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Brownian oscillations near the

boundary of a hypersurface

by L. Stoica

This paper continues the study begun in [S]. To describe what kind of problems are treated, let us simplify the hypotheses here, in the introduction, and consider the brownian motion in the strip $D = (-\infty,\infty) \times (-1,1) \subset R^2$. Let us denote by X this process endowed with its structure of standard process and set $K = \{x \in D : x^2 = 0\}$, $V^\varepsilon = \{x \in D : d(x,K) < \varepsilon\}$, for $\varepsilon \in (0,1)$. Also denote by $K_t^\varepsilon(\omega)$ = the number of times the trajectory $X_\cdot(\omega)$ hits K after wisits outside V^ε, hefore time t. The functional $K^\varepsilon = (K_t^\varepsilon)$ depends only on the behaviour of the component X_t^2 of $X_t = (X_t^1, X_t^2)$. Then, one can easily deduce, via the approximation theorem of the local time at 0 for Brownian motion in $(-1,1)$, that

$$\lim_{\varepsilon \to 0} \varepsilon K_t^\varepsilon = A_t,$$

exists a.s. The functional $A = (A_t)$ turns out to be a continuous additive functional uniformly distributed on K, in the sense that it can be represented with the (one dimensional) Lebesgue measure, μ, in K as follows

$$E^x(A_\infty) = \int_K g(x,y)\mu(dy), \quad x \in D,$$

where $g(x,y)$ is the Green function in D. Further, let us set

$$H = \{x \in K : x^1 \leq 0\} \text{ and } M_t^\varepsilon = \int_{[0,t]} 1_H(X_s) dK_s^\varepsilon.$$

Then we can show the significance of Proposition 8.1 in the text by the following relation (which is a consequence of Proposition 8.1)

$$\lim_{\varepsilon \to 0} \varepsilon M_t^\varepsilon = \int_{[0,t]} 1_H(X_s) dA_s.$$

Now let us denote by $F^{\varepsilon} = \{ x \in D : d(x,H) < \varepsilon \}$, $\varepsilon \in (0,1)$ and set $H_t^{\varepsilon}(\omega)$ = the number of times the trajectory $X_{\cdot}(\omega)$ hits H after visits outside F^{ε} , before time t. As a consequence of Proposition 9.6 we have

$$\lim_{\varepsilon \to 0} \varepsilon H_t^{\varepsilon} = \int_{[0,t]} 1_H(X_s) dA_s.$$

Thus we see that $\varepsilon M^{\varepsilon}$ and $\varepsilon H^{\varepsilon}$ behaves similarly when $\varepsilon \to 0$.

In this paper we preserve the terminology and notion adopted in [S]. Moreover the sections are numbered in continuation and we refere direct to relations and results proved in [S] just by indicating their number. The first section of this paper is Section 8. and is devoted to the proof of Proposition 8.1, mentioned above. The main result of Section 9 is Proposition 9.6. Then, in the final section we use the preceding results to treat the case of a compact hypersurface with boundary.

8. The functional with density in a piece of hyperplane

Let L be an operator of the form (1.2) with $c\equiv 0$ and $a^{ij}\in\mathcal{C}^{2+\alpha}(R^d)$, $b^i\in\mathcal{C}^{1+\alpha}(R^d)$, $i,j=1,\dots,d$. These assumptions ensure the existence of the dual L^*. As in Section 3 we set $D_1=\{x\in R^d: |x^d|<1\}$ and $K=\{x\in R^d: x^d=0\}$ and X will be an L-diffusion in D_1. Moreover we will assume that L coincides with $1/2\Delta$ outside a compact set included in D_1 (i.e. $a^{ij}=1/2\delta_{ij}$ and $b^i=0$), so that X behaves like Brownian motion outside that compact. The Green function associated to L in D_1 is denoted by g^1 and the Green potential in D_1 of a measure ν is denoted by G^1_ν. We will need also the measure λ introduced in Section 3.

We consider the numerical values $\eta'_i,\eta''_i\in[-\infty,+\infty]$ such that $\eta'_i<\eta''_i$, $i=1,\dots,d-1$ and denote by $H=\{x\in K: x^i\in[\eta'_i,\eta''_i],\ i=1,\dots,d-1\}$. With respect to K and $\varepsilon\in(0,1)$ we consider the functional A^ε given by (2.1) and the functional $A^{H,\varepsilon}$ given by

$$A^{H,\varepsilon}_t=\int_{[0,t]}1_H(X_s)\,dA^\varepsilon_s .$$

The potentials of these functionals will be denoted by u and u_H:

$$u(x)=E^x(A^\varepsilon_\infty),\quad u_H(x)=E^x(A^{H,\varepsilon}_\infty),\quad x\in D_1.$$

Our first aim in this section will be to prove the following result.

Proposition 8.1.

There exist two constants $C>0$ and $\varepsilon_0>0$ such that

$$\| u_H - G^1_{\lambda_H} \| \leq C \varepsilon^{1/6} (\ln 1/\varepsilon)^{7/6},$$

for any $\varepsilon \in (0, \varepsilon_0)$, where $\lambda_H = 1_H \cdot \lambda$. Moreover the constants C and ε_0 are independent of H (i.e. of the values η'_i, η''_i).

In order to prove the above estimate we need some preparations. We begin by establishing the following notation:

$$H_\tau = \{x \in R^d : |x^d| \leq 1/2 - \tau,\ x^i \in [\eta'_i + \tau, \eta''_i - \tau],\ i = 1, \ldots, d-1\},$$

$$N_\tau = \{x \in R^d : |x^d| \leq 1/2 - \tau, \text{ there exists } i \leq d-1 \text{ such that } x^i \in R \setminus (\eta'_i - \tau, \eta''_i + \tau)\},$$

$$M_\tau = \{x \in R^d : |x^d| \leq 1/2 - \tau,\ x \notin H_\tau \cup N_\tau\},$$

$$E_\tau = \{x \in R^d : |x^d| < 1/2 + \tau,\ x^i \in (\eta'_i - \tau, \eta''_i + \tau),\ i = 1, \ldots, d-1\}$$

$$E = \{x \in R^d : |x^d| < 1,\ x^i \in (\eta'_i - 1, \eta''_i + 1),\ i = 1, \ldots, d-1\}.$$

The number τ is arbitrary in $[0, 1/4]$, so that all these sets are contained in D_1. Since we may have $\eta'_i = -\infty$ or $\eta''_i = +\infty$ we should mention that in the above notation the usual conventions $\infty \pm \tau = \infty$, $-\infty \pm \tau = -\infty$ are in force. We also observe that the case H=K is trivial, because then estimate (3.13) produces a sharper result. Thus Proposition 8.1 is interesting only when one of the values η'_i or η''_i, $i = 1, \ldots, d-1$ is finite. Next we will establish two lemmas.

