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# DERIVATIONS ON ALGEBRAIC GROUPS, III

by

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# DERIVATIONS ON ALGEBRAIC GROUPS, III
## COMPLEMENTS

by A. BUIUM

Contents

0. Introduction

1. Descent of linear groups

2. More applications of the analytic method

3. Remarks on the case of characteristic $p>0$

4. Embeddings of f-$\Delta$-groups into algebraic groups.

## 0. Introduction

This paper is a direct continuation of $[B_1][B_2]$ from where we borrow our terminology and conventions. Our aim here is to settle some questions raised in $[B_1][B_2]$ and improve (or give alternative proofs) of some results from there.

Notably, we prove that for any irreducible linear algebraic F-group G, $F^{\Delta(G)}$ is a field of definition for G (cf. section 1) hence that any linear f-$\Delta$-group is semisplit! This result together with an improvement (cf. section 2) of a result from $[B_1]$ on abelian ideals in $\Delta(G)$ complementary to $\Delta(G,fin)$ will lead to a quite satisfactory picture of all linear f-$\Delta$-groups. The idea in sections 1-2 is to study the interplay between algebraic and analytic groups and to use analytic results of Hamm [Ha] and Hochschild-Mostow $[HM_1][HM_2]$.

In section 3 we make some remarks on the characteristic $p>0$ case. Here we always have $\Delta(G)=\Delta(G/F)$ so the moduli-theoretic problems dissapear. But new phenomena occur. First $\Delta(G/F)$ is usually infinite dimensional. Next $\Delta(G,fin)$ need not be closed under addition and many contain derivations $\delta$ for which $\log \delta \neq 0$! On the other hand one can prove that for G irreducible, solvable with commutative unipotent radical, the kernel of $\log : \Delta(G) \to W(G)$ must be contained in $\Delta(G, fin)$.

In section 4 we associate (along the lines of [J], [NW]) to any algebraic $\mathcal{U}$-group G a $\Delta$-group $\hat{G}$ (called the "prolongation of G" or the $\Delta$-group "produced" from G) such that the group $\hat{G}(\mathcal{U})$ of $\Delta$-$\mathcal{U}$- points of $\hat{G}$ identifies with the group $G(\mathcal{U})$ of $\mathcal{U}$- points of G. We observe that any f-$\Delta$- group of typical dimension $d$ can be embedded into (the prolongation of) an algebraic group of dimension $d$ as the kernel of a certain crossed $\Delta$-homomorphism of the algebraic group in a power of its Lie algebra. We also study some remarkable embeddings of f-$\Delta$-groups of typical dimension d into abelian varieties of dimension $< d$.

It should be said that $\Delta$-groups embedded into (prolongations of) algebraic groups are intimately related to derivations on proalgebraic (rather than algebraic) groups. Part of our results and methods in [$B_1$][$B_2$] extend to the proalgebraic case and hence to groups of type $> 0$. We shall come back to this question in a subsequent paper. We aknowledge our debt to Professor H. Hamm for explaining to us his results on "local systems" associated to "relative" Lie groups (cf. [Ha] ). In particular Theorem (1.3) (which is essential for our method here) and its elegant proof are due to him.

## 1. Descent of linear groups

The aim of this section is to prove the following:

(1.1) THEOREM. Let G be an irreducible linear algebraic F-group. Then $F^{\Delta(G)}$ is a field of definition for G (hence it coincides with $F_G$!).

(1.2) Remark. The above statement fails for non-linear G, cf. [$B_2$]. Our proof of (1.1) will be analogue to that of Theorem (1.1) in Chapter 2 cf [$B_o$] in the sense that we are going to use "birational quotients", a "Kodaira-Spencer map" and an analytic ingredient. In [$B_o$] the analytic ingredient was the versal deformation of a compact complex space. Here the analytic ingredient is a combination of Theorems (1.3) and (1.4) below. The first theorem is due to Hamm. To state it let's fix some notations. Assume $\pi : \mathcal{G} \to \mathcal{X}$ is an analytic family of complex connected Lie groups (i.e. a map of analytic $\mathbb{C}$- manifolds, having connected fibres, such that one is given analytic $\mathcal{X}$-maps $\mu : \mathcal{G} \times_{\mathcal{X}} \mathcal{G} \to \mathcal{G}$, $S : \mathcal{G} \to \mathcal{G}$ and a section $\varepsilon : \mathcal{X} \to \mathcal{G}$ of $\pi$ satisfying the usual axioms of comultiplication, antipode and co-unit). Assume moreover that: a) $\mathcal{X}$ is simply connected and $v_1, \dots, v_n$ are commuting vector fields on $\mathcal{X}$ giving at each point a basis of the tangent space, b) $v_1, \dots, v_n$ can be lifted to commuting vector fields $w_1, \dots, w_n$ on $\mathcal{G}$ such that $\mu, S, \varepsilon$ agree with $w_1, \dots, w_n$ in the sense that for each $w = w_i$ we have:

1) $(T_{(g_1,g_2)}\mu)(w(g_1), w(g_2)) = w(\mu(g_1,g_2))$ for any $(g_1,g_2) \in \mathcal{G} \times_{\mathcal{X}} \mathcal{G}$

2) $(T_g S)(w(g)) = w(Sg)$ for any $g \in \mathcal{G}$

3) $(T_x \varepsilon)(v(x)) = w(\varepsilon(x))$ for any $x \in \mathcal{X}$

Then we have

(1.3) THEOREM (Hamm [Ha]). Under the assumptions above there exists an analytic $\mathcal{X}$- isomorphism $\varphi : \mathcal{G}_o \times \mathcal{X} \to \mathcal{G}$ (where $\mathcal{G}_o$ is some fibre of $\pi$) which above each point of $\mathcal{X}$ is a group homomorphism and such that upon letting $v_i^*$ be the "trivial lifting" of $v_i$ from $\mathcal{X}$ to $\mathcal{G}_o \times \mathcal{X}$ we have $(T\varphi)(v_i^*) = w_i$ for all $i$.

The second theorem needed is:

(1.4) THEOREM (Hochschild - Mostow [$HM_1$]). Let $G_1, G_2$ be two connected linear algebraic $\mathbb{C}$-groups. If $G_1^{an}$ and $G_2^{an}$ are isomorphic (as analytic Lie groups) then $G_1$ and $G_2$ are isomorphic (as algebraic groups).

(1.5) Remark. The above statement fails for non-linear groups (cf. [$HM_1$] [Se]).

Theorem (1.4) is a consequence of the theory developed in [$HM_1$] and no argument will be indicated here. We shall however include the proof (due to Hamm) of (1.3) since it is quite elementary and fairly elegant:

(1.6) Proof of (1.3) (Hamm). By Frobenius, for any $x_o \in \mathcal{X}$ and any $g_o \in \pi^{-1}(x_o) = \mathcal{G}_o$ there exists a neighbourhood $\mathcal{V}_{g_o}$ of $x_o$ in $\mathcal{X}$, a neighbourhood $\mathcal{W}_{g_o}$ of $g_o$ in $\mathcal{G}_o$ and an analytic map $\psi : \mathcal{W}_{g_o} \times \mathcal{V}_{g_o} \to \mathcal{G}$ over $\mathcal{X}$ such that $T\psi$ takes $v_i^*$ (=trivial lifting of $v_i$ from $\mathcal{X}$ to $\mathcal{W}_{g_o} \times \mathcal{V}_{g_o}$) into $w_i$. A triple $(\mathcal{V}_{g_o}, \mathcal{W}_{g_o}, \psi)$ will be called a "local solution" at $g_o$. It is sufficient to show that for a given $x_o$ the various $\mathcal{V}_{g_o}$ appearing in the local solutions (with $g_o \in \mathcal{G}_o$) can be chosen to contain a fixed open neighbourhood of $x_o$. Let $e_o = \varepsilon(x_o)$ and consider the set $\Sigma$ of all $g \in \mathcal{G}_o$ such that there exists a local solution $(\mathcal{V}_g, \mathcal{W}_g, \psi)$ at $g$ with $\mathcal{V}_{e_o} \subset \mathcal{V}_g$. One easily checks that $\Sigma$ is an open subgroup of $\mathcal{G}_o$ (local solutions can be "multiplied" and "inverted" using $\mu$ and S) hence $\Sigma = \mathcal{G}_o$ since $\mathcal{G}_o$ is connected, which proves the theorem.

