# UNIMODULAR MATRICES WHOSE COMPONENTS ARE SQUARES OF A UNIMODULAR ONE

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A. DANESCU, V. VAJAITU, A. ZAHARESCU

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A. DANESCU*), V. VAJAITU*), A. ZAHARESCU*)

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*) Department of Mathematics, INCREST, Bd. Pacii 220, 79622 Bucharest, Romania.

# UNIMODULAR MATRICES WHOSE COMPONENTS

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Alexandru Dănescu, Viorel Vâjâitu, and Alexandru Zaharescu

Department of Mathematics, INCREST,

Bd. Păcii 22o, 79622 Bucharest, Romania

## Introduction

The motivation of this work arise from the
following raised by B. Gordon ( see [1] , F 28 ),
who asked for integers $a_1$ , $a_2$ ,..., $a_9$ none
of them $0$ or $\pm 1$ so that:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix} = 1 = \begin{vmatrix} a_1^2 & a_2^2 & a_3^2 \\ a_4^2 & a_5^2 & a_6^2 \\ a_7^2 & a_8^2 & a_9^2 \end{vmatrix}$$

Denoting for any matrix $A$ , by $A_2$ the
matrix whose components are the squares of the
elements of $A$ we shall prove that , more generally,
for each $k \in \mathbb{N}$ and $M \geqslant 3$ there exists
matrices $A \in M_M(\mathbb{Z})$ with $|a_{ij}| \geqslant k$ such that:

In the first section we deal with the case $M = 3$ and obtain a solution with four independent parameters. The general case is treat in the second section by a reduction procedure from $M$ to $M-2$ and by a separate treatment of the case $M = 4$.

The $M$ - dimensional solutions given by our method have $O(M^2)$ independent parameters.

In the remaining section we treat the following more general problem:

" For what $(M, k, \Delta_1, \Delta_2)$ there exists $A = (a_{ij}) \in M_M(\mathbb{Z})$ such that $|a_{ij}| > k$ for $1 \leq i, j \leq M$ $\det A = \Delta_1$ and $\det A_2 = \Delta_2$ ? "

We shall prove that for $M \geqslant 3$, the obvious necessary condition $\Delta_1 \equiv \Delta_2$ ( mod 2 ) is also sufficient.

### 1. The unimodular case $M = 3$

(i) An effective example to the Gordon's problem

We start with a matrix having the following form:

$$A = \begin{pmatrix} x & a & b \\ \alpha & p & p-1 \\ \beta & p+1 & p \end{pmatrix}$$

The equation $\det A = 1$ become:

$$x = 1 + p(\beta - \alpha)(b - a) + a\beta - b\alpha \qquad (1.1)$$

and $\det A_2 = 1$ yields:

$$x^2(p^2-1) - p^2(b^2-a^2)(\beta^2-\alpha^2) - 2p(a^2\beta^2 - \alpha^2 b^2) + \alpha^2 b^2 + a^2\beta^2 = 1 \qquad (1.2)$$

From ( 1.1 ) and ( 1.2 ) it follows:

$$\big[ 1 + p^2(\beta-\alpha)^2(b-a)^2 + (ap-b\alpha)^2 + 2p(\beta-\alpha)(b-a) + 2(ap-b\alpha)$$
$$+ 2p(ap-b\alpha)(\alpha-\beta)(b-a) \big](2p^2-1) - p^2(b^2-a^2)(\beta^2-\alpha^2) + \qquad (1.3)$$
$$+ 2p(\alpha^2 b^2 - a^2\beta^2) + \alpha^2 b^2 + a^2\beta^2 = 1$$

This equation is simplified considerably if we choose

$$\beta = \alpha \qquad , \text{ so we have:}$$

$$(1 + \alpha(a-b))^2(2p^2-1) + \alpha^2 b^2(1+2p) + \alpha^2 a^2(1-2p) = 1$$

Denoting $z = a-b$ we obtain the following equation
in the unknows $z$ , $\alpha$ , $b$ and $p$ :

$$\alpha^2\big[ z^2 p(p-1) + bz(1-2p) + b^2 \big] + \alpha z(2p^2-1) + p^2-1 = 0 \qquad (1.4)$$

