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by

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On certain automorphisms of reduced crossed products with discrete groups

by

M. Dădărlat and C. Pasnicu

For a unital C^*-algebra A and a discrete group G acting on A by automorphisms, we let $\mathrm{Aut}_A(A \rtimes_{\alpha,r} G)$ denote the topological group of all automorphisms β of the reduced crossed product $A \rtimes_{\alpha,r} G$, such that $\beta(A) = A$.

The analogue group in the framework of the von Neumann algebras was studied by I.M.Singer [13] , H.Behncke [1] and G. Zeller-Meyer [17]. Inspired by their work we give a description of $\mathrm{Aut}_A(A \rtimes_{\alpha,r} G)$, under certain assumptions on the dynamical system (A, G, α) (see Thms. 2.6. and 2.8.) . This is done in the first part of the paper.

In the second part, we analyse the topological group $\mathrm{Aut}_A(A \rtimes_{\alpha,r} G)$ from the homotopy point of view, in the case when $A = C(K)$, where K is a compact connected topological group and G is a dense subgroup of K acting on K by left translations. Using some facts of cohomology of groups we compute the homotopy groups of $\mathrm{Aut}_{C(K)}(C(K) \rtimes_{\alpha,r} G)$ in terms of the homotopy groups of K, $\mathrm{Aut}_G(K) = \{\sigma \in \mathrm{Aut}(K) : \sigma(G) = G\}$ and some "amenable" algebraic objects built from G and K (see Thm. 3.11). The computations are more precise when the abelianized of G is either free or a torsion group. These situations include the cases of the irrational rotation algebras and of the Bunce-Deddens algebras (see 3.14).

Automorphisms of the above type have been recently considered by O.Bratteli, G.A.Elliott, D.E.Evans, A.Kishimoto [2] , B.Brenken [3] and A.Kumjian [7].

§1.

Throughout this paper G will denote a discrete group with neutral element e.

If K is a group, then K^0 will denote the opposite group.

Recall that for a compact connected commutative topological group, its Pontrjagin dual is torsion free ([6]).

If K is a locally compact group, there is a natural structure of topological group on Aut (K) := the group of all continuous automorphisms of K (see [6] , §26). If G is a subgroup of K, we let $Aut_G(K)$ denote $\{\sigma \in Aut\ (K) : \sigma(G) = G\}$. We endow $Aut_G(K)$ with the topology given by : $\sigma_i \to \sigma$ in $Aut_G(K)$ iff $\sigma_i \to \sigma$ in Aut(K) and $(\sigma_{i|G}) \to (\sigma_{|G})$ in Aut (G).

If L is a topological group we let L_0 denote the path connected component of the identity. The homotopy groups of L are denoted by $\pi_n(L)$, where the base point is the identity of L.

For a unital C^*-algebra A we let U(A) denote the unitary group of A and Z(A) the center of A . Aut (A) := the group of all $*$-automorphisms of A is considered with the topology of pointwise norm convergence. Given an action $\alpha : G \to Aut\ (A)$, $Z^1\ (G,U(A))$ is by definition the space of all maps $m:G \to U(A)$ satisfying the identity;

$$m(g\cdot h) = m(g)\cdot \alpha_g(m(h)) \ ; \ g,\ h \in G.$$

We let $Z^1(G,\ U(A))$ have the product topology induced from $\prod_{g \in G} U(A)$.

We denote by k(G;A) the set of all the maps from G to A having finite support.

§2.

Under certain assumptions we shall give a description of the topological group $\mathrm{Aut}_A(A \rtimes_{\alpha,r} G)$.

Consider an injective $*$-representation $\pi : A \to B(H)$. We shall identify A with its image by π and $A \rtimes_{\alpha,r} G$ with the norm closure of $(\tilde{\pi} \times U)(l^1(G,A))$ in $B(l^2(G,H))$, where:

$$(\tilde{\pi}(a)\xi)(g) = \pi(\alpha_{g^{-1}}(a))\xi(g)$$

$$(U_h\xi)(g) = \xi(h^{-1}g)$$

for any $a \in A$, $g, h \in G$ and $\xi \in l^2(G,H)$ (see [8],Thm.7.7.5.) .

We shall need the following :

2.1 LEMMA . There is an unique linear, injective and contractive map $x \to (x_g)_{g\in G}$ from $A \rtimes_{\alpha,r} G$ to $l^\infty(G,A)$ which extends the natural inclusion $l^1(G,A) \to l^\infty(G,A)$. For any $x,y \in A \rtimes_{\alpha,r} G$ we have:

$$(x^*)_g = \alpha_g(x^*_{g^{-1}}) \quad \text{and} \quad (x\cdot y)_g = \sum_{h\in G} x_h \alpha_h(y_{h^{-1}g})$$

(strong convergence in $B(H)$). Moreover, the map $E : A \rtimes_{\alpha,r} G \to A$, given by $E(x) := x_e$, is a faithful conditional expectation and for any $x \in A \rtimes_{\alpha,r} G$ and $g \in G$, $x_g = E(xU_g^*)$.

Proof. See ([17], Thm, 4.12).

2.2 DEFINITION. We say that G acts properly outer on A, if each α_g, $g \neq e$, has the property : if $a \in A$ and $a\alpha_g(x) = xa$ for all x in A, then $a = 0$.

