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1. INTRODUCTION

In this paper we discuss the question of the first order necessary conditions for the control problem

(1.1) Minimize $\int_0^1 \{ g(y) + h(u) \} \, dt$

subject to $u \in L^2(0,1;U)$ and $y \in W^{1,2}(0,1;H)$, $y(t) \in C$ a.e. $[0,1]$, such that:

(1.2) $(y' + Ay - Bu, y - v) \leq 0, \quad \forall v \in C,$

(1.3) $y(0) = y_o$,

(1.4) $u(t) \in U_{ad}$ a.e. $[0,1]$.

Here we consider U, V, H three Hilbert spaces with norms, $|\cdot|_U$, $|\cdot|_V$, $|\cdot|_H$ and inner products $(.,.)_U$, $(.,.)_H$, $(.,.)_V$. The pairing between V and V^* (the dual space) is denoted by $(.,.)_{V \times V^*}$ and we have $V \subset H \subset V^*$ with compact imbedding, $H = H^*$, $U = U^*$.

We assume that:

- $C \subset V$ closed convex subset, $0 \in C$;
- $U_{ad} \subset U$ closed convex subset;
- $g : H \to R_+$ is convex, continuous;

- $h : U \to R_+$ is convex, continuous, coercive

(1.5) $h(u) \geq c|u|_U^2 - c_1, \quad c > 0;$

- $B : U \to H$ is linear, continuous;

- $A : V \to V^*$ is linear, continuous, symmetric and coercive

(1.6) $(Ay,y) \geq \omega|y|_V^2 - \alpha|y|_H^2, \quad \omega > 0;$

- $y_0 \in C$.

Under the above conditions, it is known that the variational inequality (1.2), (1.3) has a unique solution $y \in W^{1,2}(0,1;H) \cap L^2(0,1;V)$. Moreover, it is easy to show the existence of at least one optimal pair $[y^*,u^*]$ in $W^{1,2}(0,1;H) \times L^2(0,1;U)$ for the control problem (1.1) - (1.4), Barbu [1], Ch. 5.

The literature concerning numerical and theoretical results for control problems governed by variational inequalities or free boundary problems is very rich and we quote only the survey of V. Barbu [1] and its references. Recently, Shi Shuzhong [13] derived a more complete set of optimality conditions, in the elliptic case, by means of a new argument based on Nikaido's minimax theorem [11]. We extend this approach to parabolic problems, thus generalizing the results available in the literature. However, it seems that it is not possible to apply directly the method of Shi in the parabolic problem and we reduce it to the elliptic situation by a semidiscretization procedure. For a general discussion of the discretization and approximation of variational inequalities, we quote the books by R. Glowinski, J.L. Lions, R. Tremolieres [8] and by C.M. Elliott, J.R. Ockendon [7]. A similar semidiscretization method was used by C. Saguez [12] in the control of two-phase Stefan problems.

Since the elliptic state system obtained by discretization is nonstandard and the method proposed by Shi is very recent, we briefly recall the main steps in section 2. In section 3 we obtain the optimality conditions for the problem (1.1)-(1.4).

It turns out that these necessary conditions are exactly the same as in the case of state constrained control problems governed by linear evolution systems, V. Barbu, Th. Precupanu [2], Ch. IV. In this way, we strengthen the idea of the relationship between control problems governed by variational inequalities (without state constraints) and constrained problems, which has already appeared and been used in various forms in the works [4], [5], [9], [10], [14].

As an application, in the last section, we discuss a possible form for the dual of the problem (1.1)-(1.4) and we give an example.

Finally, we note that, if $[y^*,u^*]$ is an optimal pair for the problem (1.1)-(1.4), then it is the unique optimal pair of the problem:

$$\text{Minimize } \int_0^1 \left\{ g(y) + h(u) + \tfrac{1}{2}|u - u^*|_U^2 \right\} dt ,$$

subject to (1.2)-(1.4). This is related to the "adapted penalization method", Barbu [1], and enables us to get a characterization of all the optimal pairs of the problem (1.1)-(1.4). In the sequel, for the sake of a simpler notation we study the problem (1.1)-(1.4).

2. THE DISCRETIZED PROBLEM

Let n be a given natural number and consider the problem

$$\text{(2.1)} \quad \text{Minimize } n^{-1}\sum_{i=0}^{n-1} \left\{ g(y_{i+1}) + h(u_{i+1}) \right\}$$

over the set of all $u \in U^n_{ad}$, $y \in C^n$, such that

$$\text{(2.2)} \quad (y_{i+1} - y_i)/n^{-1} + Ay_{i+1} + \partial I_C(y_{i+1}) \ni Bu_{i+1} , \qquad i = 0,\dots,n-1,$$

where I_C is the indicator function of C in V and ∂I_C is its subdifferential.

For $y \in V^*$ and $v \in C$, we denote

$$|y|_v^C = \sup \left\{ (y,p)_{V \times V^*} ; p \in (C - v) \cap B_V \right\}$$

with B_V being the closed unit ball in V.

Then, (2.2) is equivalent with $y_{i+1} \in C$ and

$$|-ny_{i+1} - Ay_{i+1} + Bu_{i+1} + ny_i|^C_{y_{i+1}} = 0, \quad i = 0,\dots,n-1.$$

We may define the penalized problem

(2.3) $$\text{Minimize} \left\{ n^{-1}\sum_{i=0}^{n-1} [g(y_{i+1}) + h(u_{i+1})] + n^{-1}\sum_{i=0}^{n-1} N_n |-ny_{i+1} - Ay_{i+1} + Bu_{i+1} + ny_i|^C_{y_{i+1}} \right\},$$

where N_n is such that $N_n/n \to \infty$ for $n \to \infty$.