Next we need some facts about isomorphisms of Lie algebras. First "recall" the following trivial representability result:

(1.7) LEMMA. Let R be a (commutative) ring and L, L' two Lie R-algebras which are free and finitely generated as R-modules. Then the functor $\underline{\mathrm{Iso}}_{L,L'}$ from {commutative R-algebras} to {sets} defined by $\underline{\mathrm{Iso}}_{L,L'}(\tilde{R})$ = {set of $\tilde{R}$-Lie algebra isomorphisms from $L \otimes_R \tilde{R}$ to $L' \otimes_R \tilde{R}$} is representable by a finitely generated R-algebra (which we call $\mathrm{Iso}_{L,L'}$).

Exactly as in [$B_o$] pp.35-36 the above Lemma implies the following:

(1.8) LEMMA. Let $K$ be an algebraically closed field, $S$ an affine $K$-variety and $L$ a Lie $\mathcal{O}(S)$-algebra which is free and finitely generated as an $\mathcal{O}(S)$-module. Then there is a constructible subset $Z\subset S\times S$ such that for any $s_1,s_2\in S(K)$ we have $(s_1,s_2)\in Z(K)$ if and only if the Lie $K$-algebras $L\otimes_S K(s_1)$ and $L\otimes_S K(s_2)$ are isomorphic.

Next we have:

(1.9) LEMMA. Assume $K,S,L$ are as in (1.8), let $F$ be an algebraically closed extension of $Q(S)$ and assume $F_L$ (=smallest algebraically closed field of definition of $L\otimes_S F$ between $F$ and $K$, which exists by [$B_o$] p.86) equals the algebraic closure of $Q(S)$ in $F$. Then there exists an open subset $S_o\subset S$ such that for any $s_o\in S_o(K)$ the set

$$\{s\in S(K);\ L\otimes K(s)\simeq L\otimes K(s_o)\}$$

is finite.

Proof. By (1.8) and [$B_o$], (1.13) p.36 there exists an affine open set $S_1\subset S$ and a dominant morphism of affine $K$-varieties $\psi: S_1 \longrightarrow M$ such that for any $s_1\in S_1(K)$ we have $\psi^{-1}\psi(s_1)=\{s\in S_1(K);\ L\otimes K(s)\simeq L\otimes K(s_1)\}$. If $\dim M=\dim S_1$ we are done. Assume $\dim M<\dim S_1$. Then we use an argument similar to [V] p.576. Choose a closed subvariety $N\subset S_1$ with $\dim N=\dim M$, let $L_N$ be the pull-back of $L$ on $N$, let $L'$ be the pull-back of $L_N$ on the affine scheme $\tilde{S}_1=S_1\times_M N$ and let $L''$ be the pull-back of $L$ on $\tilde{S}_1$. Then for any $K$-point $x$ of $\tilde{S}_1$ one checks that $L'\otimes K(x)\simeq L''\otimes K(x)$. By representability of $\underline{\mathrm{Iso}}_{L',L''}$ (1.7) there is a generically finite dominant morphism of finite type of affine schemes $Y\to\tilde{S}_1$ with $Y$ integral such that the pull-backs of $L'$ and $L''$ on $Y$ are $Y$-isomorphic. Since $Y\to S_1$ is generically finite one can embed $Q(Y)$ over $Q(S_1)=Q(S)$ in $F$ and we get that $Q(N)$ is a field of definition for $L\otimes F$ between $K$ and $F$ contradicting our hypothesis. The lemma is proved.

Next we need a Kodaira-Spencer map for linear irreducible algebraic K-groups G (K any field containing our ground field k). Define $\Delta^2(G/K)$ to be the cohomology of the complex

$$\mathrm{Der}_K(A,A) \xrightarrow{\partial_1} \mathrm{Der}_K(A,A\otimes_K A) \xrightarrow{\partial_2} \mathrm{Der}_K(A,A\otimes_K A\otimes_K A)$$

where

$$\partial_1(d)=\mu d-(d\otimes 1+1\otimes d)\mu, \quad d\in \mathrm{Der}_K(A,A)$$
$$\partial_2(D)=(D\otimes 1)\mu-(1\otimes D)\mu+(\mu\otimes 1)D-(1\otimes\mu)D, \quad D\in \mathrm{Der}_K(A,A\otimes_K A)$$

(One can identify $\Delta^2(G/K)$ with the second Hochschild cohomology group of the adjoint representation of G, cf [DG] p.192, but we won't need this fact!). Clearly $\Delta^2(G\otimes_K K'/K')\simeq\Delta^2(G/K)\otimes_K K'$ for any field extension $K'/K$.

(1.10) LEMMA. There is an exact sequence

$$0\to\Delta(G/K)\to\Delta(G)\to \mathrm{Der}_k K \xrightarrow{\rho} \Delta^2(G/K)$$

where $\rho$ is compatible with field extensions $K'/K$.

Proof. Let's define $\rho$. Since G/K is smooth and affine, any derivation $\delta\in \mathrm{Der}_k K$ can be lifted to a k-derivation $\tilde{\delta}$ of $A=\mathcal{O}(G)$. Then one checks immediately that

$$\mu\tilde{\delta}-(\tilde{\delta}\otimes 1+1\otimes\tilde{\delta})\mu\in \mathrm{Ker}(\partial_2)\subset \mathrm{Der}_K(A,A\otimes_K A)$$

and the class of this derivation in $\Delta^2(G/K)$ does not depend on the choice of the lifting $\tilde{\delta}$; we call this class $\rho(\delta)$. We must check that

$$\mathrm{Im}(\Delta(G) \longrightarrow \mathrm{Der}_k K) = \mathrm{Ker}\,\rho$$

The inclusion "$\subset$" is clear. Conversely if $\rho(\delta)=0$ then there is a lifting $\tilde{\delta}$ of $\delta$, $\tilde{\delta} \in \mathrm{Der}_k(A,A)$ such that $\mu\tilde{\delta} = (\tilde{\delta}\otimes 1 + 1\otimes\tilde{\delta})\mu$. Then one immediately checks that

$$\mu(s\tilde{\delta}s) = [(s\tilde{\delta}s)\otimes 1 + 1\otimes(s\tilde{\delta}s)]\mu$$

Putting $\hat{\delta} = \frac{1}{2}(\tilde{\delta} + s\tilde{\delta}s)$ we see that $\hat{\delta}$ lifts $\delta$ and belongs to $\Delta(G)$ so our lemma is proved.

(1.11) Remark. If we consider the complex

$$0 \to \mathrm{Der}_K(A,A) \xrightarrow{\partial_1} \mathrm{Der}_K(A, A\otimes_K A)$$

then its cohomology (call it $\Delta^1(G/K)$) is invariant under the involution $\theta \mapsto s\theta s$ of $\mathrm{Der}_K(A,A)$ and the fixed part $\Delta^1(G/K)^S$ indentifies with our $\Delta(G/K)$. This expression for $\Delta(G/K)$ already shows that for any field extension $K'/K$ we have $\Delta(G\otimes_K K'/K') \simeq \Delta(G/K)\otimes_K K'$.

(1.12) THEOREM. Assume $G$ is an irreducible linear algebraic $F$-group and $K$ is a subfiled of $F$. Let $F_o$ be the smallest algebraically closed field of definition of G between $K$ and $F$ and $G \simeq G_o\otimes_{F_o} F$ with $G_o$ some $F_o$-group. Then the map

$$\rho_o : \mathrm{Der}_K F_o \to \Delta^2(G_o/F_o)$$

is injective.

We shall give first the proof of (1.12) in the case $K = \mathbb{C}$.

