

2) any element $s \in S$ such that

$$s = \bigvee_n (s \wedge np)$$

is quasicontinuous

3) any universally bounded element of S which is dominated by p is quasicontinuous.

<u>Proof.</u> From Theorem 2.3 we deduce that in the standard H-cone of functions S_Y of all restrictions to $Y := [p > 0]$ of the elements of S there exists a strictly positive potential.

Using Theorem 2.3 we deduce that the assertion 1) is equivalent with the fact that any universally bounded element of S_Y is nearly continuous. The assertions 2) and 3) are equivalent with the fact that any element of S_Y, respectively any universally bounded element of S_Y, is quasicontinuous. from the above remarks we finish the proof using Corollary 2.2 for the H-cone S_Y.

<u>Theorem 2.5.</u> Let $\mathcal{T}$ be a topology on X such that $\mathcal{T}_0 \subset \mathcal{T} \subset \mathcal{T}_1$ and let p be a $\mathcal{T}$-potential which is finite and $\mathcal{T}$-continuous. Then p us nearly continuous.

<u>Prof.</u> Let $(s_n)_n$ be an increasing sequence of universally continuous elements of $\mathcal{S}$ such that $\bigvee_n s_n = p$. For any $\varepsilon > 0$ and any $n \in N$ we denote by Gn the element of $\mathcal{T}$ given by

$$Gn = \left[s_n + s \cdot \sum_{m=1}^{n} \frac{\varepsilon}{2^m} > p \right].$$

where $\quad s_0 := \sum_{m=1}^{\infty} \frac{1}{2^m(1+\|s_m\|)} s_m$

Obviously $\overline{Gn}^{\tau} \subset G_{n+1}$ for any n and $\bigcup_n Gn = X$. Since p is a τ-potential we have

$$\bigwedge_n B^{X \setminus Gn}p = 0$$

and therefore, from

$$R(p-s_n) \leq s \cdot \sum_{m=1}^{n} \frac{\varepsilon}{2^m} + B^{X \setminus Gn}p$$

we deduce

$$\bigwedge_n R(p-s_n) \leq \varepsilon s, \quad \bigwedge_n R(p-s_n) = 0$$

Using ([9], proposition 5-6-1) we get that p is nearly continuous.

Cororllary. 2.6. Any potential on X which is finite and continuous is nearly continuous.

Cororllary. 2.7. Any fine potential on X which is finite is nearly continuous.

3. Potentials and superharmonic elements in a standard H-cone of functions.

In this section S will be a standard H-cone of functions on a nearly saturated set X.

Definition. An element $s \in S$ is called superharmonic if for any open subset G of X the function $B^{X \setminus G}s$ is finite and continuous on G.

Remark. It is shown ([9] Proposition 5.6.14) that if s is bounded then s is superharmonic.

Proposition 3.1. The set of all superharmonic elements of S is a solid (with respect to the natural order) convex subcone of S.

Proof. Let $s,t \in S$, $s \preceq t$ be such that t is superharmonic and let U be an open subset of X. We consider $x_0 \in U$ and V an open neighbourhood of x_0 such that $\overline{V} \subset U$.

Since $B^{X\setminus V_s} \leq B^{X\setminus V_t}$ we deduce using ([3], I, Theorem 2.1.6) that

$$B^{X\setminus U_s} = B^{X\setminus U}(B^{X\setminus V_s}), \quad B^{X\setminus U}(B^{X\setminus V_s})\big| \precsim_U B^{X\setminus U_t} \;/U$$

where $\precsim_U$ is the symbol for the specific order in the standard H-cone $S'(U)$. Since the natural topology on U given by the H-cone $S'(U)$ coincides with the restriction to U' of the natural topology on X and since the function $B^{X\setminus U_t}$ is finite and continuous on V we deduce that the function $B^{X\setminus U_s}$ is also finite and continuous on V. Hence $B^{X\setminus U_s}$ is continuous in x_o.