We remark that the functional $G_n(y,u)$ which appears in (2.3) is continuous by the properties of g, h and of the mapping $|\cdot|^C_V$. To clarify the last assertion, we take:

$$f : V^n \times (C \times U_{ad})^n \to R,$$

$$f(p,y,u) = \sum_{i=0}^{n-1} (p_{i+1}, -ny_{i+1} - Ay_{i+1} + Bu_{i+1} + ny_i)_{V \times V^*},$$

$$W : (C \times U_{ad})^n \to B_V^n,$$

$$W(y,u) = \prod_{i=0}^{n-1} (C - y_{i+1}) \cap B_V .$$

The function f is continuous with respect to the weak topology on V^n and the strong topology on $(C \times U_{ad})^n$, while the point-to-set mapping W has nonvoid, compact values in the weak topology on B_V^n and it is continuous too. The Berge maximum theorem [3] proves that the application

$$\max_{p \in W(y,u)} f(p,y,u) = \sum_{i=0}^{n-1} |-ny_{i+1} - Ay_{i+1} + Bu_{i+1} + ny_i|^C_{y_{i+1}}$$

is continuous in the strong topology on $(C \times U_{ad})^n$ and the multifunction

$$M(y,u) = \prod_{i=0}^{n-1} M^{i+1}(y,u) = \left\{ p \in B_V^n;\ f(p,y,u) = \max_{p \in W(y,u)} f(p,y,u) \right\}$$

is upper semicontinuous with respect to the strong topology on $(C \times U_{ad})^n$ and the weak topology on B_V^n and has closed, bounded convex values.

Since G_n is continuous and positive we may use the Ekeland's variational principle [6]. For any $\varepsilon_n > 0$, there is $(y_n,u_n) \in C^n \times U_{ad}^n$, such that:

$$0 \leq \inf G_n \leq G_n(y_n,u_n) \leq \inf G_n + \varepsilon_n ,$$

(2.4) $$G_n(y,u) > G_n(y_n,u_n) - \varepsilon_n(|y - y_n|^2_{V^n} + |u - u_n|^2_{U^n})^{\frac{1}{2}}, \qquad \forall\, (y,u) \neq (y_n,u_n) .$$

Let $s \in \prod_{i=0}^{n-1} (C - y_n^{i+1})$, $w \in \prod_{i=0}^{n-1} (U_{ad} - u_n^{i+1})$. For $t_k > 0$ sufficiently small $y_n + t_k s \in C^n$, $u_n + t_k w \in U_{ad}^n$ and we have

$$G_n(y_n + t_k s,\ u_n + t_k w) > G_n(y_n,\ u_n) - \varepsilon_n t_k(|s|^2_{V^n} + |w|^2_{U^n})^{\frac{1}{2}} .$$

We consider any $p_{nk}^{sw} \in M_n(y_n + t_k s,\ u_n + t_k w)$ where $M_n = N_n M$. By a detailed computation, we obtain

(2.5) $$n^{-1}\sum_{i=0}^{n-1} (ns_{i+1} + As_{i+1} - Bw_{i+1} - ns_i,\ p_{nk}^{sw,i+1})_{V^* \times V} <$$

$$< n^{-1}\sum_{i=0}^{n-1} \left[\frac{g(y_n^{i+1} + t_k s_{i+1}) - g(y_n^{i+1})}{t_k} + \frac{h(u_n^{i+1} + t_k w_{i+1}) - h(u_{i+1}^n)}{t_k}\right] +$$

$$+ \varepsilon_n(|s|^2_{V^n} + |w|^2_{U^n})^{\frac{1}{2}} .$$

Obviously (p_{nk}^{sw}) is bounded with respect to k in V^n and we may assume $p_{nk}^{sw} \rightharpoonup p_n^{sw}$ weakly in V^n. By the upper semicontinuity of the multifunction M_n, we get $p_n^{sw} \in M_n(y_n,u_n)$. Passing to the limit $k \to 0$ in (2.5), we infer

(2.6) $$n^{-1}\sum_{i=0}^{n-1}[g'(y_n^{i+1},s_{i+1}) + h'(u_n^{i+1},w_{i+1})] -$$

$$-n^{-1}\sum_{i=0}^{n-1}(ns_{i+1} + As_{i+1} - Bw_{i+1} - ns_i, p_n^{sw,i+1})_{V^* \times V} \geq$$

$$\geq - \varepsilon_n(|s|^2_{V^n} + |w|^2_{U^n})^{\frac{1}{2}},$$

for all $s \in \prod_{i=0}^{n-1}(C - y_n^{i+1})$, $w \in \prod_{i=0}^{n-1}(U_{ad} - u_n^{i+1})$.

Here g', h' are the directional derivatives of the convex mappings g, h.

