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STABILITY OF THE STEADY SECONDARY DISPLACEMENT OF OIL

by

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Abstract We study the stability of the displacement of oil, through a porous layer, by a less viscous fluid with a polymer solute as forerunner. We prove that the optimal viscozity profile, in the case of a thin region of polymer solute, is the constant one.

1. Introduction

It is well-known that the macroscopic boundary between the oil and the displacing fluid becomes unstable if the displacing fluid has a lower viscozity than the oil (for instance air or water). It was proved (see [1]) that when surface tension is present, it has a positive effect, limiting the range of unstable disturbances. Therefore it seemed a reasonable policy to use a polymer, with a high viscosity and an important surface tension against oil, as an intermediate region between the displacing fluid and oil. In fact this region is obtained by the addition of polymers in dilute quantities. As in the intermediate region the viscosity profile might be obviously controlled, it is an important problem to find the optimum situation, that is to have the smallest range of unstable disturbances and moreover without infinite growth constant.

We suppose that the total amount of additive is relatively small ; as a consequence, in non-dimensional variables, we consider $\varepsilon > 0$, the thickness of the intermediate region, as a small parameter. Thus the growth constant σ and the corresponding solu -

tion will be approximated by asymptotical expansions with respect to ε. The first-order approximation agrees with the formulas given in [1] and [2] which were obtained in the case without intermediate region. This problem was also studied in [3], where one can find a numerical solution corresponding to the most unfavourable growth constant, validated by an asymptotical analysis performed in the cases of small and large quantity of solute.

Here we prove that in order to avoid an infinite value of σ, the viscosity profile in the intermediate region would be constant. A posteriori, this result agrees with our assumption of small quantity of additive.

2. The non-dimensional equations

Let O be the origin of the coordinate system Oxyz so that negative Oz direction is the direction in which the gravitational force acts. We consider a bidimensional flow in the plane Oxy, so that positive Ox direction is the direction of the velocity at infinity: $\underline{U}_\infty = (U, 0)$ where $U > 0$ (constant). The porous medium is enclosed in the layer $y \in (0, L)$. We study the flow of two fluids of different viscosities μ_1 and μ_2 ($\mu_1 < \mu_2$) separated by an intermediate region of interfaces $x = \varphi(t,y)$ and $x = \psi(t,y)$.

The classical equations governing such phenomena (see [3]) are:

$$\left.\begin{aligned} &(2.1) \quad u_x + v_y = 0 \\ &(2.2) \quad p_x = -\frac{\mu}{q} u \\ &(2.3) \quad p_y = -\frac{\mu}{q} v \end{aligned}\right\} \quad \text{for } x \in R,\ y \in (0, L)$$

$$(2.4) \quad \mu = \mu_1 \quad \text{for } x - U\cdot t < \varphi(t,y)$$

$$(2.5) \quad \mu = \mu_2 \quad \text{for } x - U\cdot t > \psi(t,y)$$

$$(2.6) \quad \mu_t + u\,\mu_x + v\,\mu_y = 0 \text{ for } \varphi(t,y) < x - U\cdot t < \psi(t,y)$$

where (u,v) , p and q denote respectively the velocity's compo - nents, the pressure and the permeability. The indices x, y, and t de - note the corresponding partial derivation.

On the material interfaces we have :

$$\left.\begin{array}{lll} (2.7) & [(u,v)]\cdot\underline{\nu} = 0 \\ (2.8) & \varphi_t + v\cdot\varphi_y = u - U \\ (2.9) & [p] = S\cdot\varphi_{yy} \end{array}\right\} \text{ on } x - Ut = \varphi(t,y)$$

$$\left.\begin{array}{lll} (2.10) & [(u,v)]\cdot\underline{\nu} = 0 \\ (2.11) & \psi_t + v\cdot\psi_y = u - U \\ (2.12) & -[p] = T\cdot\psi_{yy} \end{array}\right\} \text{ on } x - Ut = \psi(t,y)$$

where $[\cdot]$ denotes the jump, $\underline{\nu}$ is the unit normal of the corres - ponding interface, while S and T are the corresponding surface ten- sions.