2.3 REMARKS

a) Let $G \times X \to X$, $(g,x) \to g\cdot x$, be a continuous action of G on the compact space X. The corresponding action $G \to \mathrm{Aut}(C(X))$ is properly outer iff : for each $g \neq e$, $\{x \in X : g\cdot x = x\}$ has no interior points.

b) Let L be a discrete ICC-group and $G \to \mathrm{Aut}(L)$ an action of G on L by outer automorphisms. Then the induced action $\alpha : G \to \mathrm{Aut}(C^*_{red}(L))$ is properly outer and $Z(C^*_{red}(L)) \simeq \mathbb{C}$ (see [16], 22.12)

2.4. LEMMA. Suppose that G acts properly outer on A or on Z(A) and that the spectrum of Z(A) is connected. Consider $v \in U(A \rtimes_{\alpha,r} G)$ such that $v A v^* = A$. Then, there are $a \in U(A)$ and $h \in G$, unique with the property that $v = a U_h$.

Proof. Let $(v_g)_{g\in G}$ the map associated with v, by Lemma 2.1. Fix $a \in A$. Then $b := vav^* \in A$ and $va = bv$. Hence, for any $g \in G$: $v_g\alpha_g(a) = bv_g \Leftrightarrow v_g\alpha_g(a) = vav^*v_g \Leftrightarrow v^*v_g\alpha_g(a) = av^*v_g \Leftrightarrow$ $\alpha_g(a^*)v_g^*v = v_g^*va^* \Leftrightarrow \alpha_g(a^*)v_g^*v_h = v_g^*v_h\alpha_h(a^*)$, for any $h \in G$. Hence :

$$c\, v_g^*v_h = v_g^*v_h\alpha_{hg^{-1}}(c),\ c \in A,\ g,\ h \in G. \qquad (1)$$

Since $v^*Av = A$, by (1) and Lemma 2.1. it follows that :

$$v_gv_g^* \in Z(A),\ g \in G \qquad (2)$$

From (1) and (2), we deduce :

$$c\, v_gv_g^*\, v_hv_h^* = v_gv_g^*\, v_hv_h^*\alpha_{hg^{-1}}(c),\ c \in Z(A), g, h \in G. \qquad (3)$$

By the hypothesis, (1) and (3), it follows that :

$$v_gv_g^*\, v_hv_h^* = 0,\ g \neq h \in G. \qquad (4)$$

But :

$$\sum_{g\in G} v_g v_g^* = 1 \text{ (strong covergence in } \mathcal{B}(H)) \qquad (5)$$

since $vv^* = 1$ (see Lemma 2.1). Using (4) we deduce that each $v_g v_g^*$ is a projection in $\mathcal{Z}(A)$ and hence it must be trivial ; but by (5), only one is nonzero, say $v_h v_h^*$ (=1), which implies $v = v_h U_h$ (see Lemma 2.1.).

2.5. Assume the hypothesis of Lemma 2.4. Let $B := A \rtimes_{\alpha,r} G$ and consider $\beta \in \mathrm{Aut}_A(B)$. We shall describe all such automorphisms.

By Lemma 2.4., there are $b_g \in U(A)$ and a map $\sigma : G \to G$ such that $\beta(U_g) = b_g \cdot U_{\sigma(g)}$, $g \in G$.

We shall need the following :

LEMMA. $\sigma \in \mathrm{Aut}(G)$.

Proof. a) σ is injective.

Suppose $\sigma(g) = \sigma(h)$. Then $\beta(U_{gh^{-1}}) = \beta(U_g)\beta(U_h)^* = b_g b_h^* \in A$. Since $\beta \in \mathrm{Aut}_A(B)$, it follows that $U_{gh^{-1}} \in A$, and Lemma 2.1. implies that $g = h$.

b) σ is surjective.

Suppose $(\exists)\ g_0 \in G \setminus \sigma(G)$. Then, for any $x \in A$ and $g \in G$, $E(\beta(xU_g)U_{g_0}^*) = E(\beta(x)b_g U_{\sigma(g) g_0^{-1}}) = 0$ $(\beta(x)b_g \in A,\ \sigma(g)g_0^{-1} \neq e)$ hence $E(\beta(b)U_{g_0}^*) = 0$, for any $b \in B$, a contradiction.

c) $\sigma(g\cdot h) = \sigma(g)\sigma(h)$, for $g, h \in G$.

$$b_{gh} U_{\sigma(gh)} = \beta(U_{gh}) = \beta(U_g)\beta(U_h) = b_g U_{\sigma(g)} \cdot b_h U_{\sigma(h)} =$$

$$= b_g \alpha_{\sigma(g)}(b_h) U_{\sigma(g)} U_{\sigma(h)} = b_g \alpha_{\sigma(g)}(b_h) \cdot U_{\sigma(g)\sigma(h)}.$$

Using Lemma 2.1., we obtain the desired identity.

□

By the above computations, we also obtain :

$$a_{gh} = a_g \cdot \alpha_g(a_h), \text{ for } g, h \in G$$

where, by definition, $a_g = b_{\sigma^{-1}(g)} \in U(A)$.

Notice that for any $g \in G$ and $a \in A$:

$$\beta(\alpha_g(a)) = \beta(U_g)\beta(a)\beta(U_g)^* = a_{\sigma(g)} U_{\sigma(g)} \beta(a) U_{\sigma(g)}^* a_{\sigma(g)}^* =$$
$$= \text{ad } a_{\sigma(g)} (\alpha_{\sigma(g)} (\beta(a)).$$

Hence we can define a map :

$$\phi : \mathrm{Aut}_A(B) \to \mathcal{G} \subset Z^1(G, U(A)) \times \mathrm{Aut}(A) \times \mathrm{Aut}(G)$$

where $\mathcal{G} := \{((c_g), \rho, \sigma) : \rho \circ \alpha_g = \text{ad } c_{\sigma(g)} \circ \alpha_{\sigma(g)} \circ \rho, \ g \in G\}$ by :

$$\phi(\beta) := ((a_g), \beta_{|A}, \sigma).$$

We have the following :

2.6. THEOREM. ϕ is a homeomorphism.

Proof : By the above computations it is clear that ϕ is well defined and injective ($B = C^*(A \cup U_G) \subset B(l^2(G,H))$). To prove the surjectivity of ϕ, take $((c_g), \rho, \sigma) \in \mathcal{G}$. Define $\beta : l^1(G,A) \to B(l^2(G,H))$, by $\beta((x_g)_{g\in G}) := \sum_{g \in G} \rho(x_g) \cdot c_{\sigma(g)} \cdot U_{\sigma(g)}$.