Recall from [$B_3$] that $F_G = F_{\mathcal{L}(U)}$ (=smallest algebraically closed field of definition for $\mathcal{L}(U)$ between K and F where U is the unipotent radical of G). There exist group schemes $\tilde{G} \to S$ and $\tilde{U} \to S$ (S an affine $\mathbb{C}$-variety, $\tilde{U}$ a closed subscheme of $\tilde{G}$) such that $F_o$ is the algebraic closure of $F_1 = Q(S)$, $\tilde{G} \otimes_S F_o = G_o$, $\tilde{U} \otimes_S F_o = U_o$ (=unipotent radical of $G_o$) and the fibres of $\tilde{U}/S$ are the unipotent radicals of the fibres of $\tilde{G}/S$. We may assume the relative Lie algebra $\mathcal{L}(\tilde{U}/S)$ is a free $\mathcal{O}(S)$-module. Assume $\int_o$ is not injective. Then $\int_1 : \mathrm{Der}_{\mathbb{C}} F_1 \to \Delta^2(G_1/F_1)$ is not injective (where $G_1 = \tilde{G} \otimes_S F_1$). By (1.10) there exists a derivation $\delta \neq 0$ in $\mathrm{Der}_{\mathbb{C}} F_1$ which lifts to a derivation $\tilde{\delta} \in \Delta(G_1)$. Now both $\delta$ and $\tilde{\delta}$ can be viewed as rational vector fields on the $\mathbb{C}$- varieties S and $\tilde{G}$ respectively. We may replace S by a Zariski open set such that $\delta$ and $\tilde{\delta}$ become regular everywhere. Now by (1.9) there exists a Zariski open set $S_o \subset S$ such that for any $s_o \in S_o(\mathbb{C})$ the set

$$\Sigma_{s_o} = \{ s \in S_o(\mathbb{C}) ; \mathcal{L}(U_s) \simeq \mathcal{L}(U_{s_o}) \}$$

is finite, where $U_s = \tilde{U} \otimes_S \mathbb{C}(s)$. Let $\mathfrak{X}$ be an analytic disk in $S_o^{an}$ which is an integral subvariety for $\delta$ and let $\mathcal{G} = \tilde{G}^{an} \times_{S^{an}} \mathfrak{X}$. Then $\mathcal{G}$ is an integral subvariety for $\tilde{\delta}$. By (1.3) all fibres of $\mathcal{G} \to \mathfrak{X}$ are pairwise isomorphic as complex Lie groups. By (1.4) the fibres of $\tilde{G} \to S$ above the points of $\mathfrak{X}$ are pairwise isomorphic as $\mathbb{C}$-algebraic groups; in particular all Lie algebras $\mathcal{L}(U_s)$ with $s \in \mathfrak{X}$ are pairwise isomorphic which contradicts the finitness of $\Sigma_{s_o}$ for $s_o \in S_o(\mathbb{C})$. The theorem is proved for K= $\mathbb{C}$ .

(1.13) Proof of Theorem (1.1). It is sufficient to prove that for any $\Delta \subset \Delta(G)$, $F^{\Delta}$ is a field of definition for G.

Case 1. $F^{\Delta}$ uncountable. Then we can assume $\mathbb{C} \subset F^{\Delta}$. Let $F_o$ be the smallest algebraically closed field of definition for G between $\mathbb{C}$ and $F$, $G=G_o\otimes_{F_o}F$. As in $[B_o]$ p.41 we may conclude by inspecting the diagram with exact rows and colomns (cf. (1.12), case $K=\mathbb{C}$)

$$\begin{tikzcd}
 & 0 \arrow[d] & \\
 & \mathrm{Der}_{F_o}F \arrow[d, "\psi"'] & 0 \arrow[d] \\
\Delta(G) \arrow[r, "\varphi"] & \mathrm{Der}_{\mathbb{C}}F \arrow[r, "\rho"] \arrow[d] & \Delta^2(G/F) \arrow[d] \\
0 \arrow[r] & (\mathrm{Der}_{\mathbb{C}}F_o)\otimes_{F_o}F \arrow[r, "\rho_o\otimes F"'] \arrow[d] & \Delta^2(G_o/F_o)\otimes_{F_o}F \arrow[d] \\
 & 0 & 0
\end{tikzcd}$$

that $\mathrm{Im}\,\varphi=\mathrm{Im}\,\psi$ hence that $F_o\subset F^{\Delta}$ i.e. that $G$ is defined over $F^{\Delta}$.

Case 2. $F^{\Delta}$ is countable. Then take an embedding $F^{\Delta}\subset\mathbb{C}$ and conclude exactly as in $[B_o]$ (instead of $[B_o]$ p.42, Lemma (1.19) use the fact which we already know from $[B_3]$ that the set of algebraically closed fields of definition for a linear algebraic group has a minimum element).

(1.14) Remark. Using Theorem (1.1) one can immediately prove Theorem (1.12) for arbitrary K!

The main consequence of (1.1) is:

(1.15) COROLLARY. Let $G$ be an irreducible linear algebraic $F$-group. Then we have a semidirect Lie space decomposition

$$\Delta(G)\simeq\Delta(G/F)\oplus\mathrm{Der}(F/F_G).$$

## 2. More applications of the analytic method

First a variation on Hamm's result (1.3):

(2.1) LEMMA. Let $\mathcal{G}$ be a connected Lie group and v an analytic vector field on $\mathcal{G}$ such that the multiplication $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$ and the inverse $\mathcal{G} \to \mathcal{G}$ are equivariant (with respect to the vector field v on $\mathcal{G}$ and to the vector field (v,v) on $\mathcal{G} \times \mathcal{G}$). Then there is a 1-parameter group of analytic group automorphisms $\mathbb{C} \times \mathcal{G} \to \mathcal{G}$ whose associated vector field is v.

Proof. Use Hamm's open subgroup argument (1.3) once again to show that there is a disc $0 \in B \subset \mathbb{C}$ such that for all $g \in \mathcal{G}$ there exists an analytic map $\psi_g : B \to \mathcal{G}$, $\psi_g(0) = g$ whose tangent map $T\psi : TB \to T\mathcal{G}$ takes $\frac{d}{dz}$ (z a coordinate in $\mathbb{C}$) into v. This immediately implies the lemma. We get the following improvement of [$B_1$] (2.10):

(2.2) COROLLARY. Let G be an irreducible linear algebraic $\mathbb{C}$-group. Then $\Delta(G/\mathbb{C}) \subset \mathcal{L}(\mathrm{Aut}\ G^{an})$.

The following result was proved in [$B_1$] by algebraic arguments; we provide here an analytic proof.

(2.3) PROPOSITION. Let G be an irreducible linear algebraic F-group. Then a derivation in $\Delta(G)$ belongs to $\Delta(G,\mathrm{fin})$ if and only if it preserves the unipotent radical U of G (hence if and only if it preserves $\mathcal{L}(U)$).

The basic ingredient is the following

(2.4) THEOREM. (Hochschild-Mostow [$HM_2$]). If G is a connected linear algebraic $\mathbb{C}$-group then an analytic group automorphism $\varphi \in \mathrm{Aut}\ G^{an}$ belongs to Aut G if and only if it preserves the unipotent radical of G.

Now the proof of (2.3) proceeds as follows. The "only if" part is the "easy part" cf [$B_1$] so we shall deal here only with the "if part". First assume $F=\mathbb{C}$. Let $\delta\in\Delta(G)$ preserve U, hence also $\mathcal{L}(U)$. By (1.15) $\delta=\delta^* + \theta$ where $\delta^*$ is the trivial lifting of $\delta/F$ from F to $G=G_o\otimes_{F_o} F$ ($F_o=F_G$, $G_o$ an $F_o$-group) and $\theta\in\Delta(G/F)$. Since $\delta^*$ clearly preserves $\mathcal{L}(U)$, so does $\theta$. Now by (2.1) there is a 1-parameter group of automorphisms $\mathbb{C}\to \mathrm{Aut}(G^{an})$, $t\longmapsto \varphi_t$ whose associated vector field is $\theta$. Then for each $t\in\mathbb{C}$, $\varphi_t$ preserves U. By (2.4) $\varphi_t\in \mathrm{Aut}\ G$ for each t, in particular $\theta\in\mathcal{L}(\mathrm{Aut}\ G)$ (cf. [$B_1$](1.2)). But $\mathcal{O}(G)$ is locally finite both as $\mathrm{Der}(F/F_G)$-module and as $\mathcal{L}(\mathrm{Aut}\ G)$-module. Since $\mathcal{L}(\mathrm{Aut}\ G)=\mathcal{L}(\mathrm{Aut}\ G_o)\otimes_{F_o} F$ we have $[\mathrm{Der}(F/F_G),\mathcal{L}(\mathrm{Aut}\ G)]\subset \subset\mathcal{L}(\mathrm{Aut}\ G)$, consequently $\mathcal{O}(G)$ is locally finite as a $\mathrm{Der}(F/F_G)\oplus \oplus\mathcal{L}(\mathrm{Aut}\ G)$-module hence as a $\delta$-F-vector space and our proposition is proved for $F=\mathbb{C}$. The general case easily reduces to the case $F=\mathbb{C}$.