<u>Lemma 8.2</u>

Let $p(x) = E^x(T_{H_0} < T_{D_1 \setminus E})$. Then there exists a constant $\gamma > 0$ independent of H such that

$$p(x) \leq 1 - \gamma(|x^d| - 1/2), \quad x \in E,\ |x^d| > 1/2$$

$$p(x) \leq 1 - \gamma(x^i - \eta''_i), \quad x \in E,\ x^i > \eta''_i,\ i \leq d-1$$

$$p(x) \leq 1-\gamma(\eta_i^{\prime}-x^i), \quad x\in E, \; x^i<\eta_i^{\prime}, \; i\leq d-1.$$

Proof of the lemma

The proofs of all three estimates are similar. So we are going to check only the last one. Assume that $\eta_i^{\prime}>-\infty$, for some $i\leq d-1$ and let us look at the strip $B=\{x\in R^d : x^i\in(\eta_i^{\prime}-1,\eta_i^{\prime})\}$. Let h be an L-harmonic function in B such that $h(x)=1$ if $x^i=\eta_i^{\prime}$ and $h(x)=0$ if $x^i=\eta_i^{\prime}-1$. By Lemma 4.3 we get a constant γ (independent of the strip) such that

$$h(x)\leq 1-\gamma(\eta_i^{\prime}-x^i), \quad x\in B. \tag{8.1}$$

But one easily observe that the function p, which is also L-harmonic in $E\cap B$, is dominated by h on the boundary $\partial(E\cap B)$. By the maximum principle we have $p\leq h$ in $E\cap B$. Thus (8.1) gives us the estimate in the statement of the lemma.

Lemma 8.3

There exist two families of excessive functions $\{r_\tau : \tau\in(0,1/4)\}$ and $\{r_\tau^{\prime} : \tau\in(0, 1/4)\}$ such that

$$r_\tau - P_{H_0} r_\tau \geq \gamma\tau \qquad \text{on } D_1\setminus E_\tau$$

$$r_\tau^{\prime} - P_{D_1\setminus E_0} r_\tau^{\prime} \geq \gamma\tau \qquad \text{on } H_\tau,$$

$$\|r_\tau\|\leq C, \; \|r_\tau^{\prime}\|\leq C,$$

where γ and C are strictly positive constants, independent of τ and H

Proof of the lemma

Let ν be the measure which gives the representation

$p=G_y^E$ for the function of the preceding lemma. Then we put $q=G_\nu^1$ and claim that this function is bounded by a constant independent of H and satisfy

(8.2) $$q - P_{D_1 \setminus E_\tau} q \geq \gamma \tau \qquad \text{on } H_o,$$

with γ furnished by the preceding lemma. Let us prove these facts. Let μ be the Lebesgue measure in R^d and $\mu(x,r)=1_{B(x,r)}\cdot\mu$. From the estimations (1.13) and (1.14) we get a constant $C>0$ such that

$$G_{\mu(x,r)}(y) \leq C r^d,$$

provided that $x,y \in R^d$ are such that $|x-y| \geq 1/4+r$ and

$$G_{\mu(x,r)}(x) \geq C^{-1} r^2,$$

for any $x \in R^d$. Therefore, on account of relation (1.21) we deduce

$$G^E_{\mu(x,r)}(x) \geq r^2 C^{-1}(1-C^2 r^{d-2}),$$

as soon as $r<1/4$ and $x \in H_o$. Thus we can choose a constant k such that $kG_\mu^E \geq 1$ on H_o, and hence $kG_\mu^E \geq p$. By Lemma 6.1 we have $kG_\mu^1 \geq q$. As we have seen in the proof of Lemma 5.11 the function G_μ^1 is bounded. Thus q is bounded. To prove estimation (8.2) we observe that, for $x \in H_o$, we have

$$q(x) - P_{D_1 \setminus E_\tau} q(x) = p(x) - P_{D_1 \setminus E_\tau} p(x).$$

Then by Lemma 8.2 we deduce

$$P_{D_1\setminus E_\tau}p(x)\leqslant 1-\gamma\tau .$$

Since $p=1$ on H_o, this relation may be written as

$$p(x)-P_{D_1\setminus E_\tau}p(x)\geqslant\gamma\tau ,$$

which proves (8.2).

Now we define $r_\tau=P_{D_1\setminus E_\tau}q+\gamma\tau$. Obviously these functions are uniformly bounded. Relation (8.2) shows that $q\wedge r_\tau=$ $=r_\tau$ on H_o, and hence

$$P_{H_o}r_\tau\leqslant q\wedge r_\tau .$$

On the other hand $r_\tau=q+\gamma\tau$ on $D_1\setminus E_\tau$, which implies

$$P_{H_o}r_\tau\leqslant q \text{ on } D_1\setminus E_\tau.$$

Writting $q=r_\tau-\gamma\tau$, this inequality becomes the estiamte of the statement.

To construct the functions r_τ' we begin with $p_\tau(x)=E^x(T_{H_\tau}<T_{D_1\setminus E})$. The proofs of Lemmas 4.3 and 8.2 show that

$$p_\tau\leqslant 1-\gamma\tau \qquad \text{on } E\setminus E_o,$$

with a suitable γ. Thus we have

$$P_{D_1\setminus E_o}p_\tau(x)\leqslant 1-\gamma\tau, \quad x\in E,$$

and hence

$$p_\tau - P_{D_1\setminus E_0} p_\tau \geqslant \gamma\tau \qquad \text{on } H_\tau.$$

Let ν_τ be the measure which express the function p_τ as $p_\tau = G^E_{\nu_\tau}$. Then we put $r'_\tau = G^1_{\nu_\tau}$. The preceding inequality leads to that appearing in the statement. To check the boundedness of the family $\{r'_\tau\}$ one should remark that the function p of Lemma 8.2 satisfy $p \geqslant p_\tau$ in E. Then Lemma 6.1 implies $q \geqslant r'_\tau$ in D, which completes the proof.

Proof of Proposition 8.1

Let us denote by ν the measure which represents the function u as a Green potential $u = G^1_\nu$. We know that ν is supported by the set $\{x \in R^d : |x^d| \leqslant \varepsilon\}$. We choose $\varepsilon < 1/4$, so that ν will be allways supported by the set $H_\tau \cup M_\tau \cup N_\tau$. Let B be the functional given by

$$B_t = \int_{[0,t]} 1_{K\setminus H}(X_t)\, dA^\varepsilon_t ,$$

and set $v(x) = E^x(B_\infty)$, $x \in D_1$. Since $A^\varepsilon = A^{H,\varepsilon} + B$ we have $u = u_H + v$ and consequently

$$u_H = G^1_{\varphi\cdot\nu} , \qquad v = G^1_{\psi\cdot\nu} ,$$

where φ, ψ are Borel measurable nonnegative functions such that $\varphi + \psi = 1$. Let us put $\pi = 1_{N_\tau}\cdot\varphi\cdot\nu$ and look at the potential G^1_π. We have

$$G^1_\pi - P_H G^1_\pi \leqslant u_H - P_H u_H \leqslant \varepsilon ,$$

where the last inequality is given by Lemma 2.4. Since $H \subset H_0$ we obtain $G^1_\pi - P_{H_0} G^1_\pi \leqslant \varepsilon$. Then by Lemma 8.3 we deduce

$$G^1_{\pi} - P_{H_o} G^1_{\pi} \leqslant (\varepsilon/\gamma\tau)(r_\tau - P_{H_o} r_\tau) \text{ on } D_1 \setminus E_\tau$$

and from Lemma 6.1 we get $G^1_{\pi} \leqslant (\varepsilon/\gamma\tau) r_\tau$. Finaly we may write