Since $\sum_{g \in G} \| \rho(x_g) c_{\sigma(g)} U_{\sigma(g)} \| \leq \sum_{g \in G} \| \rho(x_g) \| \cdot \| c_{\sigma(g)} U_{\sigma(g)} \| =$
$= \sum_{g \in G} \| x_g \| = \| (x_g)_{g \in G} \|_1 < +\infty$, it follows that $\beta(l^1(G,A)) \subset B$.

We have $C^*(\beta(l^1(G,A))) = B$. By ([17], Prop.2.7.), β is a unital representation of $l^1(G,A)$. Since $E : B \to A (= \rho(A))$, $E(b) = b_e$, is linear, positive and faithful, and since
$E \circ \beta_{|k(G,A)} = \beta \circ E_{|k(G,A)}$, we can apply ([17], Thm.4.22, equivalence (i)$\Leftrightarrow$(iii), with $\theta = E$) to deduce that β induces an injective $*$-homomorphism $\beta : B \to B$. Hence $\beta \in \mathrm{Aut}_A(B)$ and
$\phi(\beta) = ((c_g), \rho, \sigma)$

ϕ^{-1} is obviously continuous and for the continuity of ϕ, observe that $\| E(\beta'(U_g)\beta(U_g)^*) \| = \begin{cases} 1, & \sigma(g) = \sigma'(g) \\ 0, & \sigma(g) \neq \sigma'(g) \end{cases}$

where β (resp. β') belongs to $Aut_A(B)$ and σ (resp. σ') is the associated automorphism of G.

2.7. REMARKS. a) Assume the hypothesis of the above Theorem. If $E : B \to A$ is the conditional expectation from Lemma 2.1. , and if $\beta \in Aut(B)$, then :

$$\beta \in Aut_A(B) \Longleftrightarrow \beta \circ E = E \circ \beta \quad .$$

b) Assume that G acts properly outer on A and that Z(A) has connected spectrum. The above theorem easily implies that the topological group of the automorphisms of $A \rtimes_{\alpha,r} G$ which leave A pointwise fixed is isomorphic to $Z^1(G, UZ(A))$.

2.8. We have found a more precise description of the topological group $Aut_A(B)$ in the case when $A = C(K)$, where K is a compact connected topological group and G is a dense subgroup of K acting on K by translations (the induced action on C(K) is given by $\alpha_g(a) = a(g^{-1}\cdot))$.

In fact, this example is "generic" (at lest) for the case when $A = C(X)$, with X compact metrizable and G countable and commutative, acting freely and minimally on X, such that the action is equicontinuous relative to some metric d (i.e. $(\forall)\, \varepsilon > 0$, $(\exists)\, \delta = \delta(\varepsilon) > 0$ such that $d(x,y) < \delta \Rightarrow d(g\cdot x, g\cdot y) < \varepsilon$, $g \in G$).

For $\beta \in Aut_{C(K)}(C(K) \rtimes_{\alpha,r} G)$, Theorem 2.6. gives that :

$\beta(U_g) = a_{\sigma(g)} U_{\sigma(g)}$, $g \in G$, where $\sigma \in Aut(G)$ and $(\exists)\, \varphi \in Homeo(K)$ such that $\beta(a) = a \circ \varphi^{-1}$, $a \in C(K)$. The relation $\beta \circ \alpha_g = ada_{\sigma(g)} \circ \alpha_{\sigma(g)} \circ \beta_{|A}$, $g \in G$, is equivalent with the condition :

$$\sigma(g)\varphi(k) = \varphi(gk), \ (\forall)\ g \in G, \ (\forall)\ k \in K$$

which exactly says that : σ extends to a map (also denoted by σ) belonging to $Aut_G(K)$ and $\varphi(\cdot) = \sigma(\cdot)\varphi(e)$.

Define the homomorphism of groups $J : K^0 \rtimes_I Aut_G(K) \to$ $\to Aut(Z^1(G,U(A)))$ by $J(k,\sigma)(a_g)_g = (a_{\sigma^{-1}(g)} \circ \sigma^{-1}(\cdot k^{-1}))_g$, where the homomorphism $I : Aut_G(K) \to Aut(K^0)$ is the inclusion map. If $\Psi: Aut_{C(K)}(C(K) \rtimes_{\alpha,r} G) \to Z^1(G,UC(K)) \rtimes_J (K^0 \rtimes_I Aut_G(K))$ is given by $\Psi(\beta) := ((a_g), \varphi(e), \sigma)$, we obtain the following :

THEOREM : Ψ is an isomorphism of topological groups.

2.9. REMARK. Note that since in the above case, A is masa in B (see [17] , Prop. 4.14), we have $Aut_{C(K)}(C(K) \rtimes_{\alpha,r} G) =$

$$= \{ \beta \in Aut(C(K) \rtimes_{\alpha,r} G) : \beta(C(K)) \subset C(K) \}.$$

§3.