(2.5) Remark. Our purely algebraic proof of (2.3) in [$B_1$] has an interest in itself because it gives a hint of how the linear theory can be generalized to non-algebraic groups and to algebraic groups in characteristic $p>0$ (cf. section 3).

Before going on it is *convenient* to give the following.

(2.6) Definition. Let G be an irreducible (non-necessary linear) algebraic F-group. An ideal (of the k-Lie algebra) $\Delta(G)$ is called a representative ideal if:

a) it is an abelian ideal in $\Delta(G)$ and an F-linear subspace of $\Delta(G/F)$

b) it is an F-linear complement of $\Delta(G,\mathrm{fin})$

c) it is Aut G-invariant.

Note that representative ideals may not exist; this is the case for instance with G= "universal" extension of an elliptic curve A over F (A not defined over k!) by $G_a$, since in this case $\Delta(G/F)=0$ and $\Delta(G)\neq\Delta(G,\mathrm{fin})$.

The interest for representative ideals lies in the following result essentially proved in [$B_1$]:

(2.7) PROPOSITION. Let $\mathcal{U}$ be a universal $\Delta$-field with field of constants $\mathcal{K}$ and G an irreducible algebraic $\mathcal{U}$-group. Assume $\mathcal{K}$ is a field of definition for G and $\Delta(G)$ contains a representative ideal V. Then the set $\Gamma(G)$ of $\Delta$-isomorphism classes of f-$\Delta$-groups $\Gamma$ for which $G(\Gamma)\simeq G$ identifies with $V^{int}/\mathrm{Aut}\, G_{\mathcal{K}}$ where $G\simeq G_{\mathcal{K}}\otimes_{\mathcal{K}}\mathcal{U}$, V is viewed as a $\Delta$-$\mathcal{U}$- vector space with the rule $\delta v=[\delta^*,v]$ for all $\delta\in\Delta, v\in V$ ($\delta^*\in\Delta(G)$ is the trivial lifting of $\delta$ from $\mathcal{U}$ to G) and $V^{int}=\{(a_1,\dots,a_m)\in V^m;\ \delta_i a_j=\delta_j a_i$ for all $i,j\}$.

We shall prove here (using [$HM_2$] once again ):

(2.8) PROPOSITION. Let G be an irreducible linear algebraic F-group. Then $\Delta(G)$ contains at least one representative ideal.

(2.9) Remark. From (1.1), (2.7) and (2.8) we get a quite satisfactory "classification" of all linear f-$\Delta$-groups $\Gamma$. Indeed for any such $\Gamma$, $G=G(\Gamma)$ is defined over $\mathcal{K}$ cf.(1.1). Moreover, by (2.8) $\Delta(G)$ contains a representative ideal V. Hence by (2.7) $\Gamma(G)\simeq V^{int}/\mathrm{Aut}\, G_{\mathcal{K}}$. Of course the problem remains of describing (in special cases) a representative ideal V as above. This is done in [$B_1$] in case the radical of G is nilpotent or the unipotent radical of G is commutative. In the general case it follows from results in [$B_1$] and from (1.1) that any representative ideal is mapped isomorphically by the map log: $\Delta(G)\to W(G)$ onto an intermediate space between $W_o(G)$ (cf. [$B_1$] p.13) and $W_1(G):=\mathrm{Ker}(W(G)\to H^2(\underline{u},\underline{u}))$ (cf. [$B_1$] p.28).

To prove (2.7) the basic ingredient is the following

(2.10) THEOREM (Hochschild-Mostow [$HM_2$]). Let G be an irreducible linear algebraic C-group. Then Aut $G^{an}$ is the semidirect product of Aut G by some normal vector subgroup N of it.

We will also need the following:

(2.11) LEMMA. Let $L$ be a Lie $F$-algebra of dimension $n$, $L_1$ a Lie subalgebra of dimension $n_1$ and $A$ a locally algebraic $F$-group acting algebraically on $L$ by Lie algebra automorphisms. Assume $F_o$ is an algebraically closed subfield of $F$ over which all the above data are defined. Let $Y$ denote the subset of all $F$-points in the Grassmanian $X$ of $(n-n_1)$-subspaces of $L$ which correspond to subspaces $L'$ of $L$ enjoying the following properties:

1) $L'$ is an abelian subalgebra of $L$

2) $L'$ is an ideal in $L$

3) $L_1+L'=L$

4) $L'$ is $A$-invariant.

Then $Y$ is locally closed in $X$ in the natural $F_o$-topology of $X$.

Proof. Condition 1) is $F_o$-closed and so is 2). Indeed for 2) note that for each $x\in L$, the derivation ad $x:L\to L$ induces a vector field on $X$ (to each linear space $W\subset L$ of dimension $n-n_1$ we consider the linear map

$$W\subset L \xrightarrow{\text{ad } x} L \to L/W$$

which is an element in the tangent space to $X$ at $[W]$); the locus in $X$ of all ideals in $L$ is then given by the vanishing of ad $x_1,\dots$ $\dots$, ad $x_n$ where $x_1,\dots,x_n$ is a basis in $L$ for which the structure constants belong to $F_o$; clearly, this locus is $F_o$-closed. Condition 3) is $F_o$-open (it is given by the non-vanishing of a certain Plücker coordinate). Finally condition 4) is $F_o$-closed (since $A$ acts on $X$ by an $F_o$-rational action).

(2.12) Proof of (2.8). Put $F_o=F_G$ and let $G=G_o\otimes_{F_o}F$. It is sufficient to find an abelian ideal $I_o$ of the $F_o$-Lie algebra $\Delta(G_o/F_o)$ complementary to $\mathcal{L}(\mathrm{Aut}\ G_o)$ and Aut $G_o$-invariant; because then formula (1.15) and [$B_1$] (1.2) imply that $\Delta(G,fin)=Der(F/F_o)\oplus$ $\oplus\mathcal{L}(\mathrm{Aut}\ G)$ hence $I=I_o\otimes_F F$ will be a representative ideal in $\Delta(G)$ (use (1.11)). Now by (2.11) it is sufficient to find an abelian ideal $I$ of $\Delta(G/F)$ complementary to $\mathcal{L}(\mathrm{Aut}\ G)$ and Aut $G$-invariant. By (2.11) again we may assume (after replacing $F$ by a field extension of it or by a suitable subfield of it) that $F=\mathbb{C}$. But then (2.2) and (2.10) show that viewing $\Delta(G/\mathbb{C})$ as a subalgebra of $\mathcal{L}(\mathrm{Aut}\ G^{an})$ we have that $I=\Delta(G/\mathbb{C})\cap\mathcal{L}(N)$ satisfies our requirements ($N$ as in (2.2)).

## 3. Remarks on the case of characteristic $p>0$

(3.1) In this section only we assume char $F=p>0$ (and $F$ algebraically closed as usual). We will make some comments an how our results extend into this setting. Algebraic $F$-groups will always be assumed irreducible and reduced. If $G$ is such a group one can define $\Delta(G)$ and $\Delta(G/F)$ exactly as in [$B_1$][$B_2$]. But since $F$ is perfect any derivation on it vanishes hence these two spaces coincide; so the "moduli-theoretic" problems dissapear in characteristic $p>0$!