Now we observe that the coefficient of $\alpha^2$ can be
decomposed as the product:

$$(zp-b)(z(p-1)-b) \qquad\qquad (1.5)$$

Moreover denoting $A = zp-b$ and $B = z(p-1)-b$
the coefficient of $\alpha$ in ( 1.4 ) become $(A-B)(2p^2-1)$
and we obtain an equation which is linear in $A$ and $B$ :

$$\alpha^2 AB + \alpha(A-B)(2p^2-1) + p^2-1 = 0$$

We must have $\alpha \mid p^2-1$ and for this we choose $\alpha = p-1$
It follows:

$$A = \frac{B(2p^2-1) - (p+1)}{B(p-1) + 2p^2-1}$$

which has infinitely many integer solutions .

A solution in small numbers is for example:

$p = 3$ , $A = 10$ , $B = -58$   from which we

obtain the matrix:

$$\begin{pmatrix} 137 & 262 & 194 \\ 2 & 3 & 2 \\ 2 & 4 & 3 \end{pmatrix}$$

which has the required properties .

(ii) A four parametric solution

Here we follow the same procedure as above, this time keeping as much as possible independent parameters. Firstly, insted of the minor $\begin{vmatrix} p & p-1 \\ p+1 & p \end{vmatrix}$ we choose

the following one:

$$\begin{vmatrix} de+1 & cde+c+e \\ d & dc+1 \end{vmatrix}$$

In this case simplifications similar to those from ( $i$ ) may be obtained if we choose $\alpha = e \cdot \beta$ . So we start with the matrix:

$$\begin{pmatrix} x & a & b \\ e\beta & de+1 & cde+c+e \\ \beta & d & dc+1 \end{pmatrix} \qquad (1.6)$$

The equations ( 1.1 ) , ( 1.2 ) and ( 1.3 ) become respectively:

$$(1.7)$$

$$x + \beta(ac-b) = 1$$

$$x^2(2d^2ce + 2dc + 2de + 1) + \beta^2(2cea^2 + (2de+1)(c^2a^2 - b^2)) = 1 \quad (1.8)$$

$$\beta^2\left[(ac-b)^2(2d^2ce + 2de + 2de + 1) + 2cea^2 + (2de+1)(c^2a^2 - b^2)\right] -$$

$$- 2\beta(ac-b)(2d^2ce + 2dc + 2de + 1) +$$

$$(1.9)$$

$$+ 2d^2ce + 2dc + 2de = 0$$

Here we put $z = ac-b$ and the coefficient of $\beta^2$ in ( 1.9 ) is:

$$2c\left[z^2(ed^2 + d) + az(2de+1) + a^2e\right]$$

which is also decomposable. Denoting by:

$$A = z(de+1) + ea \quad , \quad B = zd + a$$

and observing that $A - eB = z$ the equation ( 1.9 ) becomes:

$$c\beta^2 AB + \beta(A-eB)(2d^2ce + 2dc + 2de+1) + \qquad (1.10)$$

We must have $\beta \mid ecd^2 + ed + cd$ and for this we put $d = \beta \gamma c$ . It follows:

$$A = \frac{Be\beta(2dg+1) - gd}{c\beta^2 B + \beta(2dg+1)}$$

where $g = cde + e + c$ . We observe that $A$ can be made equal to $2eg\gamma$ by choosing:

$$e B = 2eg\gamma(2dg+1) + g\gamma c = \gamma(2eg(2dg+1) + gc)$$

and we obtain an integer $B$ if we put $\gamma = ef$ .

In this way we obtain a solution with four independent parameters $e$ , $c$ , $\beta$ , $f$ :

$$x = 1 + \beta fceg + 4\beta^2 f^2 c e^3 g^2$$
$$a = 2efg + 4c\beta e^2 f^2 g^2 + fcg + f^2 c^2 e^2 g\beta + 4\beta^2 c^2 e^4 f^3 g^2$$
$$b = 3efcg + 4c^2\beta e^2 f^2 g^2 + fc^2 g + f^2 c^3 e^2 g\beta + 4\beta^2 c^3 e^4 f^3 g^2 + 4\beta c e^3 f^2 g^2$$
$$d = c\beta ef$$

where: $g = c^2 e^2 f\beta + c + e$

## 2. The $M$ - dimensional unimodular case

Proposition 1: For any $M \geqslant 3$ there exist a solution of $\det(a_{ij}) = \det(a_{ij}^2) = 1$ with $a_{ij} = $

= nonconstant polinomials with integer coefficients in $\frac{1}{2}\left(M^2 - M - 3 + 5\cdot(-1)^{M-1}\right)$ parameters .