In this section we shall analyse $\mathrm{Aut}_{C(K)}(C(K) \rtimes_{\alpha,r} G)$ from the homotopy point of view. The hypothesis is the same as in 2.8. , namely G is a dense subgroup of K(= compact connected topological group) acting on K by left translations. As a consequence of Theorem 2. 8. we have :

$$\pi_n(\mathrm{Aut}_{C(K)}(C(K) \rtimes_{\alpha,r} G)) = \pi_n(Z^1(G, UC(K))) \rtimes (\pi_n(K) \rtimes \pi_n(\mathrm{Aut}_G(K))).$$

The semidirect product structure of $\pi_n(\mathrm{Aut}_{C(K)}(C(K) \rtimes_{\alpha,r} G))$ comes from 2.8. For $n \geq 1$, this coincides with the ordinary direct product (see [14] , Ch.1, §6, Cor. 10).

The analysis of $Z^1(G, UC(K))$ requires some elements of cohomology of groups. We shall recall some definitions and standard notations.

Let M be an abelian group (written additively). We say that M is a G-module if we are given a homomorphism $G \to \mathrm{Aut}(M)$. We let ${}^g x$ denote the image of $x \in M$ under the automorphism given by $g \in G$. If M and N are G-modules, a map $f: M \to N$ is called a G-homomorphism if it is a group homomorphism which preserves the action of G or equivalently, if it is a homomorphism of G-modules. For a G-module M, we denote by M^G the submodule consisting of the elements fixed by G.

Cohomology groups (of low dimension) are easily described using standard n-cocycles $Z^n(G,M)$ and standard n-coboundaries $B^n(G,M)$, for we have $H^n(G,M) = Z^n(G,M) / B^n(G,M)$.

n=0 : $Z^0(G,M) = M^G$, $B^0(G,M) = 0$

n=1 : $Z^1(G,M)$ consists of all the maps $g \to m_g$ from G to M satisfying $m_{gh} = m_g + {}^g m_h$. $B^1(G,M)$ consists of the maps $g \to m_g$ in $Z^1(G,M)$ of the form $m_g = {}^g x - x$ for some $x \in M$.

n=2 : $Z^2(G,M)$ consists of all the maps $(g,h) \to m_{g,h}$ from $G \times G$ to M satisfying

$${}^g m_{h,k} - m_{gh,k} + m_{g,hk} - m_{g,h} = 0$$

$B^2(G,M)$ consists of all the maps in $Z^2(G,M)$ having the form

$$m_{g,h} = {}^g t_h - t_{gh} + t_g$$

for some map $g \to t_g$ from G to M.

Note that if G acts trivially on M then $H^1(G,M) = Z^1(G,M) = \mathrm{Hom}(G,M)$. For every exact sequence of G-modules :

$$0 \to M \xrightarrow{j} N \xrightarrow{q} P \to 0$$

there is a connecting homomorphism $\delta : Z^1(G,P) \to H^2(G,M)$ such that the sequence :

$$0 \to Z^1(G,M) \xrightarrow{j_*} Z^1(G,N) \xrightarrow{q_*} Z^1(G,P) \xrightarrow{\delta} H^2(G,M) \xrightarrow{j_*} H^2(G,N)$$

is exact. Moreover the connecting homomorphism depends functorialy on the given exact sequence. Let us briefly recall the definition of δ. Let $g \to p_g$ be a 1-cocycle in $Z^1(G,P)$. Since q is onto there is a map $g \to n_g$ from G to N such that $q(n_g) = p_g$ for all g in G. As $g \to p_g$ is a 1-cocycle we must have $q(n_g + {}^g n_h - n_{gh}) = 0$ and so for any $g,h \in G$ there is a unique $m_{g,h} \in M$ such that

$$j(m_{g,h}) = n_g + {}^g n_h - n_{gh}.$$

It is easy to check that the map $(g,h) \longrightarrow m_{g,h}$ defines a 2-cocycle and moreover its class of cohomology $[(m_{g,h})]$ in $H^2(G,M)$ depends only on the given 1-cocycle $g \to p_g$. By definition $\delta((p_g)) = [(m_{g,h})]$.

Let $[G,G]$ be the commutator subgroup of G and let $G_{ab} = G/[G,G]$ the abelianized of G. Let

$$0 \longrightarrow [G,G] \longrightarrow G \xrightarrow{\psi} G_{ab} \longrightarrow 0$$

be the corresponding exact sequence. It is clear that for any abelian group M the quotient map ψ induces an isomorphism of groups $\mathrm{Hom}(G_{ab},M) \longrightarrow \mathrm{Hom}(G,M)$. If $g \in G$ we shall write many times $\dot{g}$ instead of $\psi(g)$.

We shall regard the exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \xrightarrow{\exp} \mathbb{T} \longrightarrow 0 \quad (\exp(x) := e^{2\pi i x})$$

as an exact sequence of G-modules (with trivial G-actions). Therefore we have an exact sequence

$$0 \to Z^1(G,\mathbb{Z}) \to Z^1(G,\mathbb{R}) \to Z^1(G,\mathbb{T}) \xrightarrow{\delta_1} H^2(G,\mathbb{Z}) \to H^2(G,\mathbb{R}).$$

We want to find the image of δ_1. This is an easy question, however we record the answer in a lemma for later use.

3.1 LEMMA. The image of the connecting homomorphism δ_1 in the above sequence is naturally isomorphic to $\mathrm{Ext}^1_{\mathbb{Z}}(G_{ab},\mathbb{Z})$.