Theorem 3.2. Suppose that p is a superharmonic element of S. Then p is a potential on X iff for any open covering $(D_n)_n$ of X we have

$$\bigwedge_{(i_1,i_2,\dots i_n)} B^{X\setminus D_{i_1}} B^{X\setminus D_{i_2}} \dots B^{X\setminus D_{i_n}} p = 0$$

Proof. The "if" part is obvious. Suppose that p is a potential on X and let $(Dn)_n$ be a countable open covering of X. If we put

$$h := \bigwedge_{(i_1,i_2,\dots i_n)} B^{X\setminus D_{i_1}} B^{X\setminus D_{i_2}} \dots B^{X\setminus D_{i_n}} p$$

we deduce that h is a finite potential on X. Moreover, for any element Dn_o of the covering $(Dn)_n$, we deduce that

$$h = \bigwedge_{(i_1,i_2,\dots i_n)} B^{X\setminus Dn_o}(B^{X\setminus D_{i_1}} B^{X\setminus D_{i_2}} \dots B^{X\setminus D_{i_n}} p)$$

and since the family $(B^{X\setminus D_{i_1}} B^{X\setminus D_{i_2}} \dots B^{X\setminus D_{i_n}} p)_{(i_1,i_2,\dots,i_n)}$ is decreasing we get that the family

$$(B^{X\setminus D_{n_o}}(B^{X\setminus D_{i_1}} B^{X\setminus D_{i_2}} \dots B^{X\setminus D_{i_n}} p)\big|_{D_{n_o}})_{(i_1,i_2,\dots,i_n)}$$

is specifically decreasing in $S'(D_{n_o})$.

From Proposition 3.1 we deduce that h is continuous on D_{n_o} and therefore h is a finite continuous potential on X .

If we denote $Y = [p > 0]$ and we consider the standard H-cone S_Y of all restrictions to Y of the elements of S as in Theorem 2.3 we get that $h/_Y$ is a potential on Y with suspect to the H-cone S_Y and

$$h/_Y = \bigwedge_{(1, l_2, \ldots l_n)} {}^Y B^{Y \cap (X \setminus D_{l_1})} \ldots {}^Y B^{Y \cap (X \setminus D_{l_n})} (p|_Y)$$

Since Y is semisaturated with respect to the H-cone S_Y we deduce that $h/_Y$ has an empty carrier in Y. Hence, using Theorem 2.5 $h/_Y$ is a nearly continuous element of S_Y. Since any nonzero nearly continuous element of a standard H-cone of functions on a nearly saturated set has a non empty carrier on this set (see [7], Theorem 2.4) we deduce that $h/_Y = 0$ and therefore $h = 0$.

Theorem 3.3. a) Suppose that X is semisaturated and $p \in S$. If for any element $q \in S$ such that $q \leq p$ and $\mathrm{carr}_X q = \emptyset$ we have $q=0$ then p is a potential on X. Conversely if p is a superharmonic potential on X then for any $q \in S$ such that $q \leq p$ and $\mathrm{carr}_X q = \emptyset$ we have $q=0$.

b) If there exists a strictly positive potential on X then a superharmonic element $p \in S$ is a potential iff for any $q \in S$ such that $q \precsim p$ and $\mathrm{carr}_X q = \emptyset$ we have $q=0$.

Proof. a) We suppose that for any $q \in S$ such that $q \leq p$ and $\mathrm{carr}_X q = \emptyset$ we have $q=0$. Let $(Gn)_n$ be a sequence of open subset of X such that $\overline{G}_n \subset G_{n+1}$ for any $n \in N$ and $\bigcup_n Gn = X$.

If we denote $q := \bigwedge_n B^{X \setminus Gn} p$ we deduce that for any $n, m \in N$, $n < m$ we have

$$B^{X \setminus Gn} B^{X \setminus \overline{G}m} p = B^{X \setminus \overline{G}m} p$$

Since X is semisaturated we deduce that for any $x \in X$ there exists a measure $\varepsilon_x^{X \setminus Gn}$ on $X \setminus G_n$ such that $\varepsilon_x^{X \setminus Gn} s = B^{X \setminus Gn} s(x)$ for any $x \in X$ and any $s \in S$. On the other hand for any $n' \in N$, the sequence $(B^{X \setminus Gm})_{m > n'}$ is specificaly decreasing in $S'(Gn')$ and therefore for any $x \in Gn \cap [p < \infty]$ we have

$$B^{X\setminus G_n}q(x) = \mathcal{E}_x^{X\setminus G_n}(\bigwedge_{m\geq n} B^{X\setminus G_m}p) = \mathcal{E}_x^{X\setminus G_n}(\inf_m B^{X\setminus G_m}p) = \inf_{m\geq n} \mathcal{E}_x^{X\setminus G_n}(B^{X\setminus G_m}p) =$$

$$= \inf_m B^{X\setminus G_m}p(x) = q(x)$$

Hence the element $B^{X\setminus G_n}q$ coincides with the element q outside a semipolar subset of X and therefore $B^{X\setminus G_n}q = q$. The number $n\in N$ being arbitrary we get using the hypothesis, $q=0$.