As in the characteristic zero case, both Lie $F$-algebras $\Delta(G)=\Delta(G/F)$ and $\mathcal{L}(G)$ embed into $Der_F(\mathcal{O}_G)=\mathcal{L}(G)\otimes\mathcal{O}(G)$ and $[\Delta(G),\mathcal{L}(G)]\subset\mathcal{L}(G)$. Moreover if $G$ is commutative $\Delta(G)=\mathcal{L}(G)\otimes X_a(G)$ (same proof as in [$B_2$]). Note that in general $\dim_F X_a(G)=\infty$ hence $\dim_F\Delta(G)=$ $=\infty$. Note moreover that if $\theta\in\mathcal{L}(G)$ ($G$ nonnecessary commutative) and $f\in X_a(G)$ then $\theta f\in F$ (indeed $\mu\theta f=(\theta\otimes 1)\mu f=(\theta\otimes 1)(f\otimes 1+1\otimes f)=\theta f\otimes 1$ which clearly implies $\theta f\in F$). One more definition for $G$ affine (only!): a derivation $\delta\in\Delta(G)$ is called locally finite if $\mathcal{O}(G)$ is a union of finite dimensional $\delta$- stable $F$-linear subspaces. Moreover let $\Delta(G,fin)$ the set of locally finite derivations in $\Delta(G)$.

(3.2) PROPOSITION. Assume $G$ is commutative unipotent. Then $\Delta(G)=\Delta(G,fin)$.

Proof. For each $y \in \mathcal{O}(G)$ there exists an integer $N=N(y)$ such that any product of $N$ elements of $\mathcal{L}(G)$ (viewed as elements of $\mathrm{End}_F(\mathcal{O}(G))$) kills $y$ (cf. [H] pp.42 and 63-64). If $\lambda_x \in \mathrm{End}_F(\mathcal{O}(G))$ denotes the multiplication by $x \in \mathcal{O}(G)$ on $\mathcal{O}(G)$ then for any $\theta \in \mathcal{L}(G)$ we have $[\theta, \lambda_x] = \lambda_{\theta x}$; so by (3.1) if $x \in X_a(G)$ then $[\theta, \lambda_x]$ is the homotety with some scalar in F. Now pick an element $\delta = \sum \lambda_{a_i} \theta_i = \sum \theta_i \otimes a_i \in \mathcal{L}(G) \otimes X_a(G)$ where $(\theta_i)_i$ is an F-basis of $\mathcal{L}(G)$ and $a_i \in X_a(G)$. Then it is easy to check using the above remarks that the F-linear span of the set

$$\{\delta^i y;\ i \geq 0\} \subset \mathcal{O}(G)$$

is contained in the F-linear span of the set

$$\{\lambda_{a_{i_1} a_{i_2} \ldots a_{i_n}} \theta_{j_1} \theta_{j_2} \ldots \theta_{j_n};\ n \leq N-1\} \subset \mathcal{O}(G)$$

In particular $\dim_F \sum_{i \geq 0} F\delta^i y < \infty$ for all $y \in \mathcal{O}(G)$ which proves our proposition.

(3.3) Question. Is it true that $\Delta(G) = \Delta(G, \mathrm{fin})$ for any unipotent G?

(3.4) Exactly as in $[B_1]$, for any linear G and any $\delta \in \Delta(G)$ we have $\delta X_a(G) \subset X_a(G)$ and $(\log \delta)(X_m(G)) \subset X_a(G)$. In particular we dispose of an F-linear map $\log : \Delta(G) \longrightarrow W(G) = \mathrm{Hom}_{gr}(X_m(G), X_a(G))$. Unlike in characteristic zero it may happen that there exist derivations $\delta \in \Delta(G, \mathrm{fin})$ with $\log \delta \neq 0$. To construct such examples note that we can check (by direct computation) for $p=2,3,5$ and we ask whether it is true in general that:

(3.5) Question. Does the following formula hold in the polynomial ring $A = \mathbb{F}_p[x]$:

$$((x-x^p)\frac{d}{dx} + x.1_A)^{p-1}(x) = x?$$

(3.6) Assuming (3.5) above holds for a prime p (e.g. assuming $p\in\{2,3,5\}$ let $G=G_a \times G_m = \mathrm{Spec}\,[x,y,y^{-1}]$, $\mu x = x\otimes 1 + 1\otimes x$, $\mu y = y\otimes y$ and define $\delta\in \mathrm{Der}_F \mathcal{O}(G)$ by the formula

$$\delta = (x-x^p)\frac{d}{dx} + xy\,\frac{d}{dy}$$

Since $x-x^p$, $x\in X_a(G)$ and $\frac{d}{dx}$, $y\frac{d}{dy}\in\mathcal{L}(G)$ it follows by (3.1) that $\delta\in\Delta(G)$. Now (3.5) implies that $\delta^p y = \delta y$ and $\delta^p(y^{-1}) = \delta(y^{-1})$. On the other hand clearly $\delta^i x = \delta x$ for all $i\geq 2$. Consequently $\delta\in\Delta(G,\mathrm{fin})$; but clearly $\log\delta\neq 0$. There is another anomaly related to this example namely that $\Delta(G,\mathrm{fin})$ is not closed under addition. Indeed consider $\delta_1,\delta_2\in\Delta(G,\mathrm{fin})$ defined by

$$\delta_1 = (x^p - x)\frac{d}{dx} \quad \text{and} \quad \delta_2 = (x-x^p)\frac{d}{dx} + xy\frac{d}{dy}$$

Then $\delta = \delta_1 + \delta_2 = xy\frac{d}{dy}$ hence $\delta^i y = x^i y$ so $\delta\notin\Delta(G,\mathrm{fin})$! . Inspite of these anomalies the converse question of whether $\log\delta = 0$ implies $\delta\in\Delta(G,\mathrm{fin})$ may be given a positive answer in some special cases; indeed the arguments in $[B_1]$ (2.3)-(2.8) and (4.4) yeld the following:

(3.7) PROPOSITION. Let G be a solvable linear algebraic F-group.

1) Assume the unipotent radical of G is commutative. Then any derivation $\delta\in\Delta(G)$ with $\log\delta = 0$ belongs to $\Delta(G,\mathrm{fin})$.

2) Assume the unipotent radical of G is a vector group. Then $\Delta(G)$ kills the weights of G. Moreover the image of $\log:\Delta(G) \longrightarrow W(G)$ coincides with $W_o(G)$.

In the above statement the notion of "weight" and the definition of $W_o(G)$ are those of $[B_1]$. Note also that if Question (3.3) has a positive answer then the assumption in (3.7), 1) that the unipotent radical is commutative can be dropped.

(3.8) Remark. By representability of Aut G for G reductive [GD] it follows that $\Delta(G)=\Delta(G,fin)$ whenever G is reductive.

## 4. Embeddings of f-$\Delta$-groups into algebraic groups

Everywhere in this section $\Delta=\{\delta_1,\ldots,\delta_m\}$ acts by commuting derivations.

(4.1) Let F be a $\Delta$-field (once again of characteristic zero) and let $V\mapsto V^!$ be the forgetful functor

$$\{\text{reduced } \Delta\text{-schemes over } F\} \longrightarrow \{\text{reduced schemes over } F\}$$

One can construct a right adjoint $X\mapsto X^\infty$ to this functor using the usual "prolongation" procedure [J] (see also the "produced schemes" of [NW]). So, for any reduced F-scheme X and any reduced $\Delta$-scheme V over F we will have a natural bijection

$$\mathrm{Hom}_{Sch/F}(V^!,\ X)\simeq \mathrm{Hom}_{\Delta\text{-}Sch/F}(V,X^\infty)$$

"Recall" one of the possible constructions of $X\mapsto X^\infty$. We construct a sequence $\mathcal{A}_n$ $(n\geq -1)$ of sheaves of $\mathcal{O}_X$-algebras on X equiped with $\mathcal{O}_X$-algebra maps $f_n:\mathcal{A}_n\to\mathcal{A}_{n+1}$ and with $f_n$-derivations $d_n^i:\mathcal{A}_n\to$ $\to\mathcal{A}_{n+1}$ $(1\leq i\leq m)$ inductively starting with $\mathcal{A}_{-1}=F$, $\mathcal{A}_o=\mathcal{O}_X$, $f_{-1}=$ natural inclusion $F\subset\mathcal{O}_X$, $d_{-1}^i=\delta_i:F\to F\subset\mathcal{O}_X$ and then letting

$$\mathcal{A}_{n+1}=S^{\cdot}(\Omega_{\mathcal{A}_n}^{\oplus m})/J_n$$

where $J_n$ is the sheaf of ideals in the symmetric algebra of $\Omega^{\oplus m}_{\mathcal{A}_n}$ generated by elements of the form

$$(4.1.1) \qquad d^i_{n-1}a-(d(f_{n-1}(a)))e_i, \quad a\in\mathcal{A}_{n-1}$$

and elements of the form

$$(4.1.2) \qquad d(d^j_{n-1}a)e_i-d(d^i_{n-1}a)e_j, \quad a\in\mathcal{A}_{n-1}$$

where $e_1,\dots,e_m$ is the standard basis of $\Omega^{\oplus m}_{\mathcal{A}_n}$ and $d:\mathcal{A}_n\to\Omega_{\mathcal{A}_n}$ is the usual differential. Moreover we let $f_n$ be induced by the natural inclusion map $\mathcal{A}_n\to S^{\circ}(\Omega^{\oplus m}_{\mathcal{A}_n})$ and $d^i_n$ be induced by the map $\mathcal{A}_n\to\Omega^{\oplus m}_{\mathcal{A}_n}$, $b\mapsto(db)e_i$. Note that in the definitions above the modules of differentials are the absolute ones (over $\mathbb{Q}$ not over $F$!).