Proof. The exponential sequence may be seen as an injective presentation of $\mathbb{Z}$, hence there is an exact sequence

$$0 \to \mathrm{Hom}(G_{ab},\mathbb{Z}) \to \mathrm{Hom}(G_{ab},\mathbb{R}) \to \mathrm{Hom}(G_{ab},\mathbb{T}) \to \mathrm{Ext}^1_{\mathbb{Z}}(G_{ab},\mathbb{Z}) \to 0.$$

On the other hand we have the obvious identifications:

$$\begin{array}{ccccc} Z^1(G,\mathbb{Z}) & \longrightarrow & Z^1(G,\mathbb{R}) & \xrightarrow{\exp_*} & Z^1(G,\mathbb{T}) \\ \wr\| & & \wr\| & & \wr\| \\ \mathrm{Hom}(G_{ab},\mathbb{Z}) & \longrightarrow & \mathrm{Hom}(G_{ab},\mathbb{R}) & \longrightarrow & \mathrm{Hom}(G_{ab},\mathbb{T}) \end{array}$$

whence image $\delta_1 \simeq$ coker $(\exp_*) \simeq \mathrm{Ext}^1_{\mathbb{Z}}(G_{ab}, \mathbb{Z})$. □

Let $\Lambda^2 G_{ab}$ denote the second exterior power of G_{ab}.

3.2. LEMMA There is a natural isomorphism

$$\mu : H^2(G_{ab}, \mathbb{T}) \longrightarrow \mathrm{Hom}(\Lambda^2 G_{ab}, \mathbb{T})$$

which takes the class of the 2-cocycle $(m_{\dot g, \dot h})$ to the homomorphism $\dot g \wedge \dot h \longrightarrow m_{\dot g, \dot h} - m_{\dot h, \dot g}$

Proof. Let M be an abelian group considered as G-module with trivial G action. By ([4] ,Ch V, § 6, Exercise 5) there is an exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(G_{ab}, M) \longrightarrow H^2(G_{ab}, M) \xrightarrow{\mu} \mathrm{Hom}(\Lambda^2 G_{ab}, M) \to 0$$

The statement of the lemma is obtained by taking M to be the divisible group $\mathbb{T}$. □

If L is a normal subgroup of G the cohomology groups of G, L and G/L are connected by the exact sequence of Hochschild-Serre. We need the following case :

$$0 \longrightarrow H^1(G/L, M^L) \xrightarrow{\eta_1^*} H^1(G,M) \xrightarrow{\rho} H^1(L,M)^{G/L} \xrightarrow{\tau} H^2(G/L, M^L) \xrightarrow{\eta^*} H^2(G,M)$$

(see [12] , pp.118). The maps η_1^*, η^* are the inflation homomorphisms, ρ is the restriction homomorphism and τ is the transgression homomorphism. For a very concrete definition of τ see ([15], pp.215). The action of G/L on $H^1(L,M)$ is induced by the following action of G on $Z^1(L,M)$:

$$(g\cdot m)_l = {}^g m_{g^{-1}lg} \quad , \quad \text{for every 1-cocycle } \mathbf{m}=(m_l)_l \text{ in } Z^1(L,M).$$

Let us consider the above exact sequence in the case $L = [G,G]$ and $M = \mathbb{T}$ with trivial G-actions.

It is clear that $\eta_1^* : \mathrm{Hom}(G_{ab}, \mathbb{T}) \longrightarrow \mathrm{Hom}(G, \mathbb{T})$ is an isomorphism hence we get the exact sequence

$$0 \longrightarrow \mathrm{Hom}([G,G],\mathbb{T})^G \xrightarrow{\tau} H^2(G_{ab},\mathbb{T}) \xrightarrow{\eta^*} H^2(G,\mathbb{T})$$

We are interested to describe the image of the transgression homomorphism τ . This can be better done via the isomorphism exhibited in Lemma 3.2.

3.3. LEMMA Let $\varphi \in \mathrm{Hom}([G,G],\mathbb{T})^G$ and let $\hat{\varphi} \in \mathrm{Hom}(\Lambda^2 G_{ab},\mathbb{T})$ be the image of φ under the homomorphism $\mu\tau$. Then $\hat{\varphi}$ is given by the following formula

$$\hat{\varphi}\,(\dot{g}\wedge\dot{h}) = \varphi(gh\, g^{-1}h^{-1}) \quad \text{for all } g,\ h \in G$$

Proof. Using the fact that $\varphi(g^{-1}tg) = \varphi(t)$ for all $g \in G$, $t \in [G,G]$, it is easily seen that the formula

$$\varphi'(\dot{g}\wedge\dot{h}) = \varphi(ghg^{-1}h^{-1})$$

gives a well defined homomorphism $\varphi' \in \mathrm{Hom}(\Lambda^2 G_{ab},\mathbb{T})$. Choose a map $s : G_{ab} \longrightarrow G$ such that $\dot{s}(u) = u$ for each $u \in G_{ab}$ and $s(1) = 1$. The description of the transgression homomorphism given in ([15], pp.215) can be used to see that $\tau(\varphi) \in H^2(G_{ab},\mathbb{T})$ is given by the 2-cocycle $\lambda_{u,v} = \varphi(s(uv)^{-1}\, s(u)s(v))$. Consequently

$$\hat{\varphi}\,(u\wedge v) = \mu\tau(\varphi)(u\wedge v) = \lambda_{u,v} - \lambda_{v,u} = \varphi(s(uv)^{-1}s(u)s(v)s(u)^{-1}s(v)^{-1}s(vu))$$

$$= \varphi(s(u)s(v)s(u)^{-1}s(v)^{-1}) = \varphi'(u\wedge v), \text{ for all } u,v \in G_{ab}.$$