(8.3) $\quad G^1_{\pi} \leqslant C\varepsilon/\tau.$

Further we put $\pi' = 1_{H_\tau} \cdot \psi \cdot \nu$ and similarly deduce

(8.3') $\quad G^1_{\pi'} \leqslant C\,\varepsilon/\tau.$

The proof of this inequality begin with

$$G^1_{\pi'} - P_{K\setminus E_o} G^1_{\pi'} \leqslant v - P_{K\setminus E_o} v \leqslant v - P_{K\setminus H} v \leqslant \varepsilon .$$

Since $P_{D_1\setminus E_o} G^1_{\pi'} \geqslant P_{K\setminus E_o} G^1_{\pi'}$ we get

$$G^1_{\pi'} - P_{D_1\setminus E_o} G^1_{\pi'} \leqslant \varepsilon ,$$

$$G^1_{\pi'} - P_{D_1\setminus E_o} G^1_{\pi'} \leqslant (\varepsilon/\gamma\tau)(r'_\tau - P_{D_1\setminus E_o} r'_\tau) \text{ on } H_\tau .$$

Then, using Lemma 6.1 we get the estimate (8.3').

Further we choose two families of functions $\{f_\tau : \tau \in (0,1/4)\}$, $\{h_\tau : \tau \in (0,1/4)\}$ such that, $f_\tau, h_\tau \in \mathcal{C}^2(D_1)$ $f_\tau = 1$ on H_τ, $f_\tau = 0$ on N_τ, $h_\tau = 1$ on M_τ, $h_\tau = 0$ on $D_1 \setminus M_{2\tau}$, $0 \leqslant f_\tau \leqslant 1$, $0 \leqslant h_\tau \leqslant 1$ in D_1 and

$$\| f \|_2 \leqslant C\tau^{-2}, \| h_\tau \|_2 \leqslant C\tau^{-2} ,$$

with a constant C independent of τ and H. Then we may write

$$\varphi = f_\tau - 1_{H_\tau} \psi + 1_{M_\tau}(\varphi - f_\tau) + 1_{N_\tau} \cdot \varphi ,$$

$$\varphi \cdot \nu = f_\tau \cdot \nu - \pi' + \pi + 1_{M_\tau}(\varphi - f_\tau) \cdot \nu ,$$

and hence

$$|u_H - G^1_{\lambda_H}| \le |G^1_{f_\tau \cdot \nu} - G^1_{f_\tau \cdot \lambda}| + G^1_{\pi} + G^1_{\pi'} + G^1_{h_\tau \cdot \nu} + G^1_{\xi},$$

where we set $\xi = 1_{M_\tau} \cdot \lambda$. Using Lemma 8.4 from below and estimates (8.3) and (8.3') we see that the right hand side of this inequality is dominated by

$$|G^1_{f_\tau \cdot \nu} - G^1_{f_\tau \cdot \lambda}| + |G^1_{h_\tau \cdot \nu} - G^1_{h_\tau \cdot \lambda}| + C(\varepsilon\tau^{-1} + \tau \ln \tau^{-1}).$$

Then using Lemma 8.5 from below we get

$$|u_H - G^1_{\lambda_H}| \le C\tau^{-2}\sqrt{\varepsilon}(\ln \varepsilon^{-1})^{3/2} + C(\varepsilon\tau^{-1} + \tau \ln \tau^{-1}).$$

Taking $\tau = (\varepsilon \ln \varepsilon^{-1})^{1/6}$ we obtain the estimate of the statement.

Lemma 8.4

For $a \in R$, $\tau > 0$ and $i \in \{1,\dots,d-1\}$ set $A(a,\tau,i) = \{x \in R^d : x^d = 0, |x^i - a| \le \tau\}$ and $\mu(a,\tau,i) = 1_{A(a,\tau,i)} \cdot \mu$, where μ is Lebesgue measure in K. Then there exists a constant $C > 0$ independent of a, τ, such that

$$\| G^1_{\mu(a,\tau,i)} \| \le C\tau \ln \tau^{-1}, \quad \tau \in (0,1/4).$$

Proof

In order to prove the lemma we will compare the Green function g^1 with that corresponding to the operator $\tilde{L} = 1/2\Delta$. Let $\tilde{g}^1$ be the Green funciton in D_1 associated to $\tilde{L}$. We claim that

$$(8.4) \qquad g^1(x,y) \le C\tilde{g}^1(x,y), \qquad x,y \in D_1,$$

with a constant $C > 0$. Here use the fact that $L = 1/2\Delta$ outside a compact set M. First we remark that relation (1.19) allows us to estimate the function $\tilde{g}^1(x,y) \longleftrightarrow$ so that to deduce

$$|x-y|^{2-d} \leq C\tilde{g}^1(x,y)$$

for x,y in a neighbourhood of M. By estimate (1.13) we deduce that

$$g^1(x,y) \leq C\tilde{g}^1(x,y), \tag{8.5}$$

for x,y in a neighbourhood of M. For fixed x, both functions $g^1(x,y)$ and $\tilde{g}^1(x,.)$ are harmonic in $D_1 \setminus M$ and vanish at the boundary ∂D_1. Therefore, by the maximum principle we deduce that the inequality (8.5) holds for any $y \in D_1$.

Again by relation (1.19) we deduce that there exists a constant $C>0$ such that

$$|x-y|^{2-d} \leq C\tilde{g}^1(x,y)$$

for any $y \in D_1$ and any $x \in V_y$, where V_y is a neighbourhood of y. Thus we deduce

$$g^1(x,y) \leq C\tilde{g}^1(x,y), \quad y \in D_1, \quad x \in V_y.$$

Taking the best constant we deduce from this inequality and (8.5) that

$$g^1(.,y) \leq C\tilde{g}^1(.,y) \text{ on } M \cup V_y.$$

Since the adjoint L^* also coincides with $\tilde{L}^* = \tilde{L}$ in $D_1 \setminus M$, the function $g^1(.,y)$ is harmonic (as well as $\tilde{g}^1$) in $D_1 \setminus M \cup V_y$. By the maximum principle we get the estimate (8.4).