We put $\mathcal{A}^{\infty}=(\varinjlim\mathcal{A}_n)_{red}$, $d^i=(\varinjlim d^i_n)_{red}$ and $X^{\infty}=\mathrm{Spec}\,\mathcal{A}^{\infty}$. One easily checks that $X\mapsto X^{\infty}$ is the functor we are looking for (the $\Delta$- structure on $X^{\infty}$ will be given of course by $d^1,\dots,d^m$).

If $\pi:(X^{\infty})^!\to X$ is the natural map then for any open set $U\subset X$ it is easy to see that $U^{\infty}\simeq\pi^{-1}(U)$. Moreover if $X$ is affine and of finite type over $F$ then $X^{\infty}$ will be also affine and its coordinate ring $\mathcal{O}(X^{\infty})$ is $\Delta$-finitely generated over $F$ (but not finitely generated over $F$!). So if $F=\mathcal{U}$ (a universal $\Delta$-field), exactly as in the case of $\Delta$- varieties we may associate to any $\mathcal{U}$- variety the locally $\Delta$-ringed space $\hat{X}=(X^{\infty})_{\Delta}$ which will be a $\Delta$- manifold ; we get a functor $X\mapsto\hat{X}$

$$\{\mathcal{U}\text{- varieties}\}\longrightarrow\{\Delta\text{-manifolds}\}.$$

Note that we have a natural identification $X(\mathcal{U})\simeq\hat{X}(\mathcal{U})$ for any $\mathcal{U}$- variety $X$.

Coming back to an arbitrary $\Delta$-field F, universality properties immediately imply that the functor $X \mapsto X^{\infty}$ from $\{$reduced F-schemes$\}$ to $\{$reduced $\Delta$-schemes over F$\}$ commutes with direct products. So it induces in a natural way a functor $G \mapsto G^{\infty}$

$$\{\text{reduced F-group schemes}\} \to \{\text{reduced F-group schemes with } \Delta\text{- structure}\}$$

the latter being of course the group objects in

$$\{\text{reduced } \Delta\text{- schemes over F}\}$$

The functor $G \mapsto G^{\infty}$ is a right adjoint for the forgetful functor. Clearly, if G is commutative so will be $G^{\infty}$. As above we get a functor $G \mapsto \hat{G} := (G^{\infty})_{\Delta}$

$$\{\text{algebraic } \mathcal{U}\text{-groups}\} \to \{\Delta\text{-algebraic groups}\}$$

and a natural identification $G(\mathcal{U}) \simeq \hat{G}(\mathcal{U})$. Clearly $\hat{G}$ is not an f-$\Delta$-group (except if G is trivial). A morphism $H \to \hat{G}$ of $\Delta$-groups will be called an embedding if the induced morphism $H(\mathcal{U}) \longrightarrow \hat{G}(\mathcal{U}) = G(\mathcal{U})$ is injective; by above we may say that H embeds into G (rather than into $\hat{G}$).

(4.2) LEMMA. Let $G \xrightarrow{!} H$ be a morphism of algebraic $\mathcal{U}$-groups where G is an algebraic $\mathcal{U}$-group with $\Delta$- structure. The following are equivalent:

1) The induced $\Delta$-morphism $G \to H^{\infty}$ has a trivial kernel (we say simply that it is injective).

2) The kernel of $G \xrightarrow{!} H$ contains no non-trivial $\Delta$-stable algebraic subgroup.

Proof. 2) $\Longrightarrow$ 1) $\mathrm{Ker}(G \to H^{\infty})$ is a $\Delta$-stable algebraic subgroup of $\mathrm{Ker}(G \xrightarrow{!} H)$ so by 2) it is trivial.

1) $\Longrightarrow$ 2) Assume P is a $\Delta$-stable algebraic subgroup of $\mathrm{Ker}(G \xrightarrow{!} H)$. Then both the trivial $\Delta$-morphism

$$\varphi : P \to \mathrm{Spec}\,\mathcal{U} \xrightarrow{\varepsilon} H^{\infty}$$

and the $\Delta$-morphism

$$\psi : P \hookrightarrow G \to H^{\infty}$$

composed with the projection $H^{\infty} \to H$ give the same (trivial) morphism $P \to \mathrm{Spec}\,\mathcal{U} \xrightarrow{\varepsilon} H$. By universality of $H^{\infty}$, $\varphi = \psi$ hence P reduces to the identity.

(4.3) COROLLARY. Let $\Gamma$ be an f-$\Delta$-group. Then there is a natural embedding $\Gamma \to G(\Gamma)^{\wedge}$.

Proof. Apply (4.2) to the identity map $G(\Gamma) \xrightarrow{!} G(\Gamma)$ toget an injective $\Delta$-morphism $G(\Gamma) \to G(\Gamma)^{\infty}$ hence our embedding $\Gamma \to G(\Gamma)^{\wedge}$.

More about the embedding (4.3) will be proved in (4.11).

(4.4) LEMMA. Let G be an irreducible commutative algebraic F-group.

1) Any torsion point of G(F) is a $\Delta(G)$-point.

2) Any torus and any abelian variety contained in G is a $\Delta(G)$ - subvariety.

Proof. 1) If $x \in G(F)$ is an N-torsion point, consider the isogeny $\varphi_N : G \to G$, $\varphi_N(g) = Ng$. Then $\mathrm{Ker}\,\varphi_N$ is a (finite) $\Delta(G)$-subscheme of G, hence so are all its irreducible components, in particular so is x.

2) Since the torsion points are dense in tori and abelian varieties the ideal sheaf $I_T$ (respectively $I_A$) of any torus T (respectively abelian variety A) contained in G is the intersection of the ideals of the torsion points of T (respectively A), hence $I_T$ (resepctively $I_A$) is a $\Delta(G)$-ideal.

(4.5) COROLLARY. Let $\Gamma$ be an f-$\Delta$-group and $G=G(\Gamma)$. The following are equivalent:

1) There is an embedding $\Gamma \to \hat{A}$ for some abelian $\mathcal{U}$-variety A.

2) There is an injective $\Delta$-morphism $G \to A^{\infty}$ for some abelian $\mathcal{U}$-variety A.

3) The linear part of G contains no nontrivial $\Delta$- stable algebraic subgroup.

4) Any morphism from a linear f-$\Delta$-group $\Gamma'$ to $\Gamma$ is trivial.

Moreover if the above conditions hold, the linear part of G is unipotent.

Proof. Let B be the linear part of G.

3) $\Longrightarrow$2) follows from (4.2) applied to the projection $G \to G/B$.

2)$\Longrightarrow$1) is obvious.

2)$\Longrightarrow$3). We have a commutative diagram

$$\begin{tikzcd} G^{!} \arrow[rr] \arrow[dr] & & A \\ & G/B \arrow[ur] & \end{tikzcd}$$

providing a commutative diagram

$$\begin{tikzcd} G \arrow[rr, "\varphi"] \arrow[dr, "\psi"'] & & A^{\infty} \\ & (G/B)^{\infty} \arrow[ur] & \end{tikzcd}$$

Since $\varphi$ is injective so is $\psi$. Applying (4.2) to $\psi : G \to (G/B)^{\infty}$ we get our conclusion. Note that if 2) or 3) hold G is commutative so by (4.4) the maximal torus $B_m$ of B is $\Delta$- stable hence trivial so B is unipotent.

