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TWO PAPERS ON NONSTATIONARITY

by

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SOME ASPECTS OF NONSTATIONARITY

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I INTRODUCTION

It is quite clear today that the difference between stationarity and nonstationarity is not structural, but only one of complexity-see [12], [13], [7]. This will be also illustrated in the present paper. Our aim here is to develop a time-variant analogous for some basic results in Sz.-Nagy-Foias theory of contractions, as model for discrete time, time-variant linear systems. The point will be that the functional model is replaced by "marking operators" and analytic Toeplitz operators are replaced by lower triangular ones- see [10], for other aspects and applications in system theory.

We plan to tackle the following problems: in the second section we obtain a model for time-variant discrete linear systems. Then, we are faced with lower triangular representations, especially for preparing the next section, where a nonstationary variant for the lifting theorem of Sarason-Sz.-Nagy-Foias is treated in details. Of course, in this decade when the domain is dominated by the Grassmannian approach of Ball-Helton ([4]), our tentative may appear to be hopeless. The only reason to insist upon is our affiliation to "the nonstationary program of the unification of both analytic functions theory and matrices theory". In this respect we will point out now some contrctive completion problems in [5], [6], [9] fit in our approach.

We mention that the second section is taken after INCREST preprint No.60/1985.

II THE "MARKING MODEL"

In this section we are concerned with time-variant linear systems in the following state-space representation:

$$(1.1)\qquad \begin{cases} x_{n+1}=T_n^* x_n + D_{T_n} u_n \\ y_n = D_{T_n^*} x_n - T_n u_n \end{cases} \qquad n\in \mathbb{Z}$$

where $T_n \in \mathcal{L}(\mathcal{H}_{n+1}, \mathcal{H}_n)$ are contractions, $\{\mathcal{H}_n\}_{n\in\mathbb{Z}}$ is a given family of Hilbert spaces, $u_n \in \mathcal{D}_{T_n}$, $y_n \in \mathcal{D}_{T_n^*}$, $x_n \in \mathcal{H}_n$ and for a contraction $T\in\mathcal{L}(\mathcal{H},\mathcal{H}')$ we use the standard notation $D_T=(I-T^*T)^{\frac{1}{2}}$ and $\mathcal{D}_T=\overline{D_T\mathcal{H}}$. Now, we consider the positive-definite kernel $\mathcal{S}$ associated by the algorithm in [7] -Theorem 2.4 to the parameters $G_{i,i+1}=T_i$, $i\in\mathbb{Z}$ and zero in rest. Using Theorem 3.2 in [7], the Kolmogorov decomposition of $\mathcal{S}$ is simply given by the unitary operators

$$W_n : \mathcal{K}_{n+1} \longrightarrow \mathcal{K}_n$$

$$(1.2)\qquad W_n(\ldots d_{*,n}, h_{n+1}, d_{n+1}, \ldots)=$$
$$=(\ldots d_{*,n-1}, D_{T_n^*} d_{*,n} + T_n h_{n+1}, -T_n^* d_{*,n} + D_{T_n} h_{n+1}, d_{n+1}, d_{n+2}, \ldots)$$

where

$$\mathcal{K}_n = \ldots \oplus \mathcal{D}_{T_{n-2}^*} \oplus \mathcal{D}_{T_{n-1}^*} \oplus \mathcal{H}_n \oplus \mathcal{D}_{T_n} \oplus \mathcal{D}_{T_{n+1}} \oplus \ldots$$

Let us pursue by introducing the main elements of the geometrical model of (1.1). Define the spaces

$$\mathcal{K}_n^+ = \mathcal{H}_n \oplus \mathcal{D}_{T_n} \oplus \mathcal{D}_{T_{n+1}} \oplus \ldots$$

and the isometries

$$(1.3)\qquad \begin{gathered} W_n^+ : \mathcal{K}_{n+1}^+ \longrightarrow \mathcal{K}_n^+ \\ W_n^+ = W_n / \mathcal{K}_{n+1}^+ \end{gathered}$$

and use the Wold decomposition for the family $\{W_k^+\}_{k\geq n}$. We denote $\mathcal{L}_n^+ = \mathcal{K}_n^+ \ominus W_n^+ \mathcal{K}_{n+1}^+$ and according to the form (1.2) of the Kolmogorov decomposition, we have $\mathcal{L}_n^+ = W_n(\ldots \oplus 0 \oplus \mathcal{D}_{T_n^*} \oplus 0_{\mathcal{H}_{n+1}} \oplus \ldots)$. Moreover, define $\mathcal{R}_n^+ = \bigcap_{p=0}^{\infty} W_n \ldots W_{n+p} \mathcal{K}_{n+p+1}^+$ and then

$$(1.4)\qquad \mathcal{K}_n^+ = (\mathcal{L}_n^+ \oplus \bigoplus_{p=1}^{\infty} W_n^+ \ldots W_{n+p-1}^+ \mathcal{L}_{n+p}^+) \oplus \mathcal{R}_n^+ .$$