Corrolary: For any $\ell \in \mathbb{N}$ and $M \geqslant 3$ there exist $A = (a_{ij})$ , $A \in M_M(\mathbb{Z})$ such that $\det(a_{ij}) = \det(a_{ij}^2) = 1$ and $|a_{ij}| \geqslant \ell$ for $1 \leqslant i, j \leqslant M$ .

Proof of Proposition 1 : The idea is to construct a $M$ - dimensional solution $A = (a_{ij})$ such that $a_{1i} = a_{2i} = a_{3i}$ for $i = 3, 4, \ldots, M$ . Denoting $a_{11} = a$ , $a_{12} = b$ , $a_{21} = c$ , $a_{22} = d$ , $a_{31} = e$ , $a_{32} = f$ ,

$$\Delta = \begin{vmatrix} a - e & b - f \\ c - e & d - f \end{vmatrix} \quad , \quad \Delta' = \begin{vmatrix} a^2 - e^2 & b^2 - f^2 \\ c^2 - e^2 & d^2 - f^2 \end{vmatrix} ,$$

we have:

$$\det(a_{ij})_{1 \leqslant i, j \leqslant M} = \Delta \cdot \det(a_{ij})_{3 \leqslant i, j \leqslant M}$$

$$\det(a_{ij}^2)_{1 \leqslant i, j \leqslant M} = \Delta' \cdot \det(a_{ij}^2)_{3 \leqslant i, j \leqslant M}$$

Supposing the case $M - 2$ solved let $(a_{ij})_{3 \leqslant i, j \leqslant M}$ be such a solution . The components $a_{ij}$ , $4 \leqslant i \leqslant M$ , $j = 1, 2$ are clearly new independent parameters so the result easily follows if we show that the system

$$\Delta = 1 = \Delta' \tag{2.1}$$

has a three parametric solution .

Denoting $A = a - e$ , $B = b - f$ , $C = c - e$ , $D = d - f$ the two equetions become:

$$\begin{cases} AD - BC = 1 \\ AD(A+2e)(D+2f) - BC(B+2f)(C+2e) = 1 \end{cases}$$

This system is equivalent to:

$$\begin{cases} AD - BC = 1 \\ f = \dfrac{e(B^2 C - 4D^2) - AD + 1}{2e + A^2 D - BC^2} \end{cases} \qquad (2.2)$$

The second equation in ( 2.2 ) has the solution:

$$e = \tfrac{1}{2}(A^2 D - BC^2)(B^2 C - AD^2 - 1) + AD - 1$$

$$f = \tfrac{1}{2}(B^2 C - AD^2 - 1)$$

Using for the first equation in ( 2.2 ) the solution:

$$A = 2\alpha\beta + 1 \ , \ B = 2\alpha\beta\gamma + \gamma + \beta \ , \ C = 2\alpha \ , \ D = 2\alpha\gamma + 1$$

we obtain for $a$ , $b$ , $c$ , $d$ , $e$ , $f$ nonconstant polinomials in $\alpha$ , $\beta$ and $\gamma$ which have integer coefficients.

To conclude the proof of the Proposition 1 it remains to give a four dimensional solution with at least two independent parameters.