We introduce the auxiliary saddle function

$$Z : \prod_{i=0}^{n-1}(C - y_n^{i+1}) \times \prod_{i=0}^{n-1}(U_{ad} - u_n^{i+1}) \times M_n(y_n,u_n) \to R,$$

$$Z(s,w,p) = n^{-1}\sum_{i=0}^{n-1}[g'(y_n^{i+1},s_{i+1}) + h'(u_n^{i+1},w_{i+1})] + \varepsilon_n(|s|^2_{V^n} +$$

$$+ |w|^2_{U^n})^{\frac{1}{2}} - n^{-1}\sum_{i=0}^{n-1}(ns_{i+1} + As_{i+1} - Bw_{i+1} - ns_i, p^{i+1})_{V^* \times V}.$$

As $M_n(y_n,u_n)$ is weakly compact, the Nikaido minimax theorem proves the existence of $p_n \in M_n(y_n,u_n)$, such that

$$\inf_{s,w} Z(s,w,p_n) = \inf_{s,w} \max_{p \in M_n(y_n,u_n)} Z(s,w,p).$$

By (2.6), we see that $\inf_{(s,w)} Z(s,w,p_n) \geq 0$. Therefore, we obtain:

(2.7) $$n^{-1}\sum_{i=0}^{n-1} g'(y_n^{i+1},s_{i+1}) - n^{-1}\sum_{i=0}^{n-1}(ns_{i+1} + As_{i+1} - ns_i, p_n^{i+1})_{V^* \times V} \geq - \varepsilon_n |s|_{V^n},$$

$$s_0 = 0, \quad \forall s \in \prod_{i=0}^{n-1}(C - y_n^{i+1});$$

$$(2.8)\quad n^{-1}\sum_{i=0}^{n-1} h'(u_n^{i+1}, w_{i+1}) + n^{-1}\sum_{i=0}^{n-1} (Bw_{i+1}, p_n^{i+1})_{V^*\times V} \geq -\varepsilon_n |w|_{U^n},$$

$$\forall\ w \in \prod_{i=0}^{n-1} (U_{ad} - u_n^{i+1}).$$

These are the approximating first order necessary conditions for the problems (2.3) or (2.1), respectively (1.1) - (1.4).

3. OPTIMALITY CONDITIONS

We start with several basic estimates for the system (2.2). First we remark that, by (2.4), we have

$$(3.1)\quad 0 \leq G_n(y_n, u_n) \leq G_n(y_u, u) + \varepsilon_n$$

for all $u \in U^n_{ad}$ and such that $y_u \in C^n$ is the solution of (2.2) corresponding to u. Then

$$(3.2)\quad n^{-1}\sum_{i=0}^{n-1} [g(y_n^{i+1}) + h(u_n^{i+1})] + n^{-1}\sum_{i=0}^{n-1} N_n \left| -ny_n^{i+1} - Ay_n^{i+1} + Bu_n^{i+1} + \right.$$

$$\left. + ny_n^i \right|_{C_{y_n^{i+1}}} \leq n^{-1}\sum_{i=0}^{n-1} [g(y_u^{i+1}) + h(u^{i+1})] + \varepsilon_n \leq ct.$$

Since g is positive, we get

$$n^{-1}\sum_{i=0}^{n-1} h(u_n^{i+1}) \leq ct.$$

and, by (1.5), we see that

$$(3.3)\quad n^{-1}\sum_{i=0}^{n-1} |u_n^{i+1}|_U^2 \leq ct.$$

The choice of N_n implies that

$$(3.4)\quad \delta_n^{i+1} = \| -ny_n^{i+1} - Ay_n^{i+1} + Bu_n^{i+1} + ny_n^i \|^C_{y_n^{i+1}} \to 0,\ n \to \infty,$$

$i = 0,1,\dots,n-1.$

Taking into account the definition of $\|\cdot\|^C_y$, the relation (3.4) may be rewritten as

$$(3.5)\quad (ny_n^{i+1} + Ay_n^{i+1} - Bu_n^{i+1} - ny_i^n,\ z - y_n^{i+1})_{V^*xV} \geq -\delta_n^{i+1}$$

for all $z \in C$, such that $z - y_n^{i+1} \in B_V$.

Let $\theta = \max(1, \|z - y_n^{i+1}\|_V)$; then $\theta^{-1}(z - y_n^{i+1}) \in B_V \cap (C - y_n^{i+1})$ and (3.5) is valid for all $z \in C$, in the form

$$(3.6)\quad (ny_n^{i+1} + Ay_n^{i+1} - Bu_n^{i+1} - ny_n^i,\ z - y_n^{i+1})_{V^*xV} \geq -\delta_n^{i+1}\theta \geq$$

$$\geq -\delta_n^{i+1} - \delta_n^{i+1}\|z - y_n^{i+1}\|_V .$$

We fix $z = 0$ in (3.6) and we infer

$$\sum_{i=0}^{n-1}(ny_n^{i+1} - ny_n^i,\ y_n^{i+1})_H + \sum_{i=0}^{n-1}(Ay_n^{i+1},\ y_n^{i+1})_{V^*xV} \leq \sum_{i=0}^{n-1}\delta_n^{i+1} +$$

$$+\sum_{i=0}^{n-1}\delta_n^{i+1}\|y_n^{i+1}\|_V + \sum_{i=0}^{n-1}(Bu_n^{i+1}, y_n^{i+1}) .$$

By the equality

$$(3.7)\quad (w - v, w)_H = \tfrac{1}{2}\|w\|_H^2 - \tfrac{1}{2}\|v\|_H^2 + \tfrac{1}{2}\|w - v\|_H^2 ,$$

we obtain that $\|y_n^i\|_H$ is bounded for all i and n and

$$n^{-1}\sum_{i=0}^{n-1} |y_n^{i+1}|_V^2 \leq ct., \quad \forall\, n,$$

$$\sum_{i=0}^{n-1} |y_n^{i+1} - y_n^i|_H^2 \leq ct. \quad , \quad \forall\, n .$$

Now, we take $z = y_n^i$ in (3.6):

$$\sum_{i=0}^{p-1} n\,|y_n^{i+1} - y_n^i|_H^2 + \sum_{i=0}^{p-1} (Ay_n^{i+1}, y_n^{i+1} - y_n^i)_{V^*xV} \leq \sum_{i=0}^{p-1} \delta_n^{i+1} +$$

$$\sum_{i=0}^{p-1} \delta_n^{i+1} |y_n^i - y_n^{i+1}|_V + \sum_{i=0}^{p-1} (Bu_n^{i+1}, y_n^{i+1} - y_n^i)_H .$$