Conversely, suppose now that p is a superharmonic potential on X. Let $q\in S$ be such that $q\leq p$ and $\mathrm{carr}_X q = 0$, and let $(G_n)_n$ be an open covering of X for which

$$B^{X\setminus G_n}q = q \qquad (\forall)\ n\in N$$

Since q is a superharmonic potential on X, using Theorem 3.2 we get

$$q = \bigwedge_{n\in N} B^{X\setminus G_n}q = 0$$

b) We suppose that there exists a strictly positive potential on X and let p be a superharmonic element of S such that for any $q\in S$ for which $q \precsim p$ and $\mathrm{carr}_X q = 0$ we have $q=0$. We show that p is a potential on X. Let $(G_n)_n$ be sequence of open subset of X such that

$$\overline{G}_n \subset G_{n+1} \quad (\forall)\ n\in N \quad \text{and} \quad \bigcup_n G_n = X.$$

If we denote $q =: \bigwedge_m B^{X\setminus G_n}p$, by a similar argument as in the proof of assertion a), we get

$$B^{X\setminus G_n}q = q \qquad (\forall)\ n\in N$$

and therefore q is finite continuous. Hence the function $p-q$ is lower semicontinuous, de(n)sely finite on X and for any $x\in X$ and any natural neighbourhood V of x there exists $n\in N$ such that $x\in V\cap G_n$. We have, for eny open neighbourhood W of x with $\overline{W}\subset V\cap G_n$,

$$B^{X\setminus W}(p-q)\ (x) \leq p(x)-q(x)$$

and therefore using ([4], Theorem 3.5) we get $p-q\in S$, $q \precsim p$, $q=0$

<u>Definition.</u> An element s S is called <u>fine superharmonic</u> if for any fine open set G of X the function $B^{X\setminus G}s$ is finite on G

<u>Theorem 3.4.</u> A fine superharmonic element $p\in S$ is a fine potential

iff for any covering $(G_i)_{i\in I}$ of X with fine open subset we have

$$\bigwedge_{(i_1, i_2, \ldots i_n)} B^{X \setminus G_{i_1}} B^{X \setminus G_{i_2}} \ldots B^{X \setminus G_{i_n}} p = 0$$

PROOF. The "if" part of the statement is obviuos . We suppose now that p is a fine potential on X and let $(G_n)_{n \in N}$ a sequence of fine open subsets of X such that $\overline{G}_n^f \subset G_{n+1}$ for any $n \in N$ and such that $\bigcup_{n\in N} G_n = [p>0]$. The sequence $(D_n)_n$ of fine open subset of X defined by: $D_n = G_n \cup [p=0]$ has the following properties

$$\overline{D}_n^f = \overline{G}_n^f \cup [p=0] \subset D_{n+1} \qquad (\forall)\ n \in N$$

$$\bigcup_{n\in N} D_n = X$$

and therefore

$$\bigwedge_{n\in N} B^{X \setminus D_n} p = 0$$

If we consider the standard H-cone of functions on the set $Y = [p>0]$ given by the restriction to Y of the elements of S (see Theorem 2.3) we deduce that

$${}^{Y}B^{Y \setminus G_n}(p/_Y) = (B^{X \setminus D_n} p)/_Y \qquad (\forall)\ n \in N$$

and therefore $p/_Y$ is a strictly positive fine potential on Y. Let now $(G_i)_{i\in I}$ be a fine open covering of X. Obviously the family $(D_i)_{i\in I}$ defined by: $D_i = G_i \cap Y$ for any $i \in I$ is a fine open covering of Y. If we put

$$q := \bigwedge_{(i_1, i_2, \ldots i_n)} B^{X \setminus G_{i_1}} B^{X \setminus G_{i_2}} \ldots B^{X \setminus G_{i_n}} p$$

we have

$$q/_Y = \bigwedge_{(i_1, \ldots, i_n)} {}^{Y}B^{Y \setminus D_{i_1}}\, {}^{Y}B^{Y \setminus D_{i_2}} \ldots {}^{Y}B^{Y \setminus D_{i_n}}(p/_Y)$$

Since Y is semisaturated with respect to the H-cone S_Y we deduce that for any $i_0 \in I$, any $x \in D_{i_0}$ and any fine open neighbourhood V of x such that $\overline{V}^f \subset D_{i_0}$

we have

$$ {}^{Y}B^{Y \setminus V}(q/_Y) = q/_Y $$

Hence the fine carrier of $q/_Y$ on Y (with respect to the H-cone S_Y) is empty . On the other hand $q/_Y$ is a finite fine potential on Y and therefore, by Corollary 2.7, $q/_Y$ is a nearly continuous element of S_Y. Since any non zero universally continuous element of a standard H-cone of functions on a nearly saturated set has a non empty fine carrer on this set we get $q/_Y = 0$, $q=0$.