1) $\Longrightarrow$ 2) The embedding $\Gamma \to \hat{A}$ provides a $\Delta$-ring map $\mathcal{O}_{A^{\infty},o} = \mathcal{O}_{\hat{A},o} \to \mathcal{O}_{\Gamma,o} = \mathcal{O}_{G,o}$. Composing this morphism with the natural morphism $\mathcal{O}_{A,o} \to \mathcal{O}_{A^{\infty},o}$ we get a morphism $\mathcal{O}_{A,o} \to \mathcal{O}_{G,o}$ hence a rational map $\varphi : G \dashrightarrow A$ which is easily seen to agree with comultiplication generically. So $\varphi$ is an everywhere defined morphism of $\mathcal{U}$-algebraic groups and the morphism $\varphi^{\infty} : G \to A^{\infty}$ induced by it induces our morphism $\Gamma \to \hat{A}$. We are left to prove that $K = \mathrm{Ker}(G \to A^{\infty})$ is trivial. But if K is nontrivial its group $K_{\Delta}(\mathcal{U})$ of $\Delta$-$\mathcal{U}$- points is nontrivial contradicting the injectivity of $\Gamma(\mathcal{U}) \to \hat{A}(\mathcal{U}) = A(\mathcal{U})$.

3) $\Longrightarrow$ 4). If $\Gamma' \to \Gamma$ is as in 4) then the image of $G(\Gamma') \to$ $\to G(\Gamma)$ is a $\Delta$-stable subgroup of B hence trivial. So $G(\Gamma') \to G(\Gamma)$ is trivial, so $\Gamma' \to \Gamma$ is trivial.

4) $\Longrightarrow$ 3) Since $[\Gamma, \Gamma]$ is linear it is trivial so $\Gamma$ is commutative. By (4.4) $B_m$ is $\Delta$-stable. By 4) $B_m$ is trivial. Now assume 3) does not hold hence there exists a $\Delta$-stable subgroup $H \neq 0$ of B. Since H is unipotent, it is irreducible. Letting $\Gamma' = H_{\Delta}$ we get a contradiction.

(4.6) Let A an abelian $\mathcal{U}$- variety of dimension g and $\tilde{A}$ be the "universal" extension of A by $B = G_a^g$. By $[B_2]$ (5.8) the derivations of $\mathcal{U}$ uniquely lift to pairwise commuting derivations in

$\Delta(\tilde{A})$. So we may consider the f-$\Delta$-group $\tilde{A}_\Delta$; we have a natural morphism $\tilde{A}_\Delta \to \hat{A}$ induced by the projection $\tilde{A} \to A$. We are looking for a criterion for $\tilde{A}_\Delta \to \hat{A}$ to be an embedding (equivalently for $\tilde{A} \to A^\infty$ to be injective). Assume for simplicity that $\mathcal{U}$ is ordinary.

Note that if g=1 and if $\rho : \mathrm{Der}\,\mathcal{U} \to H^1(A, T_A)$ is the Kodaira-Spencer map then if $\rho(\delta) \neq 0$ then $\tilde{A}_\Delta \to \hat{A}$ is an embedding. Indeed by (4.2) it is sufficient to check that $G_a = \mathrm{Ker}(\tilde{A} \to A)$ is not $\delta$-stable; but if $G_a$ was $\delta$-stable, by [$B_2$] (3.4) $\delta$ would be locally finite on $\tilde{A}$ hence by [$B_2$] (1.6) $\tilde{A}$ would be defined over $\mathcal{K}$ hence so would be A, contradicting the fact that $\rho(\delta) \neq 0$.

The proposition below generalizes the above remark for arbitrary $g \geq 1$. First we may consider the Kodaira-Spencer map once again; identifying $H^1(A, T_A)$ via cup-product with $H^0(A, T_A) \otimes H^1(\mathcal{O}_A) = \mathrm{Hom}(H^0(\Omega^1_A), H^1(\mathcal{O}_A))$ me may consider for any element $\varphi \in H^1(A, T_A)$ its determinant

$$\det \varphi \in \mathrm{Hom}\left(\bigwedge^g H^0(\Omega^1_A), \bigwedge^g H^1(\mathcal{O}_A)\right) \simeq F$$

(4.7) PROPOSITION. In notations above assume $\det \rho(\delta) \neq 0$. Then $\tilde{A}_\Delta \to \hat{A}$ is an embedding.

Proof. We shall use notations from [$B_2$] (5.8)-(5.11) with $C = \tilde{A}$. In particular in our situation the classes $a^1, \dots, a^g \in H^1(\mathcal{O}_A)$ of the cocycles $(a^p_{ij})_{1 \leq p \leq g}$ in loc.cit. form an F-basis of $H^1(\mathcal{O}_A)$. By [$B_2$] p.48 $\delta$ is obtained by glueing together derivations of the form

$$\delta_i = \theta_i + \sum x_k v_k - \sum_k \sum_p (\alpha^p_{ik} x_k + \alpha^p_i) \frac{\partial}{\partial x_p} \tag{4.8.1}$$

where recall that $(x_k)_k$ is a basis of $E = X_a(B)$. Moreover [$B_2$] (5.10.1) implies

(4.8.2) $$\rho(\delta)=\sum a^k v_k$$

By (4.2) it is sufficient to prove that B contains no non-trivial $\delta$-stable algebraic subgroup $B_1$. Assume there exists such a $B_1$. If $B_1=B$ we conclude exactly as in the case $g=1$ (cf.(4.7)) by using (4.2) and [$B_2$] (3.4), (1.6). So we may assume $1\leq\dim B_1\leq g-1$. Now to the surjection $B\to B'=B/B_1$ there corresponds the extension $C'=C/B_1$ of A by $B'$ which is obtained by glueind together the spectra of $A_i[E']$ where $E'=X_a(B')\subset E$. We may assume $x_1,\ldots,x_r$ is a basis of $E'$, $1\leq r\leq g-1$. Since $A_i[E']$ must be $\delta_i$-subring of $A_i[E]$ formula (4.8.1) shows that $v_k=0$ for $r+1\leq k\leq g$. Then formula (4.8.2) implies $\det\rho(\delta)=0$, contradiction.

We close this paper by dicussing "logarithmic derivatives" associated to f-$\Delta$-groups.

(4.9) Let G,H be irreducible algebraic $\mathcal{U}$-groups with G acting on H by $\mathcal{U}$-group automorphism and with H commutative. By functoriality $G^\infty$ will act on $H^\infty$. By a $\Delta$-cocycle of G in H (or "crossed $\Delta$-homomorphism", cf. [$K_2$] we shall understand a morphism of $\Delta$-schemes $\varphi:G^\infty\to H^\infty$ which makes commutative the usual diagram expressing the cocycle condition (the diagram being in the category of $\Delta$-schemes over $\mathcal{U}$). Giving such a $\varphi$ is equivalent to giving for any reduced $\Delta$-algebra R over $\mathcal{U}$ a cocycle of $G^\infty(R)\simeq G(R^!)$ in $H^\infty(R)=H(R^!)$ i.e. of a map $\varphi_R:G(R^!)\to H(R^!)$ satisfying

(4.9.1) $$\varphi_R(xy)=\varphi_R(x)+x\varphi_R(y),\qquad x,y\in G(R^!)$$

The set of all $\Delta$-cocycles of G in H will be denoted by $Z^1_\Delta(G,H)$.

In particular, in the above definition we may take $H=\mathcal{L}(G)$ (viewed as a vector group) on which G acts by adjoint representation or more generally $H=\mathcal{L}(G)^N$ for some $N\geqslant 1$ with adjoint action on the components. Assume G and H as above. Then for any $\varphi\in Z^1_\Delta(G,H)$, the $\Delta$-subscheme $G_\varphi:=\varphi^{-1}(0)_{red}$ of $G^\infty$ is a group $\Delta$-subscheme of $G^\infty$; (i.e. the multiplication antipode and unit on $G^\infty$ induce a multiplication, antipode and unit on $G_\varphi$).