Similar considerations take place for the spaces

$$\mathcal{K}_n^- = \dots \oplus \mathcal{D}_{T^*_{n-1}} \oplus \mathcal{H}_n$$

and the isometries

$$W_n^- : \mathcal{K}_n^- \longrightarrow \mathcal{K}_{n+1}^-$$

(1.5)

$$W_n^- = W_n^* / \mathcal{K}_n^- \quad .$$

We use the notation $\mathcal{L}_n^- = \mathcal{K}_n^- \ominus W_{n-1}^- \mathcal{K}_{n-1}^-$ and taking again into account the form (1.2) of the Kolmogorov decomposition of $\mathcal{S}$, we get $\mathcal{L}_n^- = W_n^*(\dots 0 \oplus 0_{\mathcal{H}_n} \oplus \mathcal{D}_{T_n} \oplus 0 \oplus \dots)$. It is also usefull to denote the space $\dots \oplus \mathcal{D}_{T_n^*} \oplus 0_{\mathcal{H}_{n+1}} \oplus \dots$ by $\mathcal{D}_{T_n^*}^{(-1)}$ and $\dots 0 \oplus 0_{\mathcal{H}_n} \oplus \mathcal{D}_{T_n} \oplus 0 \oplus \dots$ by $\mathcal{D}_{T_n}^{(1)}$.

Another application of the Wold decomposition for the family $\{W_k^-\}_{k \le n-1}$ will produce a decomposition

$$\mathcal{K}_n^- = (\mathcal{L}_n^- \oplus \bigoplus_{p=1}^{\infty} W_{n-1} \dots W_{n-p} \mathcal{L}_{n-p}^-) \oplus \mathcal{R}_n^- \quad .$$

Now define the spaces:

$$\mathcal{K}_n^{out} = \bigoplus_{q=1}^{\infty} W_{n-1}^* \dots W_{n-q}^* \mathcal{D}_{T^*_{n-q-1}}^{(-1)} \oplus \mathcal{D}_{T^*_{n-1}}^{(-1)} \oplus \bigoplus_{p=0}^{\infty} W_n \dots W_{n+p} \mathcal{D}_{T^*_{n+p}}^{(-1)}$$

and

$$\mathcal{K}_n^{inp} = \bigoplus_{p=1}^{\infty} W_{n-1}^* \dots W_{n-p}^* \mathcal{D}_{T_{n-p}}^{(1)} \oplus \mathcal{D}_{T_n}^{(1)} \oplus \bigoplus_{q=0}^{\infty} W_n \dots W_{n+q} \mathcal{D}_{T_{n+q+1}}^{(1)} .$$

An usual condition in Sz.-Nagy-Foias theory (and also in system theory) is to ask $\mathcal{K}_n = \mathcal{K}_n^{inp} \vee \mathcal{K}_n^{out}$ for every $n \in \mathbb{Z}$. We have by direct computation using (1.2) that

$$\mathcal{K}_n \ominus (\mathcal{K}_n^{inp} \vee \mathcal{K}_n^{out}) =$$
$$= \{ h \in \mathcal{H}_n / \dots \|T_{n-2}T_{n-1}h\| = \|T_{n-1}h\| = \|h\| = \|T_n^* h\| = \|T_{n+1}^* T_n^* h\| = \dots \}$$

which corresponds to Theorem 3.2.I in [16] .

Finally, we define the family of characteristic operators of the system (1,1) by the formula

$$Q_n : \mathcal{K}_n^{inp} \longrightarrow \mathcal{K}_n^{out}$$

(1.6)

$$Q_n = P^{\mathcal{K}_n}_{\mathcal{K}_n^{out}} / \mathcal{K}_n^{inp}$$

We obtain a first result concerning the geometry of the space $\mathcal{K}$

2.1 THEOREM For a system (1.1) satisfying the condition $\mathcal{K}_n = \mathcal{K}_n^{inp} \vee \mathcal{K}_n^{out}$ for every $n \in \mathbb{Z}$, the following relations hold:

$$\mathcal{K}_n = \mathcal{K}_n^{out} \oplus \mathcal{Q}_n^+$$

$$\mathcal{H}_n = \mathcal{K}_n^+ \ominus \{ Q_n u \oplus (I-Q_n)u/u \quad \ldots 0 \oplus \mathcal{D}_{T_n}^{(1)} \oplus W_n \mathcal{D}_{T_{n+1}}^{(1)} \oplus \ldots \}$$

PROOF The first relation is obvious. For the second one, we remark that

$$\mathcal{K}_n^+ = \mathcal{H}_n \vee \bigvee_{p=1} W_n \ldots W_{n+p-1} \mathcal{H}_{n+p}$$

and , as $W_n \mathcal{D}_{T_n^*}^{(-1)} \oplus W_n \mathcal{H}_{n+1} = \mathcal{H}_n \oplus \mathcal{D}_{T_n}^{(1)}$ one gets

$$\mathcal{K}_n^+ \subset \mathcal{H}_n \oplus \mathcal{D}_{T_n}^{(1)} \oplus W_n \mathcal{D}_{T_{n+1}}^{(1)} \oplus \ldots .$$

The converse inclusion is clear, consequently,

$$\mathcal{H}_n = \mathcal{K}_n^+ \ominus (\mathcal{D}_{T_n}^{(1)} \oplus W_n \mathcal{D}_{T_{n+1}}^{(1)} \oplus \ldots)$$

which complets the proof. ■

Then we introduce the "marking model". The "marking operators" appear as the main elements involved by the Kolmogorov decompositio of an arbitrary positive-definite kernel. In our case, define the spaces:

$$\mathcal{M}_+ = \bigoplus_{n \in \mathbb{Z}} \mathcal{D}_{T_n^*} \quad \text{and} \quad \mathcal{M}_- = \bigoplus_{n \in \mathbb{Z}} \mathcal{D}_{T_n}$$

and the marking operators

(1.7)
$$M_n^+ : \bigoplus_{k \geq n+1} \mathcal{D}_{T_k^*} \longrightarrow \bigoplus_{k \geq n} \mathcal{D}_{T_k^*}$$
$$M_n^+(d_{*,n+1}, d_{*,n+2}, \ldots) = (0, d_{*,n+1}, d_{*,n+2}, \ldots)$$