We remark that, by (3.7) and (1.6)

$$\sum_{i=0}^{p-1} (Ay_n^{i+1}, y_n^{i+1} - y_n^i)_{V^*xV} = \sum_{i=0}^{p-1} (A^{\frac{1}{2}}y_n^{i+1}, A^{\frac{1}{2}}y_n^{i+1} - A^{\frac{1}{2}}u_n^i)_H =$$

$$= \tfrac{1}{2}|A^{\frac{1}{2}}y_n^p|_H^2 - \tfrac{1}{2}|A^{\frac{1}{2}}y_o|_H^2 + \tfrac{1}{2}\sum_{i=0}^{p-1} |A^{\frac{1}{2}}y_n^{i+1} - A^{\frac{1}{2}}y_n^i|_H^2 \geq$$

$$\geq \tfrac{1}{2}(Ay_n^p, y_n^p)_{V^*xV} - \tfrac{1}{2}(Ay_o, y_o)_{V^*xV} + \frac{\omega}{2}\sum_{i=0}^{p-1} |y_n^{i+1} - y_n^i|_V^2 -$$

$$- \frac{\alpha}{2}\sum_{i=0}^{p-1} |y_n^{i+1} - y_n^i|_H^2 .$$

Combining the above inequalities and the previous estimates, we see that

$$n^{-1}\sum_{i=0}^{n-1} \left| \frac{y_n^{i+1} - y_n^i}{n^{-1}} \right|_H^2 \leq ct., \quad \forall\, n ,$$

$$|y_n^i|_V \leq ct., \quad \forall\, n,\ i = 0,1,\ldots,n-1,$$

$$\sum_{i=0}^{n-1} |y_n^{i+1} - y_n^i|_V^2 \leq ct., \ \forall n.$$

Let y_n, u_n be the step functions obtained in [0,1] from the vectors (y_n^i), (u_n^i) and $\hat{y}_n$ be the polygonal functions obtained on [0,1] from the vector (y_n^i). That is, on $(\frac{i}{n}, \frac{i+1}{n}]$, we have

$$y_n(t) = y_n^{i+1},\ u_n(t) = u_n^{i+1},$$
$$\hat{y}_n(t) = n[(\frac{i+1}{n} - t)y_n^i + (t - \frac{i}{n})y_n^{i+1}].$$

On a subsequnce, we get that $y_n \rightarrow \tilde{y}$ weakly* in $L^{\infty}(0,1;V)$, $\hat{y}_n \rightarrow \tilde{y}$ strongly in $C(0,1;H)$, $\hat{y}_n' \rightarrow \tilde{y}'$ weakly in $L^2(0,1;H)$, $u_n \rightarrow \tilde{u}$ weakly in $L^2(0,1;U)$. The fact that y_n and $\hat{y}_n$ have the same limit is a consequence of the following equality:

$$\int_{i/n}^{(i+1)/n} |\hat{y}_n(t) - y_n(t)|_H^2 = \int_{i/n}^{(i+1)/n} (i + 1 - nt)^2 |y_{i+1} - y_i|_H^2$$

$$= (1/3n)|y_{i+1} - y_i|_H^2 .$$

Turning back to (3.6), we infer

$$(3.6)' \quad n^{-1}\sum_{i=0}^{n-1}(\frac{y_n^{i+1} - y_n^i}{n^{-1}}, y_n^{i+1} - z_n^{i+1})_H + n^{-1}\sum_{i=0}^{n-1}(Ay_n^{i+1}, y_n^{i+1} - z_n^{i+1})_{V^*xV} \leq$$

$$\leq n^{-1}\sum_{i=0}^{n-1}(Bu_n^{i+1}, y_n^{i+1} - z_n^{i+1})_H + n^{-1}\sum_{i=0}^{n-1} \delta_n^{i+1}(1 + ct.) .$$

Here $z \in L^2(0,1;V) \cap W^{1,2}(0,1;V^*)$, $z(t) \in C$ a.e. [0,1], $z(0) = y_o$ and z_n is a semidiscrete approximation of z given by the vector $(z_n^i) \in C^n$. We rewrite this inequality in the form

(3.8) $$\int_0^1 (\hat{y}'_n, y_n - z_n)_H dt + \int_0^1 (Ay_n, y_n - z_n)_{V^* \times V} dt \leq$$

$$\leq \int_0^1 (Bu_n, y_n - z_n)_H dt + n^{-1} \sum_{i=0}^{n-1} \delta_n^{i+1} (1 + ct.) .$$

Passing to the limit in (3.8), we see that $\tilde{y}$ is the solution of (1.2), (1.3) corresponding to $\tilde{u}$, since:

$$\liminf_{n \to \infty} \int_0^1 (Ay_n, y_n)_{V^* \times V} dt \geq \int_0^1 (A\tilde{y}, \tilde{y})_{V^* \times V} dt.$$

Now, we return to (3.1) and we make $n \to \infty$. Then, we obtain that $[\tilde{y}, \tilde{u}]$ is an optimal pair for the problem (1.1) - (1.4). Moreover, it is quite standard to infer that

(3.9) $$\lim_{n \to \infty} \int_0^1 h(u_n) dt = \int_0^1 h(\tilde{u}) dt.$$

By (3.9), we get

LEMMA 3.1. Assume that h is an uniformly convex function. Then $u_n \to \tilde{u}$ strongly in $L^2(0,1;U)$.