Now we estimate $\tilde{g}^1$ using relation (5.4) and obtain

$$g^1(x,y) \leqslant C\tilde{g}^1(x,y) \leqslant C|x'-y'|^{2-d}(|x'-y'|+16)^{-1}$$

where $x,y \in D_1$ and $x=(x',x^d)$, $y=(y',y^d)$. Thus we get

$$\| G^1_{\mu}(a,\tau,i) \| \leqslant C \int_{O_\tau} |y'|^{2-d}(|y'|^2+16)^{-1}dy',$$

where $O_\tau=\{y' \in R^{d-1} : |y^{d-1}|<\tau\}$. The inequality asserted by the lemma follow by direct computations.

<u>Lemma 8.5.</u>

With the notation in the proof of Proposition 8.1 we have

$$|G^1_{f,\nu} - G^1_{f,\lambda}| \leqslant C(\|f\|_2+1)\varepsilon^{1/2}(\ln \varepsilon^{-1})^{3/2},$$

for each $f \in \mathcal{C}^2(D_1)$ and $\varepsilon \in (0,\varepsilon_0)$. Here C and ε_0 are constants.

The proof of this lemma is exactly the same as that given for the inequality (7.18). It is based on the fact that the adjoint L^* is also of the form (1.2).

Now we go further and consider a functional with density $f \in \mathcal{B}_b(K)$ defined as follows:

$$A^{f,\varepsilon}_t = \int_{[0,t]} f(X_1)\,dA^\varepsilon_s \ .$$

The expression on the right hand side make sense because the functional A^ε charges only the moments t such that $X_t \in K$. If $f \geqslant 0$, the functional $A^{f,\varepsilon}$ is increasing. The potential of this functional has the expression

(8.6) $\quad E^x(A^{f,\varepsilon}_\infty)=\varepsilon\sum M^n f(x), \quad x \in D_1,$

where $Mf(x)=E^x(f(X(T_1)))$, with T_1 defined in Section 2. The proof of this formula is the same as that given for relation (3.7'). Let us recall that, for $f\in\mathcal{C}^o(K)$, the function Mf may be described analyticaly as follows: denote by h the L-harmonic function in $D_1\setminus K$ satisfying the boundary conditions

$$h(x',0)=f(x',0),\ h(x',1)=h(x',-1)=0,\ x'\in R^{d-1}.$$

Then denote by l the L-harmonic function in D_ε which satisfies the boundary condition $l(x)=h(x)$ for $x\in\partial D_\varepsilon$. The function Mf takes the values $Mf(x)=h(x)$ if $x\in D_1\setminus D_\varepsilon$ and $Mf(x)=l(x)$ if $x\in D_\varepsilon$.

Further we shall denote $N^\varepsilon=\sum_{n=1}^{\infty} M^n$ and, as it was proved at (3.8), we have a constant $\gamma>0$ such that $\|N^\varepsilon\|\leq(\gamma\varepsilon)^{-1}$. Because of the above analytical interpretation the following proposition and its corollary may be throught of as a result of partial differential equations.

Proposition 8.6.

If f is continuous with compact support in K, then there exist $C>0$ and $\varepsilon_o>0$ such that for each $\varepsilon\in(0,\varepsilon_o)$ and $\delta\in(0,1)$ the following inequality holds

$$\|\varepsilon N^\varepsilon f-G^1_{f,\lambda}\|\leq C\Gamma(f)^d\delta^{-d}\varepsilon^{1/6}(\ln\varepsilon^{-1})^{7/6}\|f\|+C\omega(f,\delta),$$

Where $\Gamma(f)$ denotes the diameter of supp f and $\omega(f,\delta)$ the oscillation: $\omega(f,\delta)=\sup\{|f(x)-f(y)| : x,y\in K,\ |x-y|<\delta\}$.

Proof

For $\delta>0$ and $k=(k^1,\dots,k^{d-1})\in Z^{d-1}$ we set

$$H(\delta,k)=\{x\in R^{d-1}: k^i\delta(d-1)^{-1/2}\leqslant x^i<(k^i+1)\delta(d-1)^{-1/2}, i=1,\dots,d-1\}$$

Then the diameter of $H(\delta,k)$ is δ and identifying $R^{d-1}\supseteq K$ we may write $K=\bigcup_k H(\delta,k)$. Let $\{H(\delta,k_j):j=1,\dots,n\}$ be the family of those cubes which have non-empty intersection with supp f. The number n is less than $(\Gamma(f)\delta^{-1}(d-1)^{1/2}+1)^d$. We put

$$\varphi=\sum_{j=1}^{n} f(\delta(d-1)^{-1/2}k_j)\cdot 1_{H(\delta,k_j)}$$

and consequently deduce $\|f-\varphi\|\leqslant\omega(f,\delta)$, $\|\varepsilon N^{\varepsilon}(f-\varphi)\|\leqslant\delta^{-1}\omega(f,\delta)$, $\|G^1_{f\cdot\lambda}-G^1_{\varphi\cdot\lambda}\|\leqslant G^1_{\lambda}\omega(f,\delta)$.

Further we intend to apply Proposition 8.1 with respect to a cube $H(\delta,k)$. In order to do so we first have to remark that the boundary of $H(\delta,k)$ is a polar set. Thus, by Proposition 8.1 we get

$$\|\varepsilon N^{\varepsilon}\varphi-G^1_{\varphi\cdot\lambda}\|\leqslant Cn\|f\|\varepsilon^{1/6}(\ln\varepsilon^{-1})^{7/6},$$

which combined with the preceding estimates leads to the inequality in the statement.

Corollary 8.7

If $f\in\mathscr{C}(R^{d-1})$ is such that $\lim_{|x|\to\infty} f(x)=0$, then the following relation holds $\lim_{\varepsilon\to 0}\|\varepsilon N^{\varepsilon}f-G^1_{f\cdot\lambda}\|=0.$

9. The case of a half-hyperplan

In this section L is supposed to be an operator satisfying the hypotheses of the preceding section. We preserve also the notation except that this time H will be a fixed half of hyperplan $H=\{x\in K: x^{d-1}\leq 0\}$. We are going to study the oscilations of the process X near H. Besides the neighbourhoods D_ε of K we consider a family $(F_\varepsilon)_{\varepsilon\in(0,1)}$ of open neighbourhoods of H which is assumed to possess the following properties:

(9.1) i) $F_\varepsilon \subset D_\varepsilon$

ii) $\{x\in R^d : x^{d-1}\leqslant 0, |x^d|<\varepsilon\}\subset F_\varepsilon$

iii) if $x\in\partial F_\varepsilon \cap D_\varepsilon$, then the distance to H, $d(x,H)$ satisfies the estimates $\tau\varepsilon\leqslant d(x,H)\leqslant\tau^{-1}\varepsilon$, with a constant $\tau\in(0,1)$ independent of ε.

iv) the boundary ∂F_ε is smooth.