(1.8)
$$M_n^- : \bigoplus_{k \geq n+1} \mathcal{D}_{T_k} \longrightarrow \bigoplus_{k \geq n} \mathcal{D}_{T_k}$$
$$M_n^-(d_{n+1}, d_{n+2}, \ldots) = (0, d_{n+1}, d_{n+2}, \ldots)$$

Our goal is to obtain identifications for $\mathcal{K}_n^{inp}$, $\mathcal{K}_n^{out}$, $\mathcal{H}_n$ and the characte_istic operators in terms of the marking operators and marking spaces. For this aim , we introduce the following unitary operators:

(1.9)
$$\Phi_n^+ : \mathcal{K}_n^{out} \longrightarrow \mathcal{M}_+$$
$$\Phi_n^+(\ldots, W_{n-1}^* d_{*,n-2}^{(-1)}, d_{*,n-1}^{(-1)}, W_n d_{*,n}^{(-1)}, \ldots) = (\ldots d_{*,-1}, d_{*,o}, d_{*,1} \ldots)$$

where $d_{*,n}^{(-1)} = (\ldots 0, d_{*,n}, 0_{\mathcal{H}_{n+1}}, \ldots) \in \mathcal{K}_{n+1}$, $d_{*,n} \in \mathcal{D}_{T_n^*}$ and

(1.10)
$$\Phi_n^- : \mathcal{K}_n^{inp} \longrightarrow \mathcal{M}_-$$
$$\Phi_n^-(\ldots W_{n-1}^* d_{n-1}^{(1)}, d_n^{(1)}, W_n d_{n+1}^{(1)}, \ldots) = (\ldots d_{-1}, d_o, d_1 \ldots)$$

where $d_n^{(1)} = (\ldots 0, 0_{\mathcal{H}_n}, d_n, 0, \ldots) \in \mathcal{K}_n$, $d_n \in \mathcal{D}_{T_n}$.

The first remark is that for every $n \in \mathbb{Z}$, we get

$$\Phi_n^+ Q_n (\Phi_n^-)^* = \Theta$$

where Θ is the transfer operator of the system (1.1)-see [10] for definitions. Θ is a lower triangular operator such that its matricial elements are $\Theta_{ij} = D_{T_j} T_{j+1} \ldots T_{i-1} D_{T_i^*}$ for $i \in \mathbb{Z}$, $j < i$ and $\Theta_{ii} = -T_i^*$, $i \in \mathbb{Z}$. Θ is a contraction and we obtain the following identification of $\mathcal{K}_n$ in the model given by the marking operators: first we define the unitary operator

(1.11)
$$\Phi_{\mathcal{R}_n^+} : \mathcal{R}_n^+ \longrightarrow \overline{D_\Theta \mathcal{M}_-}$$
$$\Phi_{\mathcal{R}_n^+}(I - Q_n) k = D_\Theta \Phi_n^- k \qquad , k \in \mathcal{K}_n^{inp}$$

then

(1.12)
$$\Psi_n : \mathcal{K}_n \longrightarrow \mathcal{M}_+ \oplus \overline{D_\Theta \mathcal{M}_-}$$
$$\Psi_n = \Phi_n^+ \oplus \Phi_{\mathcal{R}_n^+}$$

Ψ_n is a unitary operator yealding a natural identification of $\mathcal{K}_n$ in the marking model. Moreover , we have the following result which constitutes the time-variant analogous of the Sz.-Nagy-Foias functional model of a contraction.

2.2 THEOREM For a system (1.1) satisfying the condition $\mathcal{K}_n = \mathcal{K}_n^{inp} \vee \mathcal{K}_n^{out}$ for every $n \in \mathbb{Z}$, the following relations hold (through the identifications Ψ_n):

$$\mathcal{H}_n = \Big(\bigoplus_{k \geq n} \mathcal{D}_{T_k^*} \oplus \overline{D_{(H)} \mathcal{M}_-} \Big) \ominus \Big\{ (H) v \oplus D_{(H)} v / v \in \bigoplus_{k \geq n} \mathcal{D}_{T_k} \Big\}$$

$$T_n = P_{\mathcal{H}_n} (M_n^+ u_+ \oplus M_n^- v_-) \qquad , u_+ \oplus v_- \in \mathcal{H}_n.$$

PROOF From (1.4) it follows

$$\Psi_n \mathcal{K}_n^+ = \Phi_n^+ (\mathcal{L}_n^+ \oplus \bigoplus_{p=1}^{\infty} W_n^+ \dots W_{n+p-1}^+ \mathcal{L}_{n+p}^+) \oplus \overline{D_{(H)} \mathcal{M}_-}$$

and

$$\Phi_n^+ (\mathcal{L}_n^+ \oplus \bigoplus_{p=1}^{\infty} W_n^+ \dots W_{n+p-1}^+ \mathcal{L}_{n+p}^+) = \bigoplus_{k \geq n} \mathcal{D}_{T_k^*}$$

by inspection of the definitions. In a similar way,

$$\mathcal{D}_{T_n}^{(1)} \oplus W_n \mathcal{D}_{T_{n+1}}^{(1)} \oplus \dots = \bigoplus_{k \geq n} \mathcal{D}_{T_k}$$

and the first relation follows from Theorem 2.1. The second relation follows from the first one and the remark that

$$\Phi_n^+ W_n (\Phi_{n+1}^+)^* / \bigoplus_{k \geq n+1} \mathcal{D}_{T_k^*} = M_n^+$$

$$\Phi_n^- W_n (\Phi_{n+1}^-)^* / \bigoplus_{k \geq n+1} \mathcal{D}_{T_k} = M_n^- . \blacksquare$$

III LOWER TRIANGULAR REPRESENTATIONS

In this short section we translate in the nonstationary case the so-called Lemma on Fourier representation from the book [16].