REMARK 3.2. In function spaces, it is enough to assume strict convexity and some growth conditions for h, according to Visintin [15].

LEMMA 3.3. We have

$\hat{y}'_n \to \tilde{y}'$ strongly in $L^2(0,1;H)$.

Proof

Since $\hat{y}_n$ is convergent in $C(0,1;H)$, we see that $y_n^n = \hat{y}_n(1) \to \tilde{y}(1)$ strongly in

H. But (y_n^n) is bounded in V, so $y_n^n \to \tilde{y}(1)$ weakly in V, on a subsequence.

We reconsider (3.6)' with $z_n^{i+1} = y_n^i$ and we pass to **limsup**:

$$(3.10)\quad \limsup_{n\to\infty} n^{-1}\sum_{i=0}^{n-1} \left| \frac{y_n^{i+1} - y_n^i}{n^{-1}} \right|_H^2 \leq -\tfrac{1}{2}\liminf(Ay_n^n, y_n^n) + \tfrac{1}{2}(Ay_o, y_o) +$$

$$+ \lim_{n\to\infty} n^{-1}\sum_{i=0}^{n-1} \left(Bu_n^{i+1}, \frac{y_n^{i+1} - y_n^i}{n^{-1}}\right) \leq \tfrac{1}{2}(Ay_o, y_o) - \tfrac{1}{2}(A\tilde{y}(1), \tilde{y}(1)) + \int_o^1 (B\tilde{u}, \tilde{y}')dt ,$$

by the above estimates.

We know that $\tilde{y}$ is the solution of (1.2), (1.3) corresponding to $\tilde{u}$, that is:

$$\tilde{y}' + A\tilde{y} + \partial I_C(\tilde{y}) \ni B\tilde{u} \qquad \text{in } [0,1],$$

$$\tilde{y}(0) = y_o .$$

We multiply by $\tilde{y}' \in L^2(0,T;H)$. By the chain rule, we get

$$\int_o^1 |\tilde{y}'|_H^2 + \tfrac{1}{2}(A\tilde{y}(1), \tilde{y}(1)) - \tfrac{1}{2}(A\tilde{y}(0), \tilde{y}(0)) = \int_o^1 (B\tilde{u}, \tilde{y}')dt$$

and (3.10) gives

$$\limsup_{n\to\infty} n^{-1}\sum_{i=0}^{n-1} \left| \frac{y_n^{i+1} - y_n^i}{n^{-1}} \right|_H^2 \leq \int_o^1 |\tilde{y}'|_H^2$$

By a wellknown criterion for strong convergence in Hilbert spaces, we infer the desired conclusion.

LEMMA 3.4. <u>We have</u>

$y_n \to \tilde{y}$ strongly in $L^2(0,1;V)$.

Proof

Let m be another natural number. We divide [0, 1] in nm subintervals and on each one we fix $z_n = y_m^i$ in (3.8), next we reverse the indices m and n. Adding the two inequalities, we obtain

$$\int_0^1 (\hat{y}'_n - \hat{y}'_m, y_n - y_m)_H + \int_0^1 (Ay_n - Ay_m, y_n - y_m) \le$$

$$\le \int_0^1 (Bu_n - Bu_m, y_n - y_m) + n^{-1} \sum_{i=0}^{n-1} \delta_n^{i+1}(1 + ct.) + m^{-1} \sum_{i=0}^{m-1} \delta_m^{i+1}(1 + ct.).$$

The properties of (δ_n^i) and (1.6) finish the proof.

Now, we are able to give some estimates for the adjoint state p_n and next to pass to the limit in (2.7), (2.8).

We recall that $p_n \in M_n(y_n, u_n)$, therefore $N_n^{-1} p_n^{i+1} \in C - y_n^{i+1}$, $\forall$ n, $i = 0,1,\dots,n-1$. We may choose $s = p_n N_n^{-1}$, in (2.7) and we get:

$$(3.11) \quad n^{-1} \sum_{i=0}^{n-1} g'(y_n^{i+1}, N_n^{-1} p_n^{i+1}) - n^{-1} \sum_{i=0}^{n-1} (nN_n^{-1} p_n^{i+1} + N_n^{-1} Ap_n^{i+1} - nN_n^{-1} p_n^i, p_n^{i+1}) \ge$$

$$\ge - \varepsilon_n N_n^{-1} | p_n |_{Vn}.$$

Summing by parts, we infer

$$(3.12) \quad n^{-1} \sum_{i=0}^{n-1} g'(y_n^{i+1}, p_n^{i+1}) - n^{-1} \sum_{i=0}^{n-1} (Ap_n^{i+1}, p_n^{i+1})_{V^* x V} -$$

$$- n^{-1} \sum_{i=1}^{n-1} (n(p_n^i - p_n^{i+1}), p_n^i)_H - n^{-1}(np_n^n, p_n^n)_H \ge - \varepsilon_n | p_n |_{Vn}.$$

Since g is finite on H, it is locally Lipschitzian and (3.12) becomes

$$n^{-1}ct.|p_n|_{H^n} + \varepsilon_n |p_n|_{V^n} \geq n^{-1}\sum_{i=0}^{n-1}(Ap_n^{i+1},p_n^{i+1})_{V^*xV} +$$

$$+ n^{-1}\sum_{i=1}^{n-1}(n(p_n^i - p_n^{i+1}),p_n^i)_H + |p_n^n|_H^2 .$$

By (3.7) and the discrete Gronwall inequality, we obtain

LEMMA 3.5. We have:

$$|p_n^i|_H \leq ct., \quad \forall \ n, i = 0,1,\dots,n-1,$$

$$n^{-1}\sum_{i=0}^{n-1}|p_n^{i+1}|_V^2 \leq ct., \quad \forall \ n,$$

$$\sum_{i=1}^{n-1}|p_n^i - p_n^{i+1}|_H^2 \leq ct., \quad \forall \ n.$$

The step function p_n built from the vector (p_n^i) satisfies $p_n \to \tilde{p}$ weakly in $L^2(0,1;V)$ and weakly* in $L^\infty(0,1,H)$, on a subsequence.