3.4 Let $U_0C(K)$ denote the connected component of identity of the group $UC(K)$. Each element in $U_0C(K)$ has the form $\exp(2\pi ia)$ for some $a \in C(K,\mathbb{R})$. Let us denote by exp the map $a \rightarrow \exp(2\pi ia)$. Since K is connected, the kernel of exp is isomorphic to $\mathbb{Z}$. If we let G act trivially on $\mathbb{Z}$ and by left translations on $C(K,\mathbb{R})$ and $U_0C(K)$ we get an exact sequence of G-modules :

$$0 \rightarrow \mathbb{Z} \rightarrow C(K,\mathbb{R}) \xrightarrow{\exp} U_0C(K) \rightarrow 0$$

Using the Haar integral we can relate this sequence to the exponential sequence as described in the following diagram of G-modules:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{R} & \longrightarrow & \mathbb{T} & \longrightarrow & 0 \\
 & & \| & & \downarrow i & & \downarrow i' & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & C(K,\mathbb{R}) & \longrightarrow & U_0C(K) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow p & & \downarrow p' & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{R} & \longrightarrow & \mathbb{T} & \longrightarrow & 0
\end{array}$$

Here i and i' are the natural inclusions, $p(a) = \int_K a(x)\,dx$ for $a \in C(K,\mathbb{R})$ and $p'(\exp(a)) = \exp(p(a))$. Note that $pi = id_{\mathbb{R}}$ and $p'i' = id_{\mathbb{T}}$.

We denote by $\hat{K} \subset UC(K)$ the (abelian) topological group of all continuous homomorphisms $K \longrightarrow \mathbb{T}$. Recall that $\hat{K}$ is discrete since K is assumed to be compact. (when K is abelian , $\hat{K}$ is exactly the Pontrjagin dual of K). It is a result of Scheffer [11] that each continuous map $K \longrightarrow \mathbb{T}$ is homotopic to a unique homomorphism in $\hat{K}$.

The arguments given in [11] prove that one has the following:

3.5. PROPOSITION . Let K be a compact connected topological group. There is a split exact sequence of topological groups

$$0 \longrightarrow U_0C(K) \xhookrightarrow{j} UC(K) \underset{s}{\overset{q}{\rightleftarrows}} \hat{K} \longrightarrow 0$$

where $q(a)(x) = p'(a(\cdot)a(x^{-1}\cdot)^*)$ for $a \in U\cdot C(K)$ and $x \in K$. □

The section s is given by the natural inclusion $\hat{K} \hookrightarrow UC(K)$. If we let G act trivially on $\hat{K}$ and by left translations on $U_0C(K)$ and $UC(K)$ then the above sequence is an exact sequence of G-modules. Note however that the section s is not G-linear.

The exact sequence in 3.5 induces an exact sequence of groups

$$0 \longrightarrow Z^1(G,U_0C(K)) \xrightarrow{j_*} Z^1(G,UC(K)) \xrightarrow{q_*} Z^1(G,\hat{K}) \xrightarrow{\delta} H^2(G,U_0C(K))$$

If M is a topological group together with a G-action $G \to \mathrm{Aut}(M)$ of the discrete group G we let $Z^1(G,M)$ have the product topology induces from $\prod_{g \in G} M$.

Let $\Gamma_{K,G} :=$ image $q_* = \ker \delta \subset Z^1(G,\hat{K}) = \mathrm{Hom}(G,\hat{K}) = \mathrm{Hom}(G_{ab},\hat{K})$

3.6. LEMMA There is an exact sequence of topological groups

$$0 \to Z^1(G,U_0C(K)) \xrightarrow{j_*} Z^1(G,UC(K)) \xrightarrow{q_*} \Gamma_{K,G} \to 0$$

with $\Gamma_{K,G}$ totally disconnected.

Proof : The continuity of q_* follows from Proposition 3.5. $\Gamma_{K,G}$ is totally disconnected since $\hat{K}$ is discrete. □

A more precise description of $\Gamma_{K,G}$ is provided by the following:

3.7 PROPOSITION Let $\gamma : g \to \gamma_g$ be a homomorphism from G to $\hat{K}$. Then $\gamma \in \Gamma_{K,G}$ if and only if there is $\varphi \in \mathrm{Hom}([G,G],\mathbb{T})$ satisfying $\varphi(g^{-1}tg) = \varphi(t)$ for all $g \in G$, $t \in [G,G]$, such that :

$$\gamma_g(h) - \gamma_h(g) = \varphi(h^{-1}g\,h\,g^{-1}) \text{ for all } g,\ h \in G.$$

In particular, if G is abelian then :

$$\Gamma_{K,G} = \{\gamma \in \mathrm{Hom}(G,\hat{K}) : \gamma_g(h) = \gamma_h(g)\}$$

Proof. For $\gamma \in Z^1(G,\hat{K}) = \mathrm{Hom}(G,\hat{K})$ let us compute $\delta\gamma \in H^2(G,U_0C(K))$. If we regard γ as a map $G \to UC(K)$ (via the section s in 3.5) then $q(\gamma_g) = \gamma_g$ and so

$$(\delta\gamma)_{g,h} = \gamma_g(\cdot) + \gamma_h(g^{-1}\cdot) - \gamma_{gh}(\cdot) = \gamma_h(g^{-1}).$$

This computation also shows that the 2-cocycle $(g,h) \to \gamma_h(g^{-1})$ takes values in $\mathbb{T} \subset U_0C(K)$. Consequently the connecting homomorphism δ factors through the natural map $H^2(G,\mathbb{T}) \to H^2(G,U_0C(K))$.