The boundary conditions are:

(2.13) $\mu_y = v = 0$ for $y = 0$ and $y = L$

(2.14) $(u,v) \longrightarrow (U, 0)$ for $x \longrightarrow \pm\infty$

In the sequel we introduce a moving reference frame. For prac - tical reasons we also write the system (2.1)-(2.14) in dimensionless form, defining

$$x^* = \pi\cdot L^{-1}\cdot\varepsilon^{-1}\cdot(x-Ut),\ y^* = \pi L^{-1}y\ ,\ t^* = U\pi L^{-1}t$$

$$(\varphi^*, \psi^*) = \pi L^{-1}\cdot\varepsilon^{-1}\cdot(\varphi, \psi)$$

$$u^* = -1 + U^{-1}\cdot u\ ,\ v^* = U^{-1}v\ ,\ p^* = q\cdot\pi\cdot L^{-1}\cdot\mu_2^{-1}U^{-1}\cdot p$$

$$\mu^* = \mu_2^{-1}\mu,\ \alpha^* = \mu_2^{-1}\cdot\mu_1\ ,\ (S^*, T^*) = q\pi^2 L^{-2}\cdot\mu_2^{-1}U^{-1}\cdot(S,T)$$

where $\varepsilon > 0$ is a small parameter. Thus, omitting the asterisks, the system (2.1)-(2.14) takes the form

$$\left.\begin{array}{lll} (2.15) & \frac{1}{\varepsilon}\cdot u_x + v_y = 0 \\ (2.16) & \frac{1}{\varepsilon}\cdot p_x = -\mu(1 + u) \\ (2.17) & p_y = -\mu\cdot v \end{array}\right\} \text{ for } x \in R,\ y \in (0, \pi)$$

(2.18) $\mu = \alpha$ for $x < \varphi(t,y)$

(2.19) $\mu = 1$ for $x > \psi(t,y)$

(2.20) $\mu_t + \frac{1}{\varepsilon} u \cdot \mu_x + v \cdot \mu_y = 0$ for $\varphi(t,y) < x < \psi(t,y)$

(2.21) $[(u,v)] \cdot \underline{\nu} = 0$

(2.22) $\varphi_t + v \cdot \varphi_y = \frac{1}{\varepsilon} u$ for $x = \varphi(t,y)$

(2.23) $[p] = \varepsilon \cdot S \cdot \varphi_{yy}$

(2.24) $[(u,v)] \cdot \underline{\nu} = 0$

(2.25) $\psi_t + v \cdot \psi_y = \frac{1}{\varepsilon} u$ for $x = \psi(t,y)$

(2.26) $[p] = \varepsilon \cdot T \cdot \psi_{yy}$

(2.27) $\mu_y = v = 0$ for $y = 0$ and $y = \pi$

(2.28) $(u,v) \longrightarrow (0,0)$ for $x \longrightarrow \pm\infty$.

3. The stability analysis

Assuming that $(\varepsilon L/\pi)$ represents the thickness of the intermediate region, we have as basic solution of the system (2.15)-(2.28) the following

(3.1) $\varphi^B(t,y) = -1$, $\psi^B(t,y) = 0$

(3.2) $u^B = v^B = 0$

(3.3) $\mu^B(x) = \begin{cases} \alpha & \text{for } x < -1 \\ f(x) & \text{for } -1 < x < 0 \\ 1 & \text{for } x > 0 \end{cases}$ $\qquad p^B(x) = \begin{cases} -\alpha \cdot \varepsilon \cdot (x+1) & \text{for } x < -1 \\ -\varepsilon \cdot M(x) & \text{for } -1 < x < 0 \\ -\varepsilon x - \varepsilon M(0) & \text{for } x > 0 \end{cases}$

where $M(x) = \int_{-1}^{x} f(s)ds.$

If we superpose infinitesimal disturbances on the basic solution, that is if we take in (2.15)-(2.28) solutions of the form