Theorem 3.5. a) Suppose that X is semisaturated and $p \in S$. If for any element $q \in S$ which is dominated by p and has an empty fine carrier in X we have $q=0$ then p is a fine potential on X. Conversely if p is fine superharmonic and is a fine potential on X then for any $q \in S$ which is dominated by p and has an empty fine carrier in X we have $q=0$

b) If X is suslinean and there exists a strictly positive fine potential on X then an element $p \in S$ in a fine potential on X if for any $q \in S$ specifically dominated by p and having an empty fine carrier in X we have $q=0$.

Proof. a) We suppose that $p \in S$ is such that for any element $q \in S$ whi ch is dominated by p and has an empty fine carrier in X we have $q=0$. Let $(Gn)_n$ a seequence of fine open subset of X such that $\bar{G}^f_n \subset G_{n+1}$ for any $n \in N$ and such that $\bigcup_n G_n = X$. We put

$$ q := \bigwedge B^{X \setminus Gn}p $$

Since X is semisaturated and since for any $x \in [p < \infty]$ we have $q(x) = $ $= \inf_n B^{X \setminus Gn}p(x)$ we deduce that for any $n \in N$ we have $B^{X \setminus Gn}q = q$ and therefore q has an empty fine carrier in X. Hence, using the hypothesis, we get $q=0$ i.e p is a fine potential.

Conversely we suppose that p is a fine superharmonic, fine potential on X and let $q \in S$ be such that $q \leq p$ and the fine carrier of q in X is empty. For any $x \in X$ we choose a fine neighbourhood G_x of x such that $B^{X \setminus G_x} q = q$. The family $(G_x)_x$ is a fine open covering of X and therefore, q being a fine superharmonic, fine potential on X, we have by Theorem 3.4,

$$q = \bigwedge_{(x_1, x_2, \ldots, x_n)} B^{X \setminus G_{x_1}} B^{X \setminus G_{x_2}} \ldots B^{X \setminus G_{x_n}} q = 0$$

b). Suppose that $p \in S$ is such that for any element q of S having an empty fine carrier in X and $q \lessapprox p$ we have $q=0$. We want to show that p is a fine potential on X. Indeed if $(G_n)_n$ is a fine open covering of X such that $\overline{G}_n^f \subset G_{n+1}$ for any $n \in N$ and if we put $\bigwedge B^{X \setminus G_n} p = q$ we deduce that

$$B^{X \setminus G_n} q = q \qquad (\forall) \; n \in N$$

Using ([6], Theorem 2.6) we get $q \lessapprox p$ and therefore, from hypothesis $q=0$. Hence p is a fine potential on X.

<u>Remark.</u> In R^3 any Newtonian potential $x \to \int \frac{1}{|x-y|} d\mu(y)$ where μ is a nonatomic measure carried by a polar subset is a fine potential with an empty fine carrier in R^3.

<u>4. Potentials and fine potentials on a Green set</u>

In this section S will be a standard H-cone and X is a Green set associated with S. We denote by G the Green function on $X \times X$ associated with S and S (see [9], 5.5, [5])

We rememeber that an element p of S is a <u>Green potential</u> if there exists a measure μ on X (which is uniquely determined) such that

$$p(x) = \int G(x,y) d\mu(y) \qquad (\forall) \; x \in X$$

Since S and S^* are simultaneously standard H-cones of functions on X then there are four remarkable topologies on X: $\mathcal{T}_0$ (the natural topology induced by S_0), $\mathcal{T}_1$ (the fine topology conduced by S), $\mathcal{T}_0^*$ (the natural topology induced by S_0^* which is called the <u>conatural topology</u>) and $\mathcal{T}_1^*$ (the fine topology induced by S^* which is called the <u>cofine topology</u>). We have the following relations:

$\mathcal{T}_0 \subset \mathcal{T}_1 \cap \mathcal{T}_1^*$, $\mathcal{T}_0^* \subset \mathcal{T}_1 \cap \mathcal{T}_1^*$

In [5], Theorems 4.4, 4.8. we proved the followings result: Any $\mathcal{T}_0^*$-potential $p \in S$ is a Green potential; if there exists a strictly positive $\mathcal{T}_0^*$-potential $p \in S$ on X then any Green potential is a $\mathcal{T}_0^*$-potential.