Moreover since $\widehat{K}$ is abelian we have $\mathrm{Hom}(G,\widehat{K}) = \mathrm{Hom}(G_{ab},\widehat{K})$ and so δ can be factorized as in the following (commutative) diagram:

$$\begin{array}{ccc} \mathrm{Hom}(G,\widehat{K}) & \xrightarrow{\ \delta\ } & H^2(G,U_0C(K)) \\ \Big\downarrow \delta' & & \Big\uparrow i'_* \\ H^2(G_{ab},\mathbb{T}) & \xrightarrow{\ \varphi^*\ } & H^2(G,\mathbb{T}) \end{array}$$

Here i'_* is induced by the natural embeding $i' : \mathbb{T} \to U_0C(K)$, φ^* is induced by the quotient map $\varphi : G \to G_{ab}$ and $(\delta'\gamma)_{\dot{g},\dot{h}} = \gamma_h(g^{-1})$. As a consequence of 3.4 $p'_* i'_* = \mathrm{id}_{H^2(G,\mathbb{T})}$ hence i'_* is injective. In this way we find that $\ker\delta$ consists of those elements γ in $\mathrm{Hom}(G,\widehat{K})$ for which $\delta'\gamma \in \ker\varphi^*$ or equivalently $\mu\delta'\gamma \in \mathrm{image}(\mu\tau)$ (see Lemma 3.2 and the discussion before Lemma 3.3)

Finally, using Lemma 3.3, we deduce that $\gamma \in \Gamma_{K,G} = \ker\delta$ iff $\gamma_h(g^{-1}) - \gamma_{g^{-1}}(h) = \psi(gh\,g^{-1}h^{-1})$ for some $\psi \in \mathrm{Hom}([G,G],\mathbb{T})^G$. □

Having Lemma 3.6 and Proposition 3.7 we shall concentrate ourself on $Z^1(G,U_0C(K))$.

3.8 PROPOSITION. There is an exact sequence of groups

$$0 \to \mathrm{Hom}(G,\mathbb{Z}) \to Z^1(G,C(K,\mathbb{R})) \xrightarrow{\exp_*} Z^1(G,U_0C(K)) \xrightarrow{\delta_0} \mathrm{Ext}^1_{\mathbb{Z}}(G_{ab},\mathbb{Z}) \to 0$$

Proof: The commutative diagram from 3.4 induces the following commutative diagram :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z^1(G,\mathbb{Z}) & \longrightarrow & Z^1(G,\mathbb{R}) & \longrightarrow & Z^1(G,\mathbb{T}) & \xrightarrow{\ \delta_1\ } & H^2(G,\mathbb{Z}) \\ & & \| & & \Big\downarrow i_* & & \Big\downarrow i'_* & & \| \\ 0 & \longrightarrow & Z^1(G,\mathbb{Z}) & \longrightarrow & Z^1(G,C(K,\mathbb{R})) & \longrightarrow & Z^1(G,U_0C(K)) & \xrightarrow{\ \delta_0\ } & H^2(G,\mathbb{Z}) \\ & & \| & & \Big\downarrow p_* & & \Big\downarrow p'_* & & \| \\ 0 & \longrightarrow & Z^1(G,\mathbb{Z}) & \longrightarrow & Z^1(G,\mathbb{R}) & \longrightarrow & Z^1(G,\mathbb{T}) & \xrightarrow{\ \delta_1\ } & H^2(G,\mathbb{Z}) \end{array}$$

Since $p'_* i'_* = \mathrm{id}$ it follows that image δ_0 = image δ_1 .

By Lemma 3.1 image $\delta_1 \simeq \mathrm{Ext}^1_{\mathbb{Z}}(G_{ab}, \mathbb{Z})$.

$\square$

3.9 LEMMA.

i) δ_0 is constant on the path components of $Z^1(G,U_0C(K))$.

ii) $\mathrm{Hom}(G,\mathbb{Z}) \longrightarrow Z^1(G,C(K,\mathbb{R})) \xrightarrow{\exp_*} Z^1(G,U_0C(K))$ is a (Hurewicz) fibration.

Proof : i) Let $[0,1] \ni t \longrightarrow a(t) = (a_g(t))_g \in Z^1(G,U_0C(K))$ be a continuous path. Since the sequence

$\mathbb{Z} \longrightarrow C(K,\mathbb{R}) \longrightarrow U_0C(K)$ is a covering space, for each $g \in G$ we can lift the path $t \longrightarrow a_g(t) \in U_0C(K)$ to a continuous path $t \longrightarrow f_g(t) \in C(K,\mathbb{R})$ such that $\exp f_g(t) = a_g(t)$. For each t we have

$$(\delta_0 a(t))_{g,h} = f_g(t) + {}^g f_h(t) - f_{gh}(t) \in \mathbb{Z}.$$

Since the above expression depends continuously on t, we must have $\delta_0 a(0) = \delta_0 a(1)$.

ii) Let X be a topological space and let $F: X \times [0,1] \longrightarrow Z^1(G,U_0C(K))$, $f: X \times \{0\} \to Z^1(G,C(K,\mathbb{R}))$ be continuous maps satisfying $\exp_* f(x,0) = F(x,0)$ for all x in X. Our aim is to produce a continuous map $H: X \times [0,1] \longrightarrow Z^1(G,C(K,\mathbb{R}))$ which extends f and lifts F. Now since the sequence $\mathbb{Z} \longrightarrow C(K,\mathbb{R}) \xrightarrow{\exp} U_0C(K)$ is a covering space, for each $g \in G$ there is a continuous map $H'_g : X \times [0,1] \longrightarrow C(K,\mathbb{R})$ which extends f_g and lifts F_g. Let us check that for each x, t, the map $g \longrightarrow H'_g(x,t)$ belongs to $Z^1(G,C(K,\mathbb{R}))$. Define $\alpha_{g,h}(x,t) = H'_g(x,t) + {}^g H'_h(x,t) - H'_{gh}(x,t)$. It is clear that $\alpha_{g,h}(x,t) \in \mathbb{Z}$ since $\exp H'_g(x,t) = F_g(x,t)$. Also, $\alpha_{g,h}(x,0) = 0$ since $H'_g(x,0) = f_g(x,0)$. As $\alpha_{g,h}(x,t)$ depends continuously on t we must have $\alpha_{g,h}(x,t) = 0$ for all x,t. The map $(x,t) \longrightarrow (H'_g(x,t))_{g \in G}$ is a solution of the given lifting problem with initial data.