(3.4) $w(x,y,t) = w^B(x,y) + \tilde{w}(x,y) \cdot \exp(\sigma \cdot t)$

where w represents u, v, p, μ, φ and ψ , and if we neglect the disturbance products, then we obtain the following stability system:

$$
\left.\begin{array}{lll}
(3.5) & \frac{1}{\varepsilon}\tilde{u}_x + \tilde{v}_y = 0 & \\
(3.6) & \frac{1}{\varepsilon}\tilde{p}_x = -\mu^B\tilde{u} - \tilde{\mu} & \\
(3.7) & \tilde{p}_y = -\mu^B\tilde{v} &
\end{array}\right\} \text{for } x \in R,\ y \in (0,\pi)
$$

(3.8) $\tilde{\mu} = 0$ for $x \in (-\infty, -1) \cup (0, \infty)$

(3.9) $\sigma\tilde{\mu} + \frac{1}{\varepsilon}\tilde{u}\cdot\mu^B_x = 0$ for $x \in (-1, 0)$

$$
\left.\begin{array}{lll}
(3.10) & [\tilde{u}] = 0 \\
(3.11) & \sigma\cdot\tilde{\varphi} = \frac{1}{\varepsilon}\tilde{u} \\
(3.12) & [\tilde{p}] + \tilde{\varphi}[p^B_x] = \varepsilon\cdot S\cdot\tilde{\varphi}_{yy}
\end{array}\right\} \text{for } x = -1
$$

$$
\left.\begin{array}{lll}
(3.13) & [\tilde{u}] = 0 \\
(3.14) & \sigma\cdot\tilde{\psi} = \frac{1}{\varepsilon}\tilde{u} \\
(3.15) & [\tilde{p}] + \tilde{\psi}[p^B_x] = \varepsilon\cdot T\cdot\tilde{\psi}_{yy}
\end{array}\right\} \text{for } x = 0
$$

(3.16) $\tilde{\mu}_y = \tilde{v} = 0$ for $y = 0$ and $y = \pi$

(3.17) $(\tilde{u},\tilde{v}) \longrightarrow (0,0)$ for $x \longrightarrow \pm\infty$

Remark 3.1 As $\varphi = -1 + \tilde{\varphi}\exp(\sigma\cdot t)$ and neglecting the disturbance products we have

$$p^B(\text{water})\Big|_{x=\varphi} = p^B(\text{water})\Big|_{x=-1} + \tilde{\varphi}\exp(\sigma\cdot t)\cdot p^B_x(\text{water})\Big|_{x=-1}$$

$$\tilde{p}\,(\text{water})\Big|_{x=\varphi} = \tilde{p}\,(\text{water})\Big|_{x=-1}$$

from which we obtain

$$p(\text{water})\Big|_{x=\varphi} = p^B(\text{water})\Big|_{x=-1} + \exp(\sigma t)\tilde{p}(\text{water})\Big|_{x=-1} + \tilde{\varphi}\exp(\sigma t)\cdot p^B_x(\text{water})\Big|_{x=-1}$$

and consequently

$$[p]_{x=\varphi} = [\tilde{p}]_{x=-1}\cdot\exp(\sigma t) + \tilde{\varphi}\,[p^B_x]_{x=-1}\cdot\exp(\sigma t)$$

where $[\cdot]_{x=\varphi}$ denotes the jump across the interface $x = \varphi$. Thus (3.12) is obtained from (2.23). □

Since the system (3.5)-(3.17) is linear, any real solution can be decomposed into its Fourier components and analyzed separately. Taking in account (3.16),(3.5),(3.13) and(3.17) we are lead to the following form of the disturbance velocity:

$$(3.18) \qquad u = -\varepsilon k^2 g(x) \cdot \cos ky \quad , \quad v = k \cdot g'(x) \cdot \sin ky$$

where g is continous and satisfies also

$$(3.19) \qquad g(x) \longrightarrow 0 \ , \ g'(x) \longrightarrow 0 \quad \text{as} \quad x \longrightarrow \pm\infty.$$

From now on, as in (3.18)-(3.19) we omit the notation $\tilde{\ }$.