Also we have shown that generally a $\mathcal{T}_0$-potential $p \in S$ is not a Green potential on X (see [5], Remark 4.7, b)

Theorem 4.1. Any fine potential is a Green potential. If $\mathcal{T}_0^* \subset \mathcal{T}_0$ then any potential is a Green potential.

Proof. The assertion follows immediately from the above consideratio ns.

Theorem 4.2. Suppose that X is semisaturated with respect to S and S^*. Then any Green potential on X is a potential and any Green copotential on X is a copotential (i.e $\mathcal{T}_0^*$-potential). Particularly there exists a strictly positive potential and a strictly positive copotential on X.

Proof. Let p be a Green potential on X and let μ be a measure on X such that

$$p(x) = \int G(x,y)d\mu(y)$$

Let now $q \in S$, $q \leq p$ be such that $carr_X q = \emptyset$. Since X is semi saturated with respect to S^* then q is also a Green potential on X (see [5], Theorem 1.1) Hence there exists a measure ν on X such that

$$q(x) = \int G(x,y)d\nu(y)$$

On the other hand we have $carr_X q = supp\,\nu$ and therefore $\nu = 0$. From Theorem 3.3 we get that p is a potential on X.

Theorem 4.3. Suppose that there exists a strictly positive fine potential on X. Then we have:

a) X is semisaturated with respect to S and S^*

b) S and S^* satisfy axiom of polarity

c) There exists a strictly positive cofine copotential on X (i.e a $\mathcal{T}_1^*$-potential).

Proof. Since there exists a strictly positive fine potential on X then any universally bounded element of S is a fine potential and therefore from Corollary 2.7 it is nearly continuous. Hence S^* satisfies axiom of polarity. Particularly X is semisaturated with respect to S^*. On the other hand since there exists a strictly fine potential on X then X is semisaturated with respect to S. From the above considerations we deduce that any semipolar subset of X is polar and therefore S satisfies axiom of polarity or equivalently S^* satisfies axiom of nearly continuity. Hence there exists a strictly positive fine copotential on X.

Theorem 4.4. Suppose that there exists a strictly positive fine potential on X. Then the following properties are equivalent:

a) any Green potential in a fine potential

b) $\mathcal{T}_1 \subset \mathcal{T}_1^*$

Proof. a)$\Rightarrow$b) Let $a \in X$ and V be an fine open neighbourhood of a.

We consider now a sequence $(U_n)_n$ of neighbourhood of a (in the natural topology) such that

$$\overline{U}_{n+1} \subset U_n \quad (\forall) \quad n \in N$$

and $\bigcap_n U_n = \{a\}$

Let now $(V_n)_n$ be an increasing sequence of fine open subset of X such that

$$a \in V_n \subset \overline{V}_n^f \subset V_{n+1} \quad (\forall) \quad n \in N$$

and such that $\bigcup_n V_n \subset V$.

We put, for any $n \in N$

$$D_n := V_n \cup \complement \overline{U}_n$$

We have: D_n is fine open,

$$\overline{D}_n^f \subset D_{n+1} \quad (\forall) \quad n \in N$$

and $\bigcup_n D_n = X$. Since $G(\cdot, a)$ is a fine potential then

$$\bigwedge_n B^{X \setminus D_n} G(\cdot,a) = 0$$

and there exists $n_0 \in N$ such that

$$B^{X \setminus D_n} G(\cdot,a) \neq G(\cdot,a)$$

From this fact it follows (see [9], Proposition 5.5.13) that $X \setminus D_{n_0}$ is cothin at a and therefore D_{n_0} is a cofine neighbourhood of a. Since $V_{n_0} \supset U_{n_0} \cap U_{n_0}$ and since U_{n_0} is cofine open we deduce that V_{n_0} and therefore V is a cofine neighbourhood of a. b)$\Rightarrow$a). Suppose that $\mathcal{T}_1 \subset \mathcal{T}_1^*$ and let $p \in \mathcal{S}$

$$p(x) = \int G(x,y)d\mu(y) \qquad (\forall)\ x \in X$$

be a Green potential on X.