In order to obtain simplifications similar to those from section 1 we begin with a matrix of the following form:

$$A = \begin{pmatrix} x & a & b & c \\ y & p+1 & p+2 & p+3 \\ y & p & p+1 & p+2 \\ y & p & p & p+1 \end{pmatrix} \qquad (2.3)$$

The equation $\det A = 1$ becomes $x = 1 + y(c-b)$ and substituting this in $\det(a_{ij}^2) = 1$ we have:

$$y^2[(b-c)^2(4p^3 + 14p^2 + 6p + 1) - 4a^2 + (2p+1)^2(b^2 - c^2) + \qquad (2.4)$$
$$+ 2(2p+1)b^2] - 2y(b-c)(4p^3 + 14p^2 + 6p + 1) + 4p^3 + 14p^2 + 6p = 0$$

Let $z = b - c$. The coefficient of $y^2$ is:

$$z^2(4p^3 + 10p^2 + 2p) + 2bz(2p+1)^2 + 2(2p+1)b^2 - 4a^2$$

and we observe that for $a = pz$ a decomposition similar with ( 1.5 ) holds. Denoting

$$A = (p+1)z + b \qquad , \qquad B = pz + b \qquad \text{the equation } ( 2.4 )$$

become:

$$y^2(2p+1)AB - y(A-B)(2g+1) + g = 0$$

where:

$$g = 2p^3 + 7p^2 + 3p = p(2p+1)(p+3)$$

In order to have $y|g$ we choose $p=yt$ . It follows :

$$A = \frac{-By(2g+1)-g}{By^2(2p+1)-y(2g+1)} = \frac{-B(2g+1)-t(2p+1)(p+3)}{By(2p+1)-(2g+1)}$$

which equals $-2t(p+3)$ for:

$$B = -t(2p+1)(p+3)-2t(p+3)(2g+1)$$

So we obtain a solution with two independent parameters ( $y$ and $t$ ) by substituting in ( 2.3 ) :

$$x = 1-p(p+3)(2p-1+2(2g+1))$$
$$a = pt(p+3)(2p-1+2(2g+1))$$
$$b = -t(p+3)\left[2p+1+2(2g+1)+p(2p-1)+2p(2g+1)\right]$$
$$c = -t(p+3)\left[4p+4(2g+1)+p(2p-1)+2p(2g+1)\right]$$

where $p=yt$ and $g=2p^3+7p^2+3p$ . The proof of the Proposition 1 is now complete .

Remark: An example in small numbers for the four dimensional case is:

$$\begin{pmatrix} -2049 & 2050 & -3085 & -4110 \\ 2 & 3 & 4 & 5 \\ 2 & 2 & 3 & 4 \\ 2 & 2 & 2 & 3 \end{pmatrix}$$

and one for ( 2.1 ) is :

$$\begin{pmatrix} 33 & 6 \\ 32 & 5 \\ 30 & 2 \end{pmatrix}$$

Taking into account the example given in section 1 (i) we can give for any $M \geqslant 3$ an effective solution.

3. The problem for a general quadruple $(M, k, \Delta_1, \Delta_2)$

For $M \geqslant 2$, $k \geqslant 0$, $\Delta_1$ and $\Delta_2 \in \mathbb{Z}$ let $S(M, k, \Delta_1, \Delta_2) = \{ A = (a_{ij}) \in M_M(\mathbb{Z}); \det A = \Delta_1, \det A_2 = \Delta_2 \text{ and } |a_{ij}| \geqslant k \text{ for } 1 \leqslant i,j \leqslant M \}$.

In this section we shall study the following problem: Being given a quadruple $(M, k, \Delta_1, \Delta_2)$ how can we decide if $S(M, k, \Delta_1, \Delta_2)$ is empty or not ? Anf if $S(M, k, \Delta_1, \Delta_2)$ is non-empty how we can find an element of it ?

First of all we note that if $\Delta_1 \not\equiv \Delta_2$ ( mod 2 ) then $S(M, k, \Delta_1, \Delta_2) = \phi$ . In the following we shall always suppose $\Delta_1 \equiv \Delta_2$ ( mod 2 )

The reduction step from $M$ to $M-2$ used in proposition 1 show that for any $k \geqslant 0$ the following implication is true:

$$S(M, k, \Delta_1, \Delta_2) \neq \phi \Rightarrow S(M+2, k, \Delta_1, \Delta_2) \neq \phi$$

Moreover if we have an element $A$ from $S(M, k, \Delta_1, \Delta_2)$ we can construct an element $B$ from $S(M+2, k, \Delta_1, \Delta_2)$.