We take a family $\{\mathcal{E}_n\}_{n \in \mathbb{Z}}$ of Hilbert spaces and consider the marking operators

$$(3.1) \qquad M_n : \bigoplus_{k \geq n+1} \mathcal{E}_k \longrightarrow \bigoplus_{k \geq n} \mathcal{E}_k$$

$$M_n(e_{n+1}, e_{n+2}, \dots) = (0, e_{n+1}, e_{n+2}, \dots)$$

Now, Lemma 3.2 V in [16] admits the following generalization whose proof is a simple adaptation.

3.1 PROPOSITION Let $\{X_n\}_{n\in\mathbb{Z}}$ be a family of contractions, $X_n : \bigoplus_{p\geq n} \mathcal{E}'_p \longrightarrow \bigoplus_{p\geq n} \mathcal{E}_p$ such that

$$M_n X_{n+1} = X_n M'_n$$

for every $n\in\mathbb{Z}$. Then, there exists a lower triangular contraction $X\in\mathcal{L}(\bigoplus_{p\in\mathbb{Z}} \mathcal{E}'_p, \bigoplus_{p\in\mathbb{Z}} \mathcal{E}_p)$ agreeing with X_n on the corresponding subspaces. ∎

IV NONSTATIONARY LIFTING

In this section we describe a nonstationary variant for the lifting theorem of Sarason-Sz.-Nagy-Foias. The starting point is the following problem which can be viewed as an extension of the Carathéodory-Féjèr problem (see [11] for all is classical about completion problems). That is, for fixed operators $\{C_{j+r,j} / j\geq 0, 0\leq r\leq N\}$ find conditions for the existence of lower triangular contractive extensions of the family. Some other completion problems appearing in [5] - and which were solved there by using their variant of the lifting theorem - will be discussed later on. Now return to our considerations. Fix two integers $-\infty\leq M<\infty$, $-\infty<N\leq\infty$, $M\leq N$ and two families $\{T_n\}_{M\leq n\leq N}$, $\{T'_n\}_{M\leq n\leq N}$ of contractions(the extremal indices are atained only for finite M and N), $T_n\in\mathcal{L}(\mathcal{H}_{n+1},\mathcal{H}_n)$, $T'_n\in\mathcal{L}(\mathcal{H}'_{n+1},\mathcal{H}'_n)$. Let $\{A_n\}_{M\leq n\leq N+1}$ be a family of contractions, $A_n\in\mathcal{L}(\mathcal{H}_n,\mathcal{H}'_n)$ and suppose that it intertwines $\{T_n\}$ and $\{T'_n\}$, i.e.

$$T'_n A_{n+1} = A_n T_n$$

for $M\leq n\leq N$. For $\{T_n\}_{M\leq n\leq N}$ consider its associated kernel by the rule mentioned at the begining of Section 2 and let $\{W_n\}_{M\leq n\leq N}$, $W_n\in\mathcal{L}(\mathcal{K}_{n+1},\mathcal{K}_n)$ be its Kolmogorov decomposition, always written in the form (1.2). We have similar objects associated with $\{T'_n\}_{M\leq n\leq N}$. Now, the following result extends the lifting theorem of Sarason-Sz.-Nagy-Foias.

4.1 THEOREM The set

$$CID(\{A_n\}_{M\le n\le N+1})=\{\{B_n\}_{M\le n\le N+1} \ / \ B_n \text{ are contractions in } \mathcal{L}(\mathcal{K}_n^+,\mathcal{K}_n'^+),\quad W_n'^+B_{n+1}=B_nW_n^+ \text{ and } P_n'B_n=A_nP_n\}$$

is nonvoid, where P_n is the orthogonal projection of $\mathcal{K}_n^+$ onto $\mathcal{H}_n$.

PROOF Let $X_{ij}^{(n)}$ be the matrix of B_n, then writing the intertwining conditions, one gets:

$$X_{11}^{(n)}=A_n \quad , X_{1j}=0 \quad , j>1$$

$$X_{ij}^{(n)}=0 \quad j>i$$

$$X_{21}^{(n)}T_n+X_{22}^{(n)}D_{T_n}=D_{T_n'}A_{n+1}$$

$$X_{k1}^{(n)}T_n+X_{k2}^{(n)}D_{T_n}=X_{k-1,1}^{(n+1)} \quad ,k\geqslant 3$$

and

$$X_{ij}^{(n)}=X_{i-1,j-1}^{(n+1)} \quad , i,j\geqslant 3.$$

Define the operators

$$S_{k-1,n}:\mathcal{D}_{T_n^*}\longrightarrow \mathcal{D}_{T_n'}$$
(4.1)
$$S_{k-1,n}=X_{k1}^{(n)}D_{T_n^*} - X_{k2}^{(n)}T_n^*$$

such that the finite sections of B_n are contractions if and only if the operators

(4.2)
$$C_{kn}=\begin{bmatrix} A_nT_n\cdots T_{n+k-1},\ \ldots, & A_nT_nD_{T_{n+1}^*} & , & A_nD_{T_n^*} \\ & \cdots & , & S_{1n} \\ & \cdots\ , & S_{1,n+1} & , & S_{2,n+1} \\ & & & \\ D_{T'_{n+k-1}}A_{n+k},\ S_{1,n+k-1},\ldots & & , & S_{k-1,n+k-1}\end{bmatrix}$$

are also contractions.