Let $(U_n)_n$ be an increasing sequence of fine open subsets of X such that $\overline{U}_n^f \subset U_{n+1}$ and such that $\bigcup_n U_n = X$. We consider the element

$$q := \bigwedge_n B^{X \setminus U_n} p.$$

Since X is semisaturated with respect to S, we have.

$$B^{X \setminus U_n} q = q \quad (\forall)\ n \in N$$

On the other hand, X being semisaturated with respect to $\overset{*}{S}$ and since $q \leq p$, q is a Green potential. We have

$$q(x) = \int G(x,y)d\lambda(y) \qquad (\forall)\ x \in X$$

Let now ν be a measure on X such that ν charges any cofine subset of X and such that the copotential

$$^*G^\nu(x) = \int G(y,x)d\nu(y)$$

is nearly continuous with respect to $\overset{*}{S}$. We have $\int {}^*G^\nu d\lambda = \int q d\nu =$ $= \int B^{X \setminus U_n} q\, d\nu = \int {}^* B^{X \setminus U_n}\, {}^*G^\nu\, d\lambda$.

Since $\mathcal{T}_1 \subset \mathcal{T}_1^*$ it follows that U_n is cofine open and therefore (see [9], Proposition 5.5.13)

$$ {}^{*}B^{X \setminus U_n}\, {}^{*}G^{\nu} < {}^{*}G^{\nu} \quad \text{on } U_n $$

From the preceding relations we deduce that λ does not charge the set U_n. Since $\bigcup_n U_n = X$ it follows that $\lambda = 0$, $q=0$.

5. Universally potentials on standard H-cones.

In the sequel S will be standard H-cone.

Definition. A representation of S as a standard H-cone of functions on a topological space $(X, \mathcal{T})$ where $\mathcal{T}$ is a topology on X which is smaller then the fine topology $\mathcal{T}_1$ on X and greater then the natural topology $\mathcal{T}_0$ on X is called P-representation if there exists a strictly positive $\mathcal{T}$-potential on X. If there exists at least a P-representation of S then S is called a P-standard H-cone.

Definition. Let S be a P-standard H-cone. An element $p \in S$ is termed universally potential if for any P-representation of S on a topological space $(X, \mathcal{T})$, p is a $\mathcal{T}$-potential on X.

Remark. From the above definition it follows that any nearly bounded element of S is an universally potential.

Problem. Is any universally potential of S a nearly bounded element of S?

Theorem 5.1. The following assertions are equivalent:

a) S satisfies axiom of nearly continuity

b) any representation of S on a topological space $(X, \mathcal{T})$ where X is semi-saturated and $\mathcal{T}_0 \subset \mathcal{T} \subset \mathcal{T}_1$ is a P-representation ($\mathcal{T}_0$ is the natural topology; $\mathcal{T}_1$ is the fine topology)

b') any representation of S on a topological space $(X, \mathcal{T}_0)$ where X is semi-saturated and $\mathcal{T}_0$ is the natural topology is a P-representation

c) there exists a P-representation of S on a topological space $(X, \mathcal{T}_1)$ where $\mathcal{T}_1$ is the fine topology on X.

Proof. a)$\Leftrightarrow$b)$\Leftrightarrow$ c) follows from ([6], Theorem 2.3 The assertion b)$\Rightarrow$ b') is obvious.

b'$\Rightarrow$ a) Let now p be an universally bounded element of S and let u be a weak unit on S such that $p \lesssim u$. We denote by X the saturated set with respect to S such that u = 1 on X. Since by hypothesis, there exists a strictly positive potential on X (i.e a $\mathcal{T}_0$-potential) then p is also a potential on X. From Theorem 2.5 we deduce, p being finite continuous, that p is nearly continuous.

Theorem 5.2. Suppose that S satisfies axiom of nearly continuity. Then an element of S will be universally potential iff it is nearly continuous.

Proof. The if part follows from the fact that and nearly continuous element of S is nearly bounded and therefore it is an universally potential.

Suppose now that p is an universally potential of S and let u be a weak unit of S such that $p \lesssim u$. We consider a representation of S on the topological space $(X, \mathcal{T}_0)$ where X is saturated, u = 1 on X and $\mathcal{T}_0$ is the natural topology on X. Since there exists a strictly positive potential on X, then p is also a potential on X and therefore, being finite and continuous it is nearly continuous (see Theorem 2.5)

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