3.1. The case $M=2$. We have $S(M, k, \Delta_1, \Delta_2) \neq \phi$ if and only if the system:

$$\begin{cases} ad - bc = \Delta_1 \\ a^2d^2 - b^2c^2 = \Delta_2 \end{cases} \tag{3.1}$$

is solvable in integers $a$, $b$, $c$, $d$ of absolute value $\geq k$. If $\Delta_1 = 0$ then $S(2, k, \Delta_1, \Delta_2) \neq \phi$ if and only if $\Delta_2 = 0$. If $\Delta_1 \neq 0$ the solvability of the system ($3.1$) is equivalent to the following condition: The numbers

$$a_1 = \left| \frac{\Delta_2 + \Delta_1^2}{2\Delta_1} \right| \quad \text{and} \quad a_2 = \left| \frac{\Delta_2 - \Delta_1^2}{2\Delta_1} \right| \quad \text{are integers}$$

and have divisors $d_1 | a_1$, $d_2 | a_2$ such that:

$$k \leq \min\left( d_1; d_2; \frac{a_1}{d_1}; \frac{a_2}{d_2} \right)$$

Thus if $\Delta_1 \neq 0$ then $S(2, k, \Delta_1, \Delta_2)$ is finite and the above condition gives an algorithm to find all its elements.

3.2. The case $M=3$.

Suppose $\Delta_1 \neq 0$, $\Delta_2 \neq 0$. Then for each $k > 0$

$S$ ( $3$ , $k$ , $\Delta_1$ , $\Delta_2$ ) is not empty ( thus it is
infinite ) . Here we do not have a parametric solution
but we shall give an algorithm to find ( infinitely many)
solutions  via the theorem of Dirichlets on primes
in an arithmetic progresion . More exactly given  $k$ , $\Delta_1 \neq 0$
$\Delta_2 \neq 0$ ,              let us compute the number
$\Delta = \Delta_1^4 + \Delta_1^2 \Delta_2^2 + 2 \Delta_1 \Delta_2^2$ and find a prime  $q >$    max ( $k$ ; $|\Delta_1|$ ; $\Delta_2^2$ )
in the arithmetic progresion  $\{ 8 \Delta m + 1 ; m \in \mathbb{N} \}$ .

   Let  $A$    be as in ( $1.6$ ). The equations ( $1.7$ ),
( $1.8$  )   and ( $1.9$ ), ( $1.10$ ) from section 1
become:

$$X + \beta(ac - b) = \Delta_1$$
$$X^2(2u+1) + \beta^2 v = \Delta_2$$
$$\beta^2 \left[ (ac-b)^2(2u+1) + v \right] - 2\beta\Delta_1(ac-b)(2u+1) + \Delta_1^2(2u+1) - \Delta_2 = 0$$

$$c\beta^2 AB + \beta\Delta_1(A - eB)(2u+1) + u\Delta_1^2 + K = 0$$

where:

$$u = d^2ce + dc + de \quad , \quad v = 2ea^2c + (2de+1)(c^2a^2 - b^2)$$
$$A = z(de+1) + ea \;, \quad B = zd + a \;, \quad K = \frac{\Delta_1^2 - \Delta_2}{2}$$

It follows:

$$\beta \mid u\Delta_1^2 + K$$

$$(3.2)$$

$$A = \frac{eB\Delta_1(2u+1) - \frac{u\Delta_1^2 + k}{\beta}}{c\beta B + \Delta_1(2u+1)}$$

If:

$$\beta \mid e\Delta_1(2u+1) - 1 \qquad\qquad (3.3)$$

then we obtain for $A$ the integer value:

$$\frac{e\Delta(2u+1) - 1}{\beta}$$

if we put $c = 1$ and $B = (e\Delta_1^2(2u+1)^2 - \Delta_1(2u+1) + u\Delta_1^2 + k)/\beta$

From (3.2) and (3.3) it follows:

$$e\Delta_2 \equiv \Delta_1 \pmod{\beta} \qquad\qquad (3.4)$$

Now let $\beta = q$ and choose $e$ which verify ( 3.4 )
Then ( 3.2 ) and ( 3.3 ) will be valid if $d$
satisfy the following congruence:

$$d^2\Delta_1^3 + d\Delta_1^2(\Delta_2 + \Delta_1) + k\Delta_2 \equiv 0 \pmod{q}$$