But now, if $C_{on}=A_nT_n=T_n'A_{n+1}$ then there exist contractions

$X_n:\mathcal{D}_{C_{on}} \longrightarrow \mathcal{D}_{T_n'}$ and $Y_n:\mathcal{D}_{T_n^*} \longrightarrow \mathcal{D}_{C_{on}^*}$ such that $A_n D_{T_n^*} = D_{C_{on}^*} Y_n$ and $D_{T_n'} A_{n+1} = X_n D_{C_{on}}$ and using [17], [18] there exists an operator S_{1n} such that C_{1n} is a contraction. Now, the same approximating procedure as in [3] finishes the proof. ∎

We can continue the analysis of the set $CID(\{A_n\}_{M\leq n\leq N+1})$ in order to derive results generalizing those similar in [1], [3]. First of all, we obtain a one to one correspondence between the set $CID(\{A_n\}_{M\leq n\leq N+1})$ and the set of operators $\{S_{kn}\}$ such that C_{kn} are contractions. Denote by F_{kn} the nxn principal submatrix of $C_{k+1,n}$ and for a sequence of contractions $\{G_1, G_2, \ldots\}$, $G_1 \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$, $G_k \in \mathcal{L}(\mathcal{D}_{G_{k-1}}, \mathcal{H}')$, $L(G_1, G_2, \ldots)$ is the row contraction determined by these parameters (see for instance [7])-similar considerations hold for column contractions, denoted by $C(G_1, G_2, \ldots)$. Now, we can state the following nonstationary variant of the main result in [3].

<u>4.2 THEOREM</u> There exists a one to one correspondence between $CID(\{A_n\}_{M\leq n\leq N+1})$ and the set of families of contractions $\{G_{ij}\}$ such that $G_{1n} \in \mathcal{L}(\mathcal{D}_{Y_n}, \mathcal{D}_{X_n^*})$, $M\leq n\leq N$ and $G_{ij} \in \mathcal{L}(\mathcal{D}_{G_{i-1,j}}, \mathcal{D}_{G^*_{i-1,j+1}})$ for $i>2$, $M\leq j\leq N$. The correspondence is explicitely taken by the following formulas:

$$S_{kn} = L(X_n, G_{1n}, G_{2n}, \ldots G_{k-1,n}) Q_{k-1,n} \cdot$$

$$\cdot C(Y_{n-k+1}, G_{1,n-k+1}, G_{2,n-k+2}, \ldots, G_{k-1,n-1}) +$$

$$+ D_{X_n^*} D_{G_{1n}^*} \ldots D_{G^*_{k-1,n}} G_{kn} D_{G_{k-1,n-1}} \ldots D_{G_{1,n-k+1}} D_{Y_{n-k+1}},$$

where the operators Q_{kn} will be explained in the proof bellow.

<u>PROOF</u> The proof is adapted after the one indicated in [8] which actually produces a slight modified version of the algorithm in [3].

First of all, $Q_{on} = -C^*_{on}$ for $M\leq n\leq N$. We have by direct computation that

$$C_{2n}=\begin{bmatrix} F_{1n} & , & D_{F^*_{1n}}\tilde{\Omega}^*_{1n}C(Y_n,G_{1n}) \\ L(X_{n+1},G_{1,n+1})\Omega_{1n}D_{F_{1n}} & , & S_{2,n+1}\end{bmatrix}$$

where Ω_{1n} and $\tilde{\Omega}_{1n}$ are obvious identifications and using once again [17], [18] we get the desired formula for S_{2n} with

$$Q_{1n}=-\Omega_{1n}F^*_{1n}\tilde{\Omega}^*_{1n} .$$

Then we compute

$$Q_{1n}\begin{bmatrix} D_{C^*_{on}} & , & -C_{on}X^*_n \\ 0 & , & D_{X^*_n}\end{bmatrix} = -\Omega_{1n}F^*_{1n}\tilde{\Omega}^*_{1n}\begin{bmatrix} D_{C^*_{on}} & , & -C_{on}X^*_n \\ 0 & , & D_{X^*_n}\end{bmatrix}=$$

$$=-\Omega_{1n}F^*_{1n}D_{F^*_{1n}}=-\Omega_{1n}D_{F_{1n}}F^*_{1n}=$$

$$=\begin{bmatrix} D_{C_{o,n+1}} & , & -C^*_{o,n+1}Y_{n+1} \\ 0 & , & D_{Y_{n+1}}\end{bmatrix}F^*_{1n} .$$