An example, which suggests how the sets F_ε are, is obtained by taking $F_\varepsilon=\{x\in R^d: d(x,H)<\varepsilon\}$. We need the following stopping times: $T=T_K$, $S=T_{D_1\setminus D_\varepsilon}$, $R=T_H$, $Q=T_{D_1\setminus F_\varepsilon}$, $T_o=0$, $T_1=S+T\circ\theta_S$, $T_{n+1}=T_n+T_1\circ\theta(T_n)$, $R_o=0$, $R_1=S+R\circ\theta_S$, $R_{n+1}=R_n+R_1\circ\theta(R_n)$, $Q_o=0$, $Q_1=Q+R\circ\theta_Q$, $Q_{n+1}=$ $=Q_n+Q_1\circ\theta(Q_n)$. The functional we are intersected in is $C^\varepsilon=(C^\varepsilon_t)$ given by

$$C^\varepsilon_t=\varepsilon\sum_{n=1}^{\infty}1_{\{Q_n\leq t\}}.$$

Of course the new aspect of the problem is produced by the boundary of H. We will compare the functional C^ε with the functional $A^{H,\varepsilon}$ introduced in the preceding section. It turns out that both functionals have the same limit. To prove this we need another functional B^ε defined by

$$B_t^{\varepsilon}=\varepsilon\sum_{n=1}^{\infty} 1_{\{R_n\leq t\}},$$

which will help us in comparing $A^{H,\varepsilon}$ with C^{ε}. The next lemma is a deterministic result, which can be proved by direct manipulation of the stopping times using the methods of Section 2, particularly relation (2.5).

Lemma 9.1

Let ω be such that $t\to X_t(\omega)$ is continuous and $k\geq 1$ such that $X_{T_k}(\omega)\in H$. Then there exists $l\geq 1$ such that $R_l(\omega)=T_k(\omega)$.

From this lemma we immediately deduce the first inequality of (9.2) from below. The second inequality is similar:

(9.2) $$A_{\infty}^{H,\varepsilon}\leq B_{\infty}^{\varepsilon}\leq C_{\infty}^{\varepsilon} \quad \text{a.s.}$$

The reminder of this section is devoted to the proof of the asymptotic equivalence of the three functionals.

Lemma 9.2

For $a\in R$ and $\varepsilon>0$ set $D_{\varepsilon,a}=\{x\in R^d: |x^d|<\varepsilon, x^{d-1}>a\}$ and $T_a=\inf\{t>0: :X_t^{d-1}=a\}$. Then there exists C,δ and $\varepsilon_o>0$ such that

$$P^x(T_a<S)\leq C\exp(-(x^{d-1}-a)\delta\varepsilon^{-1}),$$

for each $x\in D_{\varepsilon,a}, \varepsilon\in(0,\varepsilon_o)$ and $a\in R$.

Proof

We put $f(x)=(2-(x^d\varepsilon^{-1})^2)\,\mathrm{expt}(-(x^{d-1}-a)\delta\varepsilon^{-1})$ and compute

$$Lf(x)=\varepsilon^{-1}\exp(-(x^{d-1}-a)\delta\varepsilon^{-1})\{\varepsilon^{-1}[-2a^{dd}+4\delta x^d\varepsilon^{-1}a^{d,d-1}+$$

$$+\delta(2-(x^d\varepsilon^{-1})^2)a^{d-1,d-1}] + [-2b^d x^d \varepsilon^{-1} - b^{d-1}\delta(2-(x^d\varepsilon^{-1})^2)]\}.$$

Since a^{dd} has a strictly positive lower bound and $|x^d|<\varepsilon$, it follows that we may choose δ and ε_o small enough so that $Lf\leqslant 0$ in $D_{\varepsilon,a}$ for each $\varepsilon\in(0,\varepsilon_o)$. On the boundary of $D_{\varepsilon,a}$ the function f satisfies the conditions:

— $f(x)\geqslant 1$ if $x^{d-1}=a$ and $|x^d|\leqslant\varepsilon$,

— $f(x)\geqslant 0$ for any $x\in\partial D_{\varepsilon,a}$.

The maximum principle shows that the function $u(x)=P^x(T_a<S)$ satisfies the inequality $u\leqslant f$ in $D_{\varepsilon,a}$. This implies the estimate stated by the lemma.

Further we will use the notation $H_\beta=\{x\in K: x^{d-1}\leqslant\beta\}$ for $\beta\in R$ and $H_{-\infty}=\emptyset$. If $\beta=0$ then $H_o=H$. If M is a subset of H then $B^{M,\varepsilon}$ will denote the functional

$$B_t^{M,\varepsilon} = \int_{[0,t]} 1_M(X_s)\,dB_s^\varepsilon .$$

__Lemma 9.3__

There exist the constants $C>0, \delta>0, \beta_o>0, \varepsilon_o$ (0,1) such that

$$E^x(B^{H\setminus H_\gamma,\varepsilon}) \leqslant E^x(A_\infty^{H_\beta\setminus H_{\gamma-\beta},\varepsilon}) + C\exp(-\delta\beta\varepsilon^{-1})E^x(A_\infty^{K\setminus(H_\beta\setminus H_{\gamma-\beta}),\varepsilon}),$$

for each $x\in D_1$, $\gamma\in[-\infty,0)$, $\beta\in(0,\beta_o)$, $\varepsilon\in(0,\varepsilon_o)$.

__Proof__

For each $k\geqslant 1$ we put $n_k=n(\omega,k)=:\sup\{n: T_n(\omega)\leqslant R_k(\omega)\}$. Because $R_1\geqslant T_1$ we have $n_1\geqslant 1$, a.s. Also one easily see that the following inclusion holds almost surely

(9.3) $\{R_k<\infty\}\subset\{R_k<T_{n_k+1}\leqslant R_{k+1}\}$.

By deterministic arguments, looking at the continuous trajectories of the process one can deduce the inclusion

$$\{R_k<\infty,\ X(R_k)\in H\setminus H_\gamma,\ X(T_{n_k})\in K\setminus(H\setminus H_\gamma)\}\subset$$
$$\subset\{S\circ\theta(T_{n_k})>T_{H\setminus H_\gamma}\circ\theta(T_{n_k})\},$$

which holds almost surely. From this inclusion one easily deduce (a.s.)

$$\{R_k<\infty,\ X(R_k)\in H\setminus H_\gamma\}\subset\{T_{n_k}<\infty,\ X(T_{n_k})\in H_\beta\setminus H_{\gamma-\beta}\}\cup$$
$$\cup\{T_{n_k}<\infty,\ X(T_{n_k})\in K\setminus(H_\beta\setminus H_{\gamma-\beta}),\ T_{H\setminus H_\gamma}\circ\theta(T_{n_k})<S\circ\theta(T_{n_k})\}.$$

Then this inclusion and (9.3) implies (a.s.)

$$\sum_{k=1}^{\infty}1_{\{R_k<\infty,\ X(R_k)\in H\setminus H_\gamma\}}\le\sum_{n=1}^{\infty}1_{\{T_n<\infty,\,X(T_{n_k})\in H_\beta\setminus H_{\gamma-\beta}\}}+$$
$$+\sum_{n=1}^{\infty}1_{\{T_n<\infty,\ X(T_n)\in K\setminus(H_\beta\setminus H_{\gamma-\beta}),\ T_{H\setminus H_\gamma}\circ\theta(T_n)<S\circ\theta(T_n)\}}.$$