To pass to the limit in (2.7), we rewrite it in the form

$$(3.13) \quad n^{-1}\sum_{i=0}^{n-1}g'(y_n^{i+1},z_{i+1} - y_n^{i+1}) - n^{-1}\sum_{i=0}^{n-1}(Ap_n^{i+1},z_{i+1} - y_n^{i+1})_{V^*xV} -$$

$$- \sum_{i=0}^{n-1}n^{-1}\left(\frac{z_{i+1} - z_i}{n^{-1}} - \frac{y_n^{i+1} - y_n^i}{n^{-1}},p_n^{i+1}\right)_H \geq - \varepsilon_n|z - y_n|_{V^n},$$

for all $z \in C^n$, $z_o = y_o$.

Consider any $\tilde{z} \in L^2(0,1;V) \cap W^{1,2}(0,1;V^*)$, $\tilde{z}(t) \in C$ a.e., $\tilde{z}(0) = y_o$, and let z_n

be a discretization of $\tilde{z}$ given by the vector $(z_n^i) \in V^n$. We put z_n in (3.13) and, by the above lemmas, we can pass to the limit to get

$$(3.14) \quad \int_0^1 g'(\tilde{y},\tilde{z} - \tilde{y}) - \int_0^1 (A\tilde{p}, \tilde{z} - \tilde{y}) - \int_0^1 (\tilde{p},\tilde{z}' - \tilde{y}') \geq 0.$$

for any z with the required properties.

Similarly, we may pass to the limit in (2.8) and we have

$$(3.15) \quad \int_0^1 h'(\tilde{u},\tilde{v} - \tilde{u}) + \int_0^1 (B^*\tilde{p},\tilde{v} - \tilde{u}) \geq 0$$

for all $\tilde{v} \in L^2(0,1;U)$, $\tilde{v}(t) \in U_{ad}$ a.e. [0,1].

THEOREM 3.6. Assume that h and g are Gateaux differentiable. For any optimal pair $[y^*,u^*]$ of the problem (1.1) - (1.4), there is $p^* \in L^2(0,1;V) \cap L^\infty(0,1;H)$, such that

$$\int_0^1 (\nabla g(y^*),\tilde{z} - y^*) + \int_0^1 (Ap^*,\tilde{z} - y^*) + \int_0^1 (p^*,\tilde{z}' - (y^*)') \geq 0,$$

$$\int_0^1 (\nabla h(u^*),\tilde{v} - u^*) - \int_0^1 (B^*p^*,\tilde{v} - u^*) \geq 0$$

for all $\tilde{z}$, $\tilde{v}$ as above.

Proof

We use the remark from the end of the Introduction and we denote $p^* = -\tilde{p}$.

REMARK 3.7. In order to see the significance of the optimality systsem given by Theorem 3.6, we assume that p^* is in $W^{1,2}(0,T;V^*)$ and $p^*(T) = 0$. Integrating by parts and using the definition of the subdifferential, we get

$$(p^*)' - Ap^* - \partial I_{\mathcal{C}}(y^*) \ni \nabla g(y^*),$$

$$B^*p^* \in \partial I_{\mathcal{U}_{ad}}(u^*) + \nabla h(u^*),$$

where

$$\mathcal{C} = \left\{ y \in L^2(0,1;V) \cap W^{1,2}(0,T;V^*);\ y(t) \in C\ t \in [0,1],\ y(0) = y_0 \right\},$$

$$\mathcal{U}_{ad} = \left\{ u \in L^2(0,1;U);\ u(t) \in U_{ad} \text{ a.e. } [0,1] \right\}.$$

This is just the optimality system described by Barbu and Precupanu [2], Ch. IV, in the case of state constrained control problems governed by linear evolution systems.

REMARK 3.8. The positivity and the differentiability assumptions on the mappings g,h are stronger than necessary and are imposed for the sake of simplicity. The same is valid for the condition $0 \in C$. It is also possible to take the right-hand side of (1.2) of the form $Bu + f$.

4. REMARKS ON THE DUAL PROBLEM

First, we see that

$$n^{-1}\sum_{i=0}^{n-1} \delta_n^{i+1} = n^{-1}\sum_{i=0}^{n-1} \left| -ny_n^{i+1} - Ay_n^{i+1} + Bu_n^{i+1} + ny_n^i \right|_{y_n^{i+1}}^{C} =$$

$$= n^{-1}\sum_{i=0}^{n-1} (-ny_n^{i+1} - Ay_n^{i+1} + Bu_n^{i+1} + ny_n^i, p_n^{i+1})_{V^* \times V} =$$

$$= n^{-1}\sum_{i=0}^{n-1} \left(-\frac{y_n^{i+1} - y_n^i}{n^{-1}} - Ay_n^{i+1} + Bu_n^{i+1}, p_n^{i+1}\right)_{V^* \times V} =$$

$$= \int_0^1 (-\hat{y}'_n - Ay_n + Bu_n, p_n)_{V^* \times V}\, dt \to \int_0^1 (-\tilde{y}' - A\tilde{y} + B\tilde{u}, \tilde{p})_{V^* \times V},$$

as $n \to \infty$. Therefore any optimal pair of (1.1) satisfies:

$$\int_0^1 (p^*, v^*)_{V^* \times V}\, dt = 0,$$

where $v^* \in \partial I_C(y^*)$ a.e., $v^* = -(y^*)' - Ay^* + Bu^*$.