As in [8] we find

$$Q_{1n}=-\begin{bmatrix} C^*_{on} & , & 0 \\ 0 & , & I\end{bmatrix}\begin{bmatrix} D_{Y^*_{n+1}}a_n & , & D_{Y^*_{n+1}}b_n \\ Y^*_{n+1}a_n & , & Y^*_{n+1}b_n\end{bmatrix}$$

with $a_na^*_n+b_nb^*_n=I$ and we define the operator

$$V_{on}:\mathcal{D}_{C^*_{on}}\oplus\mathcal{D}_{X^*_n}\longrightarrow\mathcal{D}_{C_{on}}\oplus\mathcal{D}_{Y_{n+1}}$$

$$V_{on}=\begin{bmatrix} D_{Y^*_{n+1}}a_n & , & D_{Y^*_{n+1}}b_n \\ -Y^*_{n+1}a_n & , & -Y^*_{n+1}b_n\end{bmatrix}$$

The rest is as in [8]. ∎

<u>4.3 REMARK</u> Using Theorem 5.2 in [2] and Theorem 4.2 above, a parametrization with lower triangular operators can be derived, together with corresponding Schur type formula as in Corollary 6.1 in [3]. ∎

V APPLICATIONS

In this section we will show the way some completion problems can be solved using Theorem 4.2. First of all, we return to the problem at the begining of Section 4. For solving it, we define for $n \geqslant 0$,

$$T_n : \bigoplus_{k=1}^{N} \mathcal{K}'_{n+k} \longrightarrow \bigoplus_{k=0}^{N-1} \mathcal{K}'_{n+k}$$

$$T_n = \begin{bmatrix} 0, & 0, & \dots, & 0 \\ I, & 0, & \dots, & 0 \\ & & \dots & \\ 0, & \dots & I, & 0 \end{bmatrix}$$

and

$$A_n : \bigoplus_{k=0}^{N-1} \mathcal{H}_{n+k} \longrightarrow \bigoplus_{k=0}^{N-1} \mathcal{K}'_{n+k}$$

$$A_n = \begin{bmatrix} C_{nn}, & 0, & \dots & & 0 \\ C_{n+1,n}, & C_{n+1,n+1}, & 0, \dots & & 0 \\ & & \dots & & \\ C_{n+N-1,n}, & \dots & & & C_{n+N-1,n+N-1} \end{bmatrix}$$

(we supposed that $C_{ij} \in \mathcal{L}(\mathcal{H}_j, \mathcal{K}'_i)$). We have $T_n A_{n+1} = A_n T_n$. Then, if A_n are contractions, we can use Theorem 4.1 and we get that there exists a family of contractions $\{B_n\}_{n \geqslant 0}$ such that

$$M_n B_{n+1} = B_n M_n \quad , \quad P_n B_n = A_n P_n ,$$

where M_n are marking operators as in Section 3. Using Proposition3.1 $\{B_n\}_{n \geqslant 0}$ gives rise to a contractive lower triangular extension of the given family $\{C_{j+r,j} / j \geqslant 0, 0 \leq r \leq N\}$. That is we obtained the following result.

5.1 PROPOSITION In order that the family $\{C_{j+r,j} / j \geqslant 0, 0 \leq r \leq N\}$ has a contractive lower triangular extension it is necessary and sufficient that A_n are contractions for $n \geqslant 0$. ∎

Moreover, Theorem 4.2 and Remark 4.3 give parametrizations for all the solutions.

But Theorem 4.1 can be used to solve completion problems with a finite number of data, those named as Nehari completions in [5] .

We indicate here (for simplicity) only the very particular case of completing

$$\begin{bmatrix} C_{oo} , & C_{o1} \\ C_{1o} , & \end{bmatrix}$$

to a contraction.

Take

$$A_o=(C_{oo},C_{o1}) \ , \ A_1=\begin{bmatrix} C_{oo} \\ C_{1o} \end{bmatrix} \ , \ T'_o=(I,0) \ , \ T_o=\begin{bmatrix} I \\ 0 \end{bmatrix}$$

then $T'_oA_1=A_oT_o$. Moreover,

$$W'^{+}_o=\begin{bmatrix} I , & 0 \\ 0 , & I \end{bmatrix} \ , \ W^{+}_o=\begin{bmatrix} I \\ 0 \end{bmatrix}$$

and if A_o and A_1 are supposed to be contractions , then Theorem 4.1 asserts the existence of a contraction $\begin{bmatrix} C_{oo}, & C_{o1} \\ C_{21}, & C_{22} \end{bmatrix}$ such that

$$\begin{bmatrix} I , & 0 \\ 0 , & I \end{bmatrix}\begin{bmatrix} C_{oo} \\ C_{1o} \end{bmatrix}=\begin{bmatrix} C_{oo}, & C_{o1} \\ C_{21}, & C_{22} \end{bmatrix}\begin{bmatrix} I \\ 0 \end{bmatrix}.$$

Consequently, $C_{21}=C_{1o}$ and $\begin{bmatrix} C_{oo}, & C_{o1} \\ C_{1o}, & C_{22} \end{bmatrix}$ is a contractive completion of the given $\{C_{oo}, C_{o1}, C_{1o}\}$. This shows that Theorem 4.1 (together with the parametrization in Theorem 4.2) for M=N=0 is equivalent with [17], [18]. [redacted] . The last application here is an extension of Theorem 5 [5] and of a similar result in [14].

<u>5.2 PROPOSITION</u> Let A and B be two lower triangular operators, $A\in\mathcal{L}(\bigoplus_{n\in\mathbb{Z}}\mathcal{H}_n , \bigoplus_{n\in\mathbb{Z}}\mathcal{H}''_n)$, $B\in\mathcal{L}(\bigoplus_{n\in\mathbb{Z}}\mathcal{H}_n , \bigoplus_{n\in\mathbb{Z}}\mathcal{H}'_n)$. Then a necessary and sufficient condition for the existence of a lower triangular contraction $C\in\mathcal{L}(\bigoplus_{n\in\mathbb{Z}}\mathcal{H}'_n , \bigoplus_{n\in\mathbb{Z}}\mathcal{H}''_n)$ such that $A=CB$ is that $A^*A\leq B^*B$.