Taking the expectation we get

$$E^x(B_\infty^{H\setminus H_\gamma,\varepsilon})\le E^x(A_\infty^{H_\beta\setminus H_{\gamma-\beta},\varepsilon})+\Sigma,$$

where we write Σ for the sum

$$\sum_{n=1}^{\infty}P^x(\{T_n<\infty,X(T_n)\in K\setminus(H_\beta\setminus H_{\gamma-\beta}),T_{H\setminus H_\gamma}\circ\theta(T_n)<S\circ\theta(T_n)\}).$$

By the strong Markov property, the general term of this sum may be written as

$$E^x(P^{X(T_n)}(T_{H\setminus H_\gamma}<S);T_n<\infty,\ X(T_n)\in K\setminus(H_\beta\setminus H_{\gamma-\beta})).$$

[illegible] 9.2 we deduce

$$P^x(T_{H\setminus H_\gamma}<S)\leq C\exp(-\beta\delta\varepsilon^{-1}),$$

for each $x\in K\setminus(H_\beta\setminus H_{\gamma-\beta})$. Therefore the general term of Σ is dominated by

$$C\exp(-\beta\delta\varepsilon^{-1})P^x(T_n<\infty, X(T_n)\in K\setminus(H_\beta\setminus H_{\gamma-\beta})).$$

This leads to the inequality stated by the lemma.

Lemma 9.4

There exist two constants $\theta\in(0,1)$ and $\varepsilon_o\in(0,1)$ such that

$$P^x(R<S)\leq\theta,$$

for each $x\in F_\varepsilon\cap D_\varepsilon$ and $\varepsilon\in(0,\varepsilon_o)$.

Proof

We put $u_\varepsilon(x)=P^x(S<R)$, $x\in D_\varepsilon$. One easily see that the inequality we have to prove is equivalent to

(9.4) $0<\inf\{u_\varepsilon(x):x\in\partial F_\varepsilon\cap D_\varepsilon,\varepsilon\in(0,\varepsilon_o)\}$.

The function u_ε is L - harmonic in $D_\varepsilon\setminus H$ and satisfies the boundary conditions

$$u_\varepsilon(x)=1 \text{ if } |x^d|=\varepsilon \quad\text{and}\quad u_\varepsilon(x)=0 \text{ if } x\in H.$$

To prove relation (9.4) we introduce the sets

$$E'=\{x\in R^d: |x^d|<1, |x^i|<2\tau^{-1},\ i=1,\dots,d-1\},$$

$$\Gamma=\{x\in R^d: x^i=0,\ i=1,\dots,d-2,\ 0\leq x^{d-1}, |x^d|\leq 1, \tau\leq |x|\leq \tau^{-1}\},$$
$$\Lambda=\{x\in R^d: |x^d|=1,\ |x'|\leq \tau^{-1}\},$$

where τ is the constant appearing in (9.1.iii). Then we choose a domain E with boundary of class $\mathcal{C}^\infty$ such that $E\subset E'$, $E\cap H=\emptyset$, $\Gamma\cap E=\Gamma\cap D_1$ and $\Lambda\subset\partial E$. Then we have $\Gamma\cap\partial E=\Gamma\cap\partial D_1\subset\Lambda$. Now for a point $z\in\partial F^\varepsilon\cap D_\varepsilon$ we set $y=(z'',0,0)$, $x=(0,\varepsilon^{-1}z^{d-1},\varepsilon^{-1}z^d)$ and conseauently we have $z=\varepsilon x+y$. Condition (9.1.iii) implies $x\in\Gamma$. Further we introduce the functions

$$v_\varepsilon^y(x)=u_\varepsilon(\varepsilon x+y),\ x\in E,$$

with y of the form $y=(y'',0,0)$, $y''\in R^{d-2}$. Such a function is L_ε^y - harmonic with L_ε^y given by

$$L_\varepsilon^y u(x)=\sum_{ij=1}^{d} a^{ij}(\varepsilon x+y)D_{ij}u(x)+\varepsilon\sum_{k=1}^{d} b^k(\varepsilon x+y)D_k u(x).$$

Each function v_ε^y satisfies the following boundary conditions

$$v_\varepsilon^y\geq 0 \text{ on } \partial E \text{ and } v_\varepsilon^y=1 \text{ on } \Lambda.$$

Now, let f be a fixed function $f\in\mathcal{C}^\infty(\partial E)$ such that f=1 on $\Gamma\cap\partial E$ and f=0 on $\partial E\setminus\Lambda$. We define the function w_ε^y such that it is L_ε^y-harmonic in E and $w_\varepsilon^y=f$ on ∂E. Since $f\leq v_\varepsilon^y$ on ∂E, we get $w_\varepsilon^y\leq v_\varepsilon^y$ in E. Therefore relation (9.4) follows once we have proved that

(9.5) $0<\inf\{w^y(x): x\in\Gamma, \varepsilon\in(0,\varepsilon_0), y=(y'',0,0), y''\in R^{d-2}\}$.

In order to check this relation we are going to approximate each function w_ε^y by a function w^y which is choosen to be L^y-harmonic in E with

$$L^y = \sum_{i,j=1} a^{ij}(y) D_{ij}$$

(which has constant coefficients) and to verify the boundary condition $v^y = f$ on ∂E. Then we assert that there exists a constant C independent of y and ε such that

$$(9.6) \qquad |w_\varepsilon^y - w^y| \leqslant C\varepsilon$$

Now let us prove it. First we remark that the family of operators (L_ε^y) possess the same constant for the Schauder estimates and hence

$$\| w_\varepsilon^y \|_2 \leqslant \| w_\varepsilon^y \|_{2+\alpha} \leqslant C.$$

Then computing

$$L^y(w_\varepsilon^y - w^y)(x) = \sum_{i,j=1}^{d} (a^{ij}(y) - a^{ij}(y+\varepsilon x)) D_{ij} w_\varepsilon^y(x) + \varepsilon \sum_{k=1}^{d} b^k(y+\varepsilon x) D_k w_\varepsilon^y(x)$$

we deduce

$$|L^y(w_\varepsilon^y - w^y)(x)| \leqslant C\varepsilon .$$

Further we take $h(x) = a(1-(x^d)^2)$ and compute

$$L^y h = -2aa^{dd}(y).$$

Choosing a large enough we obtain $L^y h \leqslant -1$ in D_1. By the maximum principle one gets $|w_\varepsilon^y - w^y| \leqslant C\varepsilon h$, which implies the estimate (9.6).

Now we observe that, on account of inequality (9.6), relation (9.5) follows, with a suitable ε , from the next relation

(9.7) $0<\inf\{w^y(x): x\in\Gamma,\ y=(y'',0,0),\ y''\in R^{d-2}\}$.