For the sake of simplicity, we assume that $y_o = 0$. We define the dual problem by:

(4.1) Minimize $\left\{ (G + I_{\mathscr{C}})^*(-w) + (F + I_{\mathcal{U}_{ad}})^*(B^*p) \right\}$

over all the pairs $[p,w] \in L^2(0,1;V) \times Z^*$, such that:

(4.2) $\int_0^1 (p, z' + Az)_{V \times V^*} dt = (w, z)_{Z^* \times Z}, \quad \forall \quad z \in Z$,

$z(0) = 0$

and under the state constraint

(4.3) $\int_0^1 (p, v^*)_{V \times V^*} dt \geq 0$.

Here, we denote:

$-z = L^2(0,1;V) \cap W^{1,2}(0,1;V^*)$,

$-G(z) = \int_0^1 g(z)dt$,

$-F(u) = \int_0^1 h(u)dt$

and $(G + I_{\mathscr{C}})^*$, $(F + I_{\mathcal{U}_{ad}})^*$ are the Fenchel conjugates of the mappings $G + I_{\mathscr{C}}$, $F + I_{\mathcal{U}_{ad}}$ on the spaces $Z \times Z^*$ and $L^2(0,1;U)$ repectively.

LEMMA 4.1. For any $w \in Z^*$ there is a unique $p \in L^2(0,1;V)$ such that (4.2) is satisfied.

Proof

The equation

$z' + Az = q, \qquad z(0) = 0$

has a unique solution $z \in S = \{ z \in Z;\ z(0) = 0 \}$ for all $q \in L^2(0,1;V^*)$. The mapping $\mathcal{A}: z \to q$ is bijective and bicontinuous between S with the induced topology and $L^2(0,1;V^*)$. Therefore $\mathcal{A}^{-1} : L^2(0,1;V^*) \to Z$ is linear, continuous and one to one.

The equation (4.2) may be rewritten equivalently

$$\int_0^1 (p,q)_{V \times V^*} dt = (w, \mathcal{A}_q^{-1})_{Z \times Z^*} .$$

The right-hand side defines a linear, continuous functional on $L^2(0,1;V^*)$ and we obtain the existence and uniqueness of the solution $p \in L^2(0,1;V)$, as claimed.

REMARK 4.2. In fact $p = (\mathcal{A}^{-1})^* w$ and, since $\mathcal{A}^{-1}$ is one to one, with closed range, then $(\mathcal{A}^{-1})^*$ is onto. So, the adjoint optimal state p^* is the solution of (4.2) for some $w^* \in Z^*$. It also satisfies the state constraint (4.3), therefore it is an admissible state for the problem (4.1).

THEOREM 4.3. The pair $[p^*, w^*]$ is optimal for the problem (4.1) - (4.3) and we have

$$(G + I_{\mathcal{C}})^*(-w^*) + (F + I_{\mathcal{U}_{ad}})^*(B^*p^*) = -G(y^*) - F(u^*) ,$$

for any optimal pair $[y^*,u^*]$ of the problem (1.1)-(1.4).

Proof

We have:

$$(G + I_{\mathcal{C}})^* + (F + I_{\mathcal{U}_{ad}})^*(B^*p) + G(y^*) + I_{\mathcal{C}}(y^*) + F(u^*) + I_{\mathcal{U}_{ad}}(u^*) \geq$$

$$\geq (-w, y^*)_{Z^* \times Z} + \int_0^1 (B^*p, u^*) = -\int_0^1 (p, (y^*)' + Ay^*) + \int_0^1 (Bu^*, p) =$$

$$= -\int_0^1 (p, (y^*)' + Ay^* - Bu^*) = \int_0^1 (p, v^*) \geq 0 ,$$

for all the admissible pairs $[p,w]$ of the problem (4.1)-(4.3).

On the other side, the optimality conditions from the Theorem 3.6. give

$$\int_0^1 (\nabla g(y^*), z - y^*) + \int_0^1 (p^*, z' + Az - (y^*)' - Ay^*) \geq 0$$

for all $z \in \mathcal{C}$, so

$$\int_0^1 (\nabla g(y^*), z - y^*) + (w^*, z - y^*)_{Z^* \times Z} \geq 0, \qquad \forall z \in \mathscr{C},$$

or equivalently

(4.4) $-w^* \in \nabla G(y^*) + \partial I_{\mathscr{C}}(y^*) = \partial(G + I_{\mathscr{C}})(y^*)$.

Similarly, we obtain

(4.5) $B^*p^* \in \nabla F(u^*) + I_{\mathcal{U}_{ad}}(u^*) = \partial(F + I_{\mathcal{U}_{ad}})(u^*)$.

Then, by (4.4), (4.5) we have equality in the inequality of Young and we infer

$$(G + I_{\mathscr{C}})^*(-w^*) + (F + I_{\mathcal{U}_{ad}})(B^*p^*) + G(y^*) + I_{\mathscr{C}}(y^*) + F(u^*) + I_{\mathcal{U}_{ad}}(u^*) =$$

$$= (-w^*, y^*)_{Z^* \times Z} + \int_0^1 (B^*p^*, u^*)_{L^2(0,1;U)} dt =$$

$$= -\int_0^1 (p^*, (y^*)' + Ay^* - Bu^*) dt = \int_0^1 (p^*, v^*) dt = 0 .$$

REMARK 4.4. In the case $y_0 \neq 0$, we assume that $Ay_0 \in H$ and we introduce the unknown state $y - y_0$. The general dual problem has the form

(4.1)' Minimize $\left\{ (G + I_{\mathscr{C}})^*(-w) + (F + I_{\mathcal{U}_{ad}})^*(B^*p) + (w, y_0)_{Z^* \times Z} - \int_0^1 (Ay_0, p) dt \right.$

subject to (4.2), (4.3).