<u>PROOF</u> Follows the one in [5] and [14] . Take $A_n=A/\bigoplus_{k\geq n}\mathcal{H}_k$, $B_n=B/\bigoplus_{k\geq n}\mathcal{H}_k$

and M_n, M'_n, M''_n the marking operators as in Section 3, then we have:

$$M'_n A_{n+1} = A_n M_n, \quad M''_n B_{n+1} = B_n M_n.$$

Since A and B are lower triangular, then $A_n^* A_n \leq B_n^* B_n$ for $n \in \mathbb{Z}$ and there exist uniquely determined contractions $X_n : \overline{\text{Range} B_n} \longrightarrow \overline{\text{Range} A_n}$ such that $A_n = X_n B_n$. From now on we follow [14] in order to find the position in which Theorem 4.1 is applicable and using also Proposition 3.1 we obtain the desired C. ∎

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ON A PROBLEM OF I.GOHBERG AND S.LEVIN

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I INTRODUCTION

In the paper [5] the authors raised the following problem connected with the First Szegö Theorem. Let $\{A_n\}_{n\in\mathbb{Z}}$ be a family of rxr matrices and considering the Toeplitz matrices

$$T_n=(A_{j-k})_{j,k=o}^{n} \quad , n=0,1,\ldots$$

compute the matrix

$$\lim_{n\to\infty}(T_n^{-1})_{oo}$$

in terms of the symbol $a(z)=\sum_{j=-\infty}^{\infty} A_j z^j \quad , |z|=1.$

In [5] there are presented some statements based on the projection method and in the recent paper [4] another formula is derived in terms of a realization of the symbol.

Our purpose is to derive formulas based on the Schur analysis of block-matrices, as developed in [1] and [2], and having as starting point the classical work of I.Schur in [6]. We will approach two cases: in the second section we obtain an operatorial variant for a positive-definite kernel on the set of integers and in the last section we analyse the finite dimensional case of arbitrary signature.

II POSITIVE-DEFINITE KERNELS

We take into account a family of Hilbert spaces $\{\mathcal{H}_n\}_{n\in\mathbb{Z}}$ and an application $\mathcal{S}$ defined on $\mathbb{Z}\times\mathbb{Z}$ such that $\mathcal{S}(i,j)=S_{ij}\in\mathcal{L}(\mathcal{H}_j,\mathcal{H}_i)$ for every $i,j\in\mathbb{Z}$ and the operators

$$M_{ij}(\mathcal{S})=M_{ij}:\bigoplus_{k=i}^{j}\mathcal{H}_k \longrightarrow \bigoplus_{k=i}^{j}\mathcal{H}_k$$

$$M_{ij}=(S_{mn})_{i\leq m,n\leq j}$$

are all positive for $i,j\in\mathbb{Z}$, $i\leq j$. We suppose $S_{ii}=I_{\mathcal{H}_i}$, without restricting generality. A result in [2] associated with such an object a family $\mathcal{G}=\{G_{ij}/i,j\in\mathbb{Z},\ i\leq j\}$ of contractions, where $G_{ii}=$

$=0_{\mathcal{H}_i}$, $i\in\mathbb{Z}$ and for $i<j$, $G_{ij}\in\mathcal{L}(\mathcal{D}_{G_{i+1,j}},\mathcal{D}_{G^*_{i,j-1}})$- for a contraction $T\in\mathcal{L}(\mathcal{H},\mathcal{H}')$, we use the standard notation $D_T=(I-T^*T)^{\frac{1}{2}}$ and $\mathcal{D}_T=\overline{D_T\mathcal{H}}$.

Suppose that $D_{G_{ij}}$ are invertible operators for all $G_{ij}\in\mathcal{G}$, this yealding the invetibility of M_{om} for every $m\geqslant 0$. Define the operator

$$H_\infty=(\underset{n\to\infty}{s\text{-lim}}(D_{G^*_{o1}}D_{G^*_{o2}}\dots D^2_{G^*_{on}}\dots D_{G^*_{o2}}D_{G^*_{o1}}))^{\frac{1}{2}}$$

and we can get the result of this section.

<u>2.1 PROPOSITION</u> If H_∞ is also invertible, then there exists the strong operatorial limit:

$$\underset{m\to\infty}{s\text{-lim}}(M^{-1}_{om})_{oo}=H^{-2}_\infty .$$

<u>PROOF</u> It is a consequence of some formulas derived in [2] . First of all we have the Cholesky factorization

$$M_{om}=F^*_{om}F_{om}$$

and using Lemma 1.2 in [2], which asserts that

$$F_{om}=V_{om}\begin{bmatrix} X^*_{om} & ,F_{1m} \\ D_{G^*_{om}}\dots D_{G^*_{o1}} & ,\ 0\end{bmatrix}$$

where V_{om} is a unitary operator and

$$X_{om}=(G_{o1},D_{G^*_{o1}}G_{o2},\dots,D_{G^*_{o1}}D_{G^*_{o2}}\dots D_{G^*_{o,m-1}}G_{om})$$