So we have to prove this relation. Let $M(C)$ be the set of all constant matrices $U=(u^{ij})$ with d columns and d rows which satisfy the relation

$$C^{-1}|\xi|^2 \leqslant \sum_{ij} u^{ij}\xi^i\xi^j \leqslant C|\xi|^2,\quad \xi\in R^d,$$

with a constant $C>1$. The set $M(C)$ is compact, as a subset of R^{d^2}. For each $U\in M(C)$ we denote by $w(x,U)$ the function which is L^U-harmonic in E with respect to the operator

$$L^U = \sum_{i,j=1}^{d} u^{ij} D_{ij}\ ,$$

and satisfies the boundary condition $w(x,U)=f(x)$ for $x\in\partial E$. A standard argument (as above) shows that $w:\bar{E}\times M(C)\longrightarrow R$ is a continuous function. Also, we have $w(x,U)=f(x)=1$ for each $U\in M(C)$ and $x\in\Gamma\cap\partial E$. Moreover $w(x,U)>0$ provided that $x\in E$, because E is connex. We conclude that w is strictly positive on $\Gamma\times M(C)$. Since this set is compact we deduce

$$0<\inf\{w(x,U): x\in\Gamma,\ U\in M(C)\}\ ,$$

which implies (9.7). The proof is complete.

<u>Lemma 9.5</u>

There exist the constants $\theta\in(0,1)$, $\delta>0$, $C>0$, $\varepsilon_o\in(0,1)$ and $\beta_o>0$ such that

$$E^x(C_\infty^\varepsilon)\leqslant E^x(B_\infty^\varepsilon) +$$
$$+\theta(1-\theta)^{-1}(\varepsilon+E^x(B_\infty^{H\setminus H_{-\beta},\varepsilon})+C\exp(-\delta\beta\varepsilon^{-1})E^x(B_\infty^{H_{-\beta},\varepsilon}))$$

for each $x \in D_1, \varepsilon \in (0,\varepsilon_o)$, $\beta \in (0,\beta_o)$.

Proof

In order to distinguish the functionals B^ε and C^ε we introduce the stopping time Q_1' defined by $Q_1'(\omega)=Q_1(\omega)$ if $Q_1(\omega)<S(\omega)$ and $Q_1'(\omega)=\infty$ if $Q_1(\omega)\geq S(\omega)$. Then we set $Q_{n+1}'=Q_n'+Q_1'\circ\theta(Q_n')$ and assert that for each ω such that the trajectory $t\to X_t(\omega)$ is continuous the following equality holds

$$\{Q_n(\omega), n\geq 1\} = \{R_n(\omega): n\geq 1\} \cup \bigcup_{k=0}^{\infty} \{R_k(\omega)+Q_n'\circ\theta(R_k)(\omega): n\geq 1\}.$$

We do not insist on the proof of this deterministic equality. We remark instead that the sets appearing in the right side are mutually disjoint. This shows that, almost surely, we have

$$C_\infty^\varepsilon = B_\infty^\varepsilon + \varepsilon \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} 1_{\{R_k+Q_n'\circ\theta(R_k)<\infty\}} . \tag{9.8}$$

Before going further and estimate the double sum in the right hand side, we will prove the following estimates .

$$P^x(Q_1'<\infty)\leq\theta , \quad x\in D_1 \tag{9.9}$$

$$P^x(Q_1'<\infty)\leq\theta C \exp(-\delta\beta\varepsilon^{-1}), \quad x\in H_{-\beta}, \tag{9.10}$$

where $\theta\in(0,1)$ is a constant. We may write

$$P^x(Q_1'<\infty)=P^x(Q + R\circ\theta_Q<S; Q<S).$$

On the set $\{Q<S\}$ we have $S=Q+S\circ\theta_Q$ and $X_Q\in\partial F_\varepsilon\cap D_\varepsilon$, which implies

$$P^x(Q'<\infty)=E^x(P^{X(Q)}(R<S); Q<S)\leq\theta P^x(Q<S).$$

For the last inequality we have used the preceding lemma. Relation (9.9) is proved. If $x \in H_{-\beta}$ we can dominate the last expression by using Lemma 9.2 so that we get relation (9.10).

Now let us calculate the expectation of the general term appearing in the double sum in the right side of (9.8):

$$(9.11) \qquad P^x(R_k + Q'_n \circ \theta(R_k) < \infty) =$$
$$= E^x(P^{X(R_k)}(Q'_n < \infty); R_k < \infty, X(R_k) \in H \setminus H_{-\beta}) +$$
$$+ E^x(P^{X(R_k)}(Q'_n < \infty); R_k < \infty, X(R_k) \in H_{-\beta}).$$

From the inequality (9.9) it follows

$$P^x(Q'_n < \infty) \leq \theta^n, \quad x \notin D_1$$

and from (9.10)

$$P^x(Q'_n < \infty) \leq \theta^n C \exp(-\delta \beta \varepsilon^{-1}), \text{ if } x \in H_{-\beta}.$$

Therefore the expression (9.11) is dominated by

$$\theta^n(P^x(R_k < \infty, X(R_k) \in H \setminus H_{-\beta}) + C \exp(-\delta \beta \varepsilon^{-1}) P^x(R_k < \infty, X(R_k) \in H_{-\beta})).$$

Then, taking the expectation in the equality (9.8) and using the above expression to dominate the general term of the sum, we obtain the inequality asserted by the lemma.

Lemma 9.5 and Lemma 9.3 together with relation (9.2) show that C^ε is asymptotic equivalent to $A^{H,\varepsilon}$. This can be seen in the proof of the following proposition. We denote by $u(x) = E^x(C^\varepsilon_\infty)$ and $\lambda_H = 1_H \cdot \lambda$, with λ introduced in Section 3.

Proposition 9.6

There exist two constants $C>0$ and $\varepsilon_o>0$ such that

$$\| u-G^{1}_{\lambda_H} \| \leqslant C\varepsilon^{1/6}(\ln \varepsilon^{-1})^{7/6},$$

for each $\varepsilon\in(0,\varepsilon_o)$.