This congruence ( of degree 2 ) has solutions because
the discriminant is $\Delta_1^2 \cdot \Delta$ which is a square ( mod $q$ )
from our choice of $q$ ( If $\Delta = \pm q_1^{K_1} \dots q_\Delta^{K_\Delta}$
then $\left(\frac{\Delta}{q}\right) = \prod_{i=1}^{\Delta}\left(\frac{q_i}{q}\right)^{\alpha_i}$ ; but for $q_i$ odd we have
$\left(\frac{q_i}{q}\right) = \left(\frac{q}{q_i}\right) = 1$ and also $\left(-\frac{1}{q}\right) = \left(\frac{2}{q}\right) = 1$ )
Now it is clear that for $e$ and $d$ large
enough ( which verify (3.2) and (3.3) ) the
absolute value of any element of $A$ will be $\geqslant k$ ,

and this complete the proof in this case.

Observe that this proof does not work if $\Delta_1 = 0$ or $\Delta_2 = 0$ so we must treat separately these two cases.

Suppose now $\Delta_1 = 0$. In this case we choose:

$$A = \begin{pmatrix} a & b & a+b \\ c & d & c+d \\ e & f & e+f \end{pmatrix}$$

Then $\det A = 0$ and $\det A_2 = 2 \cdot \left| \begin{array}{cc} a & b \\ c & d \end{array} \right| \left| \begin{array}{cc} a & b \\ e & f \end{array} \right| \cdot \left| \begin{array}{cc} e & f \\ c & d \end{array} \right|$

If $a$, $b$, $c$, $d$ are choosen so that $ad - bc = 1$ then for $e = a + \frac{\Delta_2}{2} \cdot c$, $f = b + \frac{\Delta_2}{2} \cdot d$ we obtain $\det A_2 = \Delta_2$.

Suppose now $\Delta_2 = 0$. It is sufficient to consider the case $\Delta_1 = 2$. The proof is based on the idea from section 1 (i) and we begin with a matrix $A$ in the form :

$$A = \begin{pmatrix} x+1 & a & b \\ \alpha & p & p-1 \\ \alpha & p+1 & p \end{pmatrix}$$

From $\det A = 2$ we have $x - \alpha (a - b) = 1$ and substituting this in $\det A_2 = 0$ we obtain :

$$\alpha^2 \left[ a^2 (p^2 - p) - ab(2p^2 - 1) + b^2 (p^2 + p) \right] + 2\alpha(a-b)(2p^2 - 1) + 2(2p^2 - 1) = 0$$

Denoting $A = ap - bp - b$, $B = ap - bp - a$ we obtain:

$$\alpha^2 AB + 2\alpha (A - B)(2p^2 - 1) + 2(2p^2 - 1) = 0$$

and by puting $A = 2A_1$ for $\alpha$ for which
$2p^2 - 1 = \alpha^2$ we have :

$$A_1 B + \alpha(2A_1 - B) + 1 = 0$$

An integer solution for this equation is:

$$A_1 = \alpha - 1 \quad , \quad B = 2\alpha^2 - 2\alpha + 1$$

so we obtain : $a - b = -2\alpha^2 + 2\alpha - 2$ , $x = \alpha(-2\alpha^2 + 2\alpha - 1) + 1$

Observing that $a - b \approx O(\alpha^2)$ and $p \approx O(\alpha)$ it
results that $(a-b)p \approx O(\alpha^3)$ . But $A \approx O(\alpha)$
and $B \approx O(\alpha^2)$ so that $a \approx O(\alpha^3)$ and $b \approx O(\alpha^3)$

Noteing that the equation $2p^2 - 1 = \alpha^2$ has infinitely
many integers solutions we conclude that $|a_{ij}| \geq k$
for $\alpha$ sufficiently large .

From the reduction step it follows now that the
case $M$ odd , $M \geqslant 5$ is also settled.

3.3 The case $M = 4$ .

We shall prove that $S(4, k, \Delta_1, \Delta_2) \neq \phi$
and we will show how to find elements of it.