Indeed, it is sufficient to check that for any reduced $\Delta$-algebra R over $\mathcal{U}$ the set $G_\varphi(R^!)$ is a subgroup of $G(R^!)$; but $G_\varphi(R^!)=\{x\in G(R);\ \varphi_R(x)=0\}$ which clearly is a subgroup by (4.9.1).

The proposition below shows in particular that any f-$\Delta$-group $\Gamma$ can be canonically realized as the kernel $G_\varphi$ of some suitable $\Delta$-cocycle $\varphi$ of $G=G(\Gamma)$ in $\mathcal{L}(G)^m$:

(4.10) PROPOSITION. Let G be an irreducible algebraic $\mathcal{U}$-group. There exists a natural injective map

$$\ell:\Delta(G)^{int}\longrightarrow Z^1_\Delta(G,\mathcal{L}(G)^m)\,,$$

assigning to any m-uple $\delta=(\delta_1,\ldots,\delta_m)\in\Delta(G)^{int}$ of pairwise commuting elements in $\Delta(G)$ lifting the derivations of $\mathcal{U}$ a $\Delta$-cocycle $\ell\delta=(\ell\delta_1,\ldots,\ell\delta_m)$ whose kernel $G_{\ell\delta}$ is isomorphic (as a group scheme with $\Delta$-action) with $(G,\delta)$ (i.e. with G equiped with derivations $\delta_1,\ldots,\delta_m$).

Proof. Let $\delta=(\delta_1,\ldots,\delta_m)\in\Delta(G)^{int}$ be as in the statement of (4.10). To define $\ell\delta_i$ we must define for any reduced $\Delta$-algebra R over $\mathcal{U}$, cocycles $(\ell\delta_i)_R:G(R^!)\to\mathcal{L}(G)\otimes_{\mathcal{U}}R$ behaving functorially in R. We define them by the formula

(4.10.1) $(\ell\delta_i)_R(x)=L_x\delta_i^R L_x^{-1}-\delta_i^R\ ,\ x\in G(R^!)$

where for any $x\in G(R^!)$ we denote by $\delta_i^R$ the derivation on $\mathcal{O}_{G\otimes R}$ deduced from G and R and $L_x:\mathcal{O}_{G\otimes R}\to\mathcal{O}_{G\otimes R}$ is induced by left translation with x. That $(\ell\delta_i)_R(x)\in\mathcal{L}(G)\otimes R$ follows from the fact that

$$\begin{aligned}\mathrm{Der}_R(\mathcal{O}_{G\otimes R},\mathcal{O}_{G\otimes R})&=\mathrm{Hom}_{G\otimes R}(\Omega_{G\otimes R/R},\mathcal{O}_{G\otimes R})=H^0(\check{\Omega}_{G\otimes R/R})=\\&=H^0(\check{\Omega}_{G/\mathcal{U}})\otimes_{\mathcal{U}}R=\mathrm{Der}_{\mathcal{U}}(\mathcal{O}_G,\mathcal{O}_G)\otimes_{\mathcal{U}}R\end{aligned}$$

from identification of $\mathcal{L}(G)$ with the right invariant members of $\mathrm{Der}_{\mathcal{U}}(\mathcal{O}_G,\mathcal{O}_G)$ and from the following computation (with $\mu^R=\mu\otimes 1_R$)

$$\begin{aligned}\mu^R(L_x\delta^R L_x^{-1}-\delta^R)&=\mu^R L_x\delta^R L_x^{-1}-\mu^R\delta^R=\\&=(L_x\otimes 1)\mu^R\delta^R L_x^{-1}-\mu^R\delta^R=\\&=(L_x\otimes 1)(\delta^R\otimes 1+1\otimes\delta^R)\mu^R L_x^{-1}-(\delta^R\otimes 1+1\otimes\delta^R)\mu^R=\\&=((L_x\otimes 1)(\delta^R\otimes 1+1\otimes\delta^R)(L_x^{-1}\otimes 1)-(\delta^R\otimes 1+1\otimes\delta^R))\mu^R=\\&=((L_x\delta^R L_x^{-1}-\delta^R)\otimes 1)\mu^R\end{aligned}$$

The fact that $(\ell\delta_i)_R$ are indeed cocycles follows by immediate computation. To check injectivity of $\ell$ assume $\ell\delta_i=\ell\delta'_i$, $1\le i\le m$ for some $(\delta_1,\dots,\delta_m)$, $(\delta'_1,\dots,\delta'_m)\in\Delta(G)^{int}$. Then if $\theta_i=\delta_i-\delta'_i$ we get that $L_x\theta_i=\theta_i L_x$ for all $x\in G(\mathcal{U})$. Since the $\theta_i$'s are $\mathcal{U}$- linear we get that $\theta_i$ is a left-invariant vector field on G vanishing at the identity of G, hence $\theta_i=0$ for all i and injectivity of $\ell$ follows.
To check that $G_{\ell\delta}\simeq(G,\delta)$ it is sufficient to show that for all reduced $\Delta$-algebra R over $\mathcal{U}$ the sequence of pointed sets

$$1\to\mathrm{Hom}_{\Delta\text{-}Sch}(\mathrm{Spec}\ R,(G,\delta))\xrightarrow{i}\mathrm{Hom}_{Sch}(\mathrm{Spec}\ R,G)=G(R^!)\xrightarrow{\ell\delta_R}$$
$$\longrightarrow\mathcal{L}(G)^m\otimes R$$

is exact.

If $x:\mathrm{Spec}\, R \to G$ is a $\Delta$-morphism, clearly the left translation $G\otimes R \to G\otimes R$ defined by $x$ is a $\Delta$-morphism, equivalently $(\ell\delta_i)_R(x)=$ $=0$ for all $i$. Conversely if the latter happens, since the unit $\mathrm{Spec}\, R \to G\otimes R$ is a $\Delta$-morphism so will be its composition with the left translation by $x$ which is precisely $x$. Our proposition is proved.

(4.11) COROLLARY. Let $\Gamma$ be an f-$\Delta$-group. Then there exists a natural morphism of $\Delta$-manifolds $\ell_\Gamma : \hat{G} \to (\mathcal{L}(G)^m)^\wedge$ (where $G=G(\Gamma)$) such that $\Gamma \simeq \ell_\Gamma^{-1}(0)$ (isomorphism of $\Delta$-manifolds). In particular there is an exact sequence of pointed sets

$$1 \to \Gamma(\mathcal{U}) \to G(\mathcal{U}) \xrightarrow{\ell_\Gamma(\mathcal{U})} \mathcal{L}(G)^m$$

Moreover the image of $\ell_\Gamma(\mathcal{U})$ equals the set of all m-uples $(\theta_1,\dots,\theta_m) \in \mathcal{L}(G)^m$ such that $\delta_i\theta_j = \delta_j\theta_i$ for all $i,j$.

Proof. Everything but the last assertion follows from (4.10). The last assertion follows by arguments similar to those in $[B_o]$ p.51.

(4.12) Remark. If $\Gamma$ is a split f-$\Delta$-group the map $\ell_\Gamma$ above is of course Kolchin's logarithmic derivative in $[K_1]$. Moreover if for instance $\Gamma = \check{A}_o$ with $A_o$ an abelian $\mathcal{K}$-variety then the logarithmic derivative $\ell_\Gamma$ has a nice "geometric" interpretation (cf $[B_2]$, section 2): if we let $A = A_o \otimes \mathcal{U}$ then $\ell_\Gamma(\mathcal{U}) : A(\mathcal{U}) \to \mathcal{L}(A)$ is induced by logarithmic derivative of cocycles $H^1(A^o, \mathcal{O}^*) \to$ $\to H^1(A^o, \mathcal{O})$ where $A^o$ is the dual abelian variety of $A$. It would be interesting to give such "geometric" interpretations of $\ell_\Gamma$ for f-$\Delta$-groups $\Gamma$ which are not split (or even non-semisplit). In particular it is reasonable to believe that if $\Gamma = \tilde{A}_\Delta$ (cf. (4.6)) then the map $\ell_{\Gamma}$ can be expressed in terms of the "multiplicative analogue" of the Gauss-Manin connection (cf. [$B_2$]).

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