Now, we assume that $B : H \to H$ is the identity operator and $C \subset H$ is a closed convex cone. Moreover we ask that $h : H \to]-\infty, +\infty]$ is non decreasing with respect to the order on H induced by the cone C^0 (the polar of C), where we redefine h by $h \to h + I_{U_{ad}}$, in order to include the control constraints.

LEMMA 4.5. Under the above assumptions, the problem (1.1)-(1.4) has at least one optimal pair $[y^*, u^*]$ such that $0 = v^* = \partial I_C(y^*) = u^* - (y^*)' - Ay^*$.

Proof

We remark that $w \in \partial I_C(y)$ iff $(w,y) = 0$ and $(w,z) \leq 0$, $\forall\ z \in C$, since C is a cone. Therefore $\partial I_C(y) \subset C^o$ for all $y \in H$.

Let $[y^*,u^*]$ be any optimal pair for (1.1)-(1.4) and $v^* \in \partial I_C(y^*) \subset C^o$, $v^* = u^* - (y^*)' - Ay^*$.

Then, the pair $[y^*,u^* - v^*]$ is admissible for the problem (1.1)-(1.4) and $g(y^*) + h(u^* - v^*) \leq g(y^*) + h(u^*)$ by the monotonicity assumption. So $[y^*,u^* - v^*]$ is an optimal pair with $0 \in \partial I_C(y^*)$.

COROLLARY 4.6. Under the above assumptions, the dual problem is given by (4.1)', (4.2) since (4.3) is automatically fulfilled.

REMARK 4.7. In this special case, the dual problem is unconstrained and its definition is self contained.

REMARK 4.8. It is known by a result of Bonnans and Tiba [5], that any optimal pair of the problem

(P) $\quad$ Minimize $\int_0^1 \{g(y) + h(u)\}\, dt$,

$y' + Ay = u, \quad y(0) = y_0,$

$y(t) \in C \quad \text{in } [0,1], \quad u(t) \in U_{ad} \text{ a.e. } [0,1],$

is also optimal for the problem (1.1)-(1.4), under the above assumptions. By Remark 3.7, since the necessary conditions for the problem (P) are also sufficient, we see that the form of the necessary conditions for the problem (1.1)-(1.4), given by Theorem 3.6, is quite sharp.

We close this section with the following example

(4.6) $\quad$ Minimize $\int_0^1 \left\{ \frac{1}{2} |y - y_d|_H^2 + \frac{1}{2} |u|_H^2 \right\} dt$,

(4.7) $\quad y_t - \Delta y + \beta(y) \ni u \qquad$ a.e. Q,

$y(0,x) = 0 \qquad$ a.e. Ω,

$y(t,x) = 0 \qquad$ a.e. $\partial\Omega \times [0,1]$,

$|u(t,x)| \leq 1 \qquad$ a.e. Q.

Here $H = U = L^2(\Omega)$, $Q =]0,1[\times \Omega$ (Ω is a finite dimensional, bounded domain), $y_d \in L^2(Q)$ and

$$\beta(y) = \begin{cases}]-\infty,0] & y = 0, \\ 0 & y > 0, \\ \emptyset & y < 0 . \end{cases}$$

It is possible to apply the same argument as in Lemma 4.5 and to see that any optimal pair $[y^*,u^*]$ of the problem (4.6), (4.7) satisfies $\beta(y^*) = 0$ a.e. Q. Since we also have $\bar{y}_0 = 0$, the dual problem is given by (4.1), (4.2). We compute it explicitly for $w \in L^2(Q)$:

$$(F + I_{\mathcal{U}_{ad}})^*(h) = \frac{1}{2} \int_{|h(t,x)|<1} h^2 dxdt + \int_{|h(t,x)| \geq 1} \left\{ |h| - \frac{1}{2} \right\} dxdt$$

$$(G + I_{\mathcal{C}})^*(-w) = \frac{1}{2} \int_Q (y_d - w)_+^2 dxdt - \frac{1}{2} \int_Q y_d^2$$

Therefore, for $w \in L^2(Q)$, the dual problem of (4.6), (4.7) is

$$\text{Minimize} \left\{ \frac{1}{2} \int_Q (y_d - w)_+^2 dxdt + \frac{1}{2} \int_{|p(t,x)|<1} p^2 dxdt + \right.$$

$$\left. + \int_{|p(t,x)| \geq 1} \left\{ |p| - \frac{1}{2} \right\} dxdt \right\},$$

subject to

$$p_t + \Delta p = -w \qquad \text{a.e. } Q,$$

$$p(T,x) = 0 \qquad \text{a.e. } \Omega,$$

$$p(t,x) = 0 \qquad \text{a.e. } \partial\Omega \times [0,T],$$

since the unique solution of the above equation is also the unique solution of (4.2).

In the general case, by the Fenchel duality theorem, we have

$$(G + I_{\mathcal{C}})^*(-w) = -\frac{1}{2} \int_Q y_d^2 + \min \left\{ \frac{1}{2} \int_Q (y_d - p)^2 ; p \in (w + \mathcal{C}_0) \cap L^2(Q) \right\},$$

where $\mathcal{C}_0$ is the polar cone of $\mathcal{C}$ in $Z \times Z^*$, and the state equation should be considered in the form (4.2).

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