Proof

On account of Proposition 8.1, the estimate follows from the next

$$\left| E^x(A^{H,\varepsilon}_{\infty})-E^x(C^{\varepsilon}_{\infty})\right| \leqslant C\varepsilon^{1/6}(\ln\varepsilon^{-1})^{7/6}, \quad x\in D_1 .$$

Then, because of relation (9.2), to prove this inequality it suffices to estimate

$$E^x(B^{\varepsilon}_{\infty})-E^x(A^{H,\varepsilon}_{\infty}) \text{ and } E^x(C^{\varepsilon}_{\infty})-E^x(B^{\varepsilon}_{\infty}).$$

Lemma 9.3 with $\gamma=-\infty$ $(H_\gamma=\emptyset)$ and arbitrary β yields

$$E^x(B^{\varepsilon}_{\infty})\leqslant E^x(A^{H_\beta,\varepsilon}_{\infty}) + C\exp(-\delta\beta\varepsilon^{-1}),$$

because $E^x(A^{K\setminus H_\beta,\varepsilon}_{\infty})\leqslant E^x(A^{\varepsilon}_{\infty})\leqslant C$. This leads to

$$E^x(B^{\varepsilon}_{\infty})-E^x(A^{H,\varepsilon}_{\infty})\leqslant E^x(A^{H_\beta\setminus H,\varepsilon}_{\infty})+C\exp(-\delta\beta\varepsilon^{-1}).$$

Then we use again Proposition 8.1 to estimate

$$E^x(A^{H_\beta\setminus H,\varepsilon}_{\infty})\leqslant G^{1}_{\lambda(\beta)} + C\varepsilon^{1/6}(\ln\varepsilon^{-1})^{7/6},$$

where $\lambda(\beta)=1_{H_\beta\setminus H}\cdot\lambda$. The potential $G^1_{\lambda(\beta)}$ may be estimated by Lemma 8.4 so that we get

$$(9.12)\quad E^x(B^\varepsilon_\infty)-E^x(A^{H,\varepsilon}_\infty)\leq C(\beta\ln\beta^{-1}+\varepsilon^{1/6}(\ln\varepsilon^{-1})^{7/6}+\exp(-\delta\beta\varepsilon^{-1}).$$

Further we use Lemma 9.5 and obtain

$$E^x(C^\varepsilon_\infty)-E^x(B^\varepsilon_\infty)\leq C(\varepsilon+E^x(B^{H\setminus H_{-\beta},\varepsilon}_\infty)+\exp(-\delta\beta\varepsilon^{-1})E^x(B^{H_{-\beta},\varepsilon}_\infty))$$

Then we apply Lemma 9.3 to evaluate the last term

$$E^x(B^{H_{-\beta},\varepsilon}_\infty)\leq E^x(B^\varepsilon_\infty)\leq C,$$

and also

$$E^x(B^{H\setminus H_{-\beta},\varepsilon}_\infty)\leq E^x(A^{H_\beta\setminus H_{-2\beta},\varepsilon}_\infty)+C\exp(-\delta\beta\varepsilon^{-1}).$$

The right side of this inequality is evaluated by Proposition 8.1 and Lemma 8.4, and hence we obtain

$$E^x(B^{H\setminus H_{-\beta},\varepsilon}_\infty)\leq C(\varepsilon^{1/6}(\ln\varepsilon^{-1})^{7/6}+\beta\ln\beta^{-1}+\exp(-\delta\beta\varepsilon^{-1})).$$

From these estimates we conclude

$$E^x(C^\varepsilon_\infty)-E^x(B^\varepsilon_\infty)\leq C(\varepsilon^{1/6}(\ln\varepsilon^{-1})^{7/6}+\beta\ln\beta^{-1}+\exp(-\delta\beta\varepsilon^{-1})).$$

Taking $\beta=\varepsilon^{1/2}$ in this estimate and in (9.12) we get

$$E^x(C^\varepsilon_\infty)-E^x(A^{H,\varepsilon}_\infty)\leq C\varepsilon^{1/6}(\ln\varepsilon^{-1})^{7/6}.$$

This completes the proof.

10. The case of a hypersurface with boundary

The estimate obtained in Proposition 9.6 allows us to treat a hypersurface with boundary by the method used in establishing Theorem 7.2. The details are omited. Here is the result.

Theorem 10.1

Let L be an operator of the form (1.2) in $R^d (d\geq 3)$, such that $a^{ij}\in \mathscr{C}^{2+\alpha}(R^d)$, $b^i\in \mathscr{C}^{1+\alpha}(R^d)$, $i,j=1,\dots,d$ and $c\equiv 0$. Let K be a compact hypersurface with boundary of class $\mathscr{C}^{3+\alpha}$ and let us denote

$$a(x)=2\sum_{i,j=1}^{d} a^{ij}(x)n^i(x)n^j(x),\quad x\in K\setminus\partial K,$$

where $n^i(x)$, $i=1,\dots,d$ are the components of a unit vector normal to the hypersurface at x. Let μ be the surface area in $K\setminus\partial K$ and $\lambda=a.\mu$. Assume that X is an L-diffusion in R^d and A^{ε} is the functional defined by (2.1) for each $\varepsilon>0$. Then there exists a continuous additive functional, A, and two constants, $C>0$, $\varepsilon_0>0$, such that

(10.1) $$\lim_{\varepsilon\to 0}\sup_t |A_t^{\varepsilon}-A_t|=0,\quad \text{a.s.},$$

(10.2) $$E^x(\sup_t |A_t^{\varepsilon}-A_t|^2)^{1/2}\leq C\varepsilon^{1/12}(\ln\varepsilon^{-1})^{13/12},\ x\in R^d, \varepsilon\in(0,\varepsilon_0),$$

(10.3) $$E^x(A_\infty)=\int_K g(x,y)\lambda(dy),\quad x\in R^d.$$

The proof of this theorem follows from the next estimate of the function $u(x)=E^x(A_\infty^{\varepsilon})$, $x\in R^d$,

(10.4) $$|u-G_\lambda|\leq C\varepsilon^{1/6}(\ln\varepsilon^{-1})^{13/6},\ \varepsilon\in(0,\varepsilon_0).$$

This can be obtained by repeating step with step the reasoning whic leads to the estimate of Lemma 7.6. Instead of estimate (3.13) used in the proof of Lemma 7.4 one should use Proposition 9.6. Of course, diffeomorphisms like those defined above Lemma 7.3 are needed again. However, in the case when the domain of a diffeomorphism contains a piece of the boundary ∂K, then one should take care to transport that piece onto a piece of the boundary of the semi-hyperplan. The relation (7.13′) do not hold near the boundary. Instead, the conditions (9.1) are fulfilled.

Finally we mention the following theorem which can be stated as a purely analytic result. It can be proved by the same method as the preceding. Under the assumptions of the preceding theorem we set $V^{\varepsilon}=\{x\in R^d : d(x,K)<\varepsilon\}$ and for $f\in\mathcal{C}(K)$ define h to be L-harmonic in $R^d\setminus K$ satisfying the boundary conditions $h=f$ on K and $\lim_{|x|\to\infty} h(x)=0$; also we define l to be L-harmonic in V^{ε} with boundary condition $l=h$ on ∂V^{ε}. Then we define $Mf: R^d \to R$ as follows: $Mf(x)= =h(x)$ if $x\in R^d\setminus V^{\varepsilon}$ and $Mf(x)=l(x)$ if $x\in V^{\varepsilon}$. Thus $M:\mathcal{C}(K)\to\mathcal{C}^0(R^d)$ is a linear operator and, as one can easily see, $\|M\|<1$. Then the operator $N^{\varepsilon}=\sum_{n=1} M^n$ is well defined.

Theorem 10.2

For each $f\in\mathcal{C}(K)$, one has

$$\lim_{\varepsilon\to 0} \|\varepsilon N^{\varepsilon} f - G^{1}_{f.2}\| = 0.$$

References

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