, one gets

$$(M^{-1}_{om})_{oo}=\left[\begin{bmatrix}0\ , & D^{-1}_{G^*_{o1}}\dots D^{-1}_{G^*_{om}}\\ * \ , & *\end{bmatrix}\begin{bmatrix}0 & ,\\ D^{-1}_{G_{om}}\dots D^{-1}_{G_{o1}} & ,\end{bmatrix}\right]_{oo}=$$

$$=D^{-1}_{G^*_{o1}}\dots D^{-2}_{G^*_{om}}\dots D^{-1}_{G^*_{o1}}$$

(the star-marked entries does not matter).If H_∞ is invertible, we obtain that

$$\underset{m\to\infty}{s\text{-lim}}(M^{-1}_{om})_{oo}=H^{-2}_\infty \quad . \blacksquare$$

2.2 REMARKS (1) When $\mathcal{S}$ is Toeplitz, H_∞ measures the so-called error-prediction operator of $\mathcal{S}$ (combine [2] and [7]) and in this sense , Proposition above keeps the same meaning as in the scalar case- [5] .

(2) Restrict again to the scalar Toeplitz case, let $\{\varphi_n\}_{n\geq o}$ be the orthogonal polynomials of $\mathcal{S}$ and $\varphi_n^T(z)=z^n\overline{\varphi}(1/z)$, $\overline{\varphi}_n(z)=\overline{\varphi_n(\overline{z})}$ then another classical result (see [3]) asserts that φ_n^T converges uniformly on compact subsets of the unit circle to the inverse of the spectral factor of $\mathcal{S}$. The convergence in 0 means the first Szegö theorem and as the orthogonal polynomials appear as columns in the inverses of the Cholesky factors, the problem of Gohberg and Levin naturally extends to the assymptotics of the diagonal elements of M_{om}^{-1} . A few remarks are immediately available in the operatorial case. Consider $\{G_k\}_{k\geq o}$ the parameters of $\mathcal{S}$ and consider $\mathcal{S}^{(n)}$ the positive-definite Toeplitz kernel given by the family $\{G_1, G_2,\ldots,G_n,0,0,\ldots\}$ for every $n>0$. Let $\mathcal{F}$ be the maximal spectral factor of $\mathcal{S}$ ([7]) and $\mathcal{F}^{(n)}$ the maximal spectral factor of $\mathcal{S}^{(n)}$. Suppose again that D_{G_n} are invertible operators, consequently let $\mathcal{G}^{(n)}$ be the inverse of $\mathcal{F}^{(n)}$ and let $G_k^{(n)}$ its coefficients.It is plain that for fixed k,

$$(M_{om}^{-1})_{kk}=G_k^{(m)}G_k^{(m)*} \qquad , m\geq k .$$

That is , all is reduced to the question whether $G_k^{(m)}G_k^{(m)*}$ has limit or not. When H_∞ is invertible, there exists $\mathcal{G}$ the inverse of $\mathcal{F}$ and we have seen in Proposition 2.1 that this is all we need for handling the case k=0. For k > 0 the difficulties are similar with those encountered in the construction of the filter of prediction-a manageable situation is that treated in [8] . ■

III MATRICES

We take again into cosideration an application $\mathcal{S}$ defined on $\mathbb{Z}\times\mathbb{Z}$ such that $S_{ij}\in\mathcal{L}(\mathcal{H}_j,\mathcal{H}_i)$, but with $\mathcal{H}_i$ of finite dimension,

M_{ij} of arbitrary signature and for simplicity, $\mathcal{S}$ is Toeplitz. Also suppose that M_{ij} are invertible and let $\{G_n\}_{n\geq 0}$ be the family of parameters associated in [1] to $\mathcal{S}$. The connections between S_n and G_n are the same as in the positive-definite case, but now G_n are arbitrary matrices. We have

$$(M_{om}^{-1})_{oo}=|S_o|^{-\frac{1}{2}}D_{G_1^*}^{-1}J_{G_1^*}D_{G_2^*}^{-1}\dots J_{G_{m-1}^*}D_{G_m^*}^{-1}J_{G_m^*}D_{G_m^*}^{-1}J_{G_{m-1}^*}\dots J_{G_1^*}D_{G_1^*}^{-1}|S_o|^{-\frac{1}{2}}$$

where

$$J_{G_n}=\operatorname{sgn}(J_{G_{n-1}}-G_n^*J_{G_{n-1}^*}G_n) \quad \text{for } n>1,\ J_{G_1}=\operatorname{sgn}(J_{S_o}-G_1^*J_{S_o}G_1),$$

$$D_{G_n}=|J_{G_{n-1}}-G_n^*J_{G_{n-1}^*}G_n|^{\frac{1}{2}} \quad \text{for } n>1,\quad D_{G_1}=|J_{S_o}-G_1^*J_{S_o}G_1|^{\frac{1}{2}},\ J_{S_o}=\operatorname{sgn}S_o$$

A convenient condition is to suppose that $\mathcal{S}$ has k negative squares in the sense that the block-matrices M_{om} have k negative squares for $m>o$. Then , from a certain rank , G_n become contractions. Szegö type results hold ([1]) and the sequence

$$|S_o|^{\frac{1}{2}}D_{G_1^*}J_{G_1^*}\dots D_{G_m^*}J_{G_m^*}D_{G_m^*}\dots J_{G_1^*}D_{G_1^*}|S_o|^{\frac{1}{2}}$$

converges. When its limit is invertible, $(M_{om}^{-1})_{oo}$ converges to this inverse.

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