For a $4 \times 3$ matrix $T = (t_{ij})$ $1 \leq i \leq 4, 1 \leq j \leq 3$
denote by $\tilde{T} = (\tilde{t}_{ij})$ $1 \leq i, j \leq 4$ the $4 \times 4$ matrix
given by $\tilde{t}_{ij} = t_{ij}^2$ if $1 \leq i \leq 4$ and $1 \leq j \leq 3$
and $\tilde{t}_{i4} = t_{i2} \cdot t_{i3}$ for $1 \leq i \leq 4$ . We consider
now the following unimodular matrix:

$$B = \begin{pmatrix} p^2+p-1 & p+1 & p \\ p^2-p-1 & p & p-1 \\ 2p^2-1 & 2p+1 & 2p-1 \end{pmatrix}$$

It has the following interesting property: If we adjoint a forth row $(a, b, b)$ to obtain a $4 \times 3$ matrix $C$ then $\det \tilde{C} = a^2 - 3b^2$. So if $a + b\sqrt{3} = (2+\sqrt{3})^M$ then $\tilde{C}$ is unimodular. Denoting by $C'$ and $C''$ the matrices obtained from $C$ by changing the first two columns ( and respectively the first with the third ) then $\det \tilde{C}'$ and $\det \tilde{C}''$ are non-null polinomials in $a$, $b$ and $p$. Fix now $M$ and $p$ ( large enough ) such that:

$$a \geqslant k, \quad b \geqslant k, \quad p \geqslant k+1, \quad \det \tilde{C}' \neq 0, \quad \det \tilde{C}'' \neq 0$$

We adjoint now to $C$ a fourth column $(x_0, y_0, z_0, t_0)$ to obtain a matrix $D$ such that:

$$\det D = \Delta_1$$

This clearly may be done because $B$ is unimodular. Adding to the fourth column of $D$ the first three columns multiplied by $\alpha$, $\beta$ and $\gamma$ respectively we obtain a matrix $A$ :

$$A = \begin{pmatrix} p^2 + p - 1 & p+1 & p & x \\ p^2 - p - 1 & p & p-1 & y \\ 2p^2 - 1 & 2p+1 & 2p-1 & z \\ a & b & b & t \end{pmatrix}$$

which verifies:

$$\det A = \Delta_1$$

We want to choose $\alpha$, $\beta$ and $\gamma$ such that:

$$\det A_2 = \Delta_2 \qquad (3.5)$$

and:

$$|x|, |y|, |z|, |t| \geqslant \ell \qquad (3.6)$$

The equation ( 3.5 ) is of the form :

$$2\alpha\beta a_1 + 2\alpha\gamma a_2 + 2\beta\gamma a_3 + 2\alpha b_1 + 2\beta b_2 + 2\gamma b_3 +$$
$$+ \det D_2 = \Delta_2$$

where $a_1 = -\det \tilde{c}'' \neq 0$ , $a_2 = -\det \tilde{c}' \neq 0$ , $a_3 = \det \tilde{c} = 1$

It follows:

$$\gamma = \frac{-a_1 \alpha\beta - b_1 \alpha - \beta b_2 + \dfrac{\Delta_2 - \det D_2}{2}}{\beta + \alpha a_2 + b_3}$$

which gives an integer $\gamma$ if we choose $\beta = -\alpha a_2 - b_3 + 1$
( note that $\det D_2 \equiv \det D = \Delta_1 \equiv \Delta_2$ ( mod 2 ))

In this manner we obtain an element $A$ of
$S(4, k, \Delta_1, \Delta_2)$ ( From $q_1 \neq 0$, $q_2 \neq 0$ it
follows that $x$, $y$, $z$, $t$ are polinomials in $\alpha$
of degree exact two so we can choose $\alpha$ large enough
such that $(3.6)$ holds ).

This conclude the case $M = 4$ and from the
reduction step the case $M$ even, $M \geq 6$.

We can now state the following proposition:

Proposition 2: For any $M \geq 3$, $k \in \mathbb{N}$, $\Delta_1, \Delta_2 \in \mathbb{Z}$
$\Delta_1 \equiv \Delta_2$ ( mod 2 ) it can be find a matrix $A = (a_{ij}) \in M_M(\mathbb{Z})$
such that:

1) $\det A = \Delta_1$

2) $\det A_2 = \Delta_2$

3) $|a_{ij}| \geq k$ for $1 \leq i, j \leq M$.