3.10 PROPOSITION

$$\pi_n(Z^1(G,U_0C(K))) = \begin{cases} \mathrm{Ext}^1_{\mathbb{Z}}(G_{ab},\mathbb{Z}) & \text{for } n = 0 \\ \mathrm{Hom}(G_{ab},\mathbb{Z}) & \text{for } n = 1 \\ 0 & \text{for } n \geq 2 \end{cases}$$

Proof : Let Z^1_0 be the path component of 1 in $Z^1(G,U_0C(K))$. Since the group $Z^1(G,C(K,\mathbb{R}))$ is contractible at the zero cocycle (use the obvious homotopy $H((a_g)_g,t) = (ta_g)_g$) it follows that in the sequence 3.8. $\ker \delta_0 = \mathrm{image}(\exp_*) \subset Z^1_0$. On the other hand, by Lemma 3.9(i) , $Z^1_0 \subset \ker \delta_0$. Thus $Z^1_0 = \mathrm{image}(\exp_*)$ and $\pi_0(Z^1(G,U_0C(K))) = Z^1(G,U_0C(K)) \,/\, Z^1_0 \simeq \mathrm{Ext}^1_{\mathbb{Z}}(G_{ab},\mathbb{Z})$. As a consequence of Lemma 3.9 (ii) the sequence

$0 \longrightarrow \mathrm{Hom}(G,\mathbb{Z}) \longrightarrow Z^1(G,C(K,\mathbb{R})) \xrightarrow{\exp_*} Z^1_0 \longrightarrow 0$ defines a fibration and we have seen that its total space is contractible and the fiber is totally disconnected. The homotopy sequence of the fibration gives us $\pi_n(Z^1_0) = \begin{cases} \mathrm{Hom}(G,\mathbb{Z}) & \text{for } n = 1 \\ 0 & \text{for } n \geq 2 \end{cases}$. □

The results of this section can be collected in the following:

3.11. THEOREM. Let K be a compact connected topological group and let G be a dense subgroup of K acting on K by left translations. Then :

$$\pi_n(\mathrm{Aut}_{C(K)}(C(K) \rtimes_{\alpha,r} G)) = \begin{cases} \pi_0(Z^1(G,UC(K)) \rtimes (\pi_0(K) \rtimes \pi_0(\mathrm{Aut}_G(K))) & \text{for } n=0. \\ \mathrm{Hom}(G,\mathbb{Z}) \times \pi_1(K) \times \pi_1(\mathrm{Aut}_G(K)) & \text{for } n=1. \\ \pi_n(K) \times \pi_n(\mathrm{Aut}_G(K)) & \text{for } n \geq 2. \end{cases}$$

$\pi_0(Z^1(G,UC(K)))$ fits into the following exact sequence :

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(G_{ab},\mathbb{Z}) \xrightarrow{i_*} \pi_0(Z^1(G,UC(K))) \xrightarrow{q_*} \Gamma_{K,G} \longrightarrow 0$$

where $\Gamma_{K,G}$ is described by Proposition 3.7.

Proof : The theorem follows from Lemma 3.6. and Proposition 3.10.

3.12 REMARK. If G is abelian or countable, then $Aut_G(K)$ is totally path-wise-disconnected, hence $\pi_n(Aut_G(K)) = \begin{cases} Aut_G(K) & \text{for } n=0. \\ 0 & \text{for } n \geq 1. \end{cases}$

There are explicitely formulae for the maps i_* and q_* from above : if $[\chi] \in Ext^1_{\mathbb{Z}}(G_{ab}, \mathbb{Z})$ is represented by $\chi \in Hom(G, \mathbb{T})$ then $i_*[\chi]$ is the component of the 1-cocycle $g \longrightarrow \chi(g)$; if $a = (a_g)_g \in Z^1(G, UC(K))$ then $q_*[a]$ is the homomorphism $\gamma \in Hom(G, \hat{K})$ given by $\gamma_g(x) = p'(a_g(\cdot)a_g(x^{-1}\cdot)^*)$ (see 3.5).

3.13 COROLLARY. Under the hypothesis of the above theorem we have:

a) if G_{ab} is free, then :

$$\pi_0(Aut_{C(K)}(C(K) \rtimes_{\alpha,r} G)) = \Gamma_{K,G} \rtimes (\pi_0(K) \rtimes \pi_0(Aut_G(K))).$$

b) if G_{ab} is a torsion group, then :

$$\pi_0(Aut_{C(K)}(C(K) \rtimes_{\alpha,r} G)) = Hom(G, \mathbb{T}) \rtimes (\pi_0(K) \rtimes \pi_0(Aut_G(K)))$$

3.14 REMARK. For the case of the Bunce-Deddens algebras, i. e. $K = \mathbb{T}$ and G is an infinite torsion subgroup of $\mathbb{T}$, the above corollary gives:

$$\pi_n(Aut_{C(\mathbb{T})}(C(\mathbb{T}) \rtimes_{\alpha} G)) = \begin{cases} Hom(G, \mathbb{T}) \rtimes \mathbb{Z}_2 & \text{for } n = 0 \\ \mathbb{Z} & \text{for } n = 1 \\ 0 & \text{for } n \geq 2. \end{cases}$$

where $\mathbb{Z}_2$ acts on $Hom(G, \mathbb{T})$ by conjugation.

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