Using (3.6)-(3.7) we obtain in a straightforward manner :

$$\left.\begin{array}{ll} (3.20) & p = C_1 \alpha k \varepsilon \exp(\varepsilon k(x+1)) + C_3 \\ (3.21) & g = C_1 \exp(\varepsilon k(x+1)) \end{array}\right\} \text{for} \quad x < -1$$

$$\left.\begin{array}{ll} (3.22) & p = - C_2 \varepsilon \cdot k \cdot \exp(-\varepsilon k \cdot x) + C_4 \\ (3.23) & g = C_2 \cdot \exp(-\varepsilon k \cdot x) \end{array}\right\} \text{for} \quad x > 0$$

For the intermediate region,with (3.3)and (3.18), from (3.9) , (3.11) and (3.14) we obtain:

$$\left.\begin{array}{ll} (3.24) & \mu = \sigma^{-1} k^2 f'(x) \cdot g(x) \cdot \cos ky \\ (3.25) & \varphi = -\sigma^{-1} k^2 \ C_1 \cdot \cos ky \\ (3.26) & \psi = -\sigma^{-1} k^2 \ C_2 \cdot \cos ky \end{array}\right\} \text{for} \quad x \in (-1, 0)$$

Using again (3.6)-(3.7) we get

$$(3.27) \qquad p = f(x) \ g'(x) \ \cos ky + C_5 \qquad \text{for} \quad x \in (-1, 0)$$

where g satisfies

$$(3.28) \qquad (f \ g')' = \varepsilon^2 k^2 \cdot f \cdot g - \sigma^{-1} k^2 \varepsilon f' \cdot g \qquad \text{for} \quad x \in (-1,0).$$

The jump conditions (3.12) and (3.15) yield:

$$(3.29) \qquad C_3 = C_4 = C_5$$

$$(3.30) \qquad f(-1) g_r'(-1) = C_1(\sigma^{-1} k^2 \varepsilon (\alpha - f(-1)) + \alpha k \varepsilon + \sigma^{-1} k^4 \cdot \varepsilon \cdot S)$$

$$(3.31) \qquad f(0) \ g_\ell'(0) = C_2(\sigma^{-1} k^2 \varepsilon (1 - f(0)) - k\varepsilon - \sigma^{-1} k^4 \cdot \varepsilon \cdot T)$$

where the subscripts r and ℓ indicate the right and left derivatives.

It is obvious that we get the solution of (3.5)-(3.17), corresponding to the wave number k,as soon as we solve the eigenvalue problem consisting of the differential equation (3.28) subjected to the continuity of g and the boundary conditions (3.30)-(3.31). For this we use an asymptotic expansion with respect to the small parameter $\varepsilon > 0$:

(3.32) $$g = \sum_{m \geq 0} \varepsilon^m \cdot g_m \quad , \quad \sigma^{-1} = \sum_{m \geq 0} \varepsilon^m \cdot \eta_m .$$

Introducing (3.32) in (3.28), (3.30) and (3.31), and equating the coefficients of the same powers of ε we get a sequence of problems from which g_m and η_m are succesively obtained. The first problem(ε^0 - order) yields

(3.33) $$g_o(x) = C_o \quad , \quad \text{for} \quad x \in (-1 , 0) .$$

Taking in account that g is continous on R , then (3.31), (3.23) and (3.33) imply

(3.34) $$C_o = C_1 = C_2$$

The second problem(ε^1- order) is the following:

(3.35) $$f(x)g_1''(x)+f'(x)g_1'(x) + \eta_o k^2 C_o f'(x) = 0 \quad \text{for} \quad x \in (-1,0)$$

(3.36) $$g_1(-1) = g_1(0) = 0$$

(3.37) $$f(-1)g_1'(-1) = C_o(\eta_o k^2 (\alpha - f(-1)) + \alpha k + \eta_o k^4 S)$$

(3.38) $$f(0) g_1'(0) = C_o(\eta_o k^2 (1- f(0)) - k - \eta_o k^4 T) .$$

Now, let us notice that (3.35) is equivalent to

(3.39) $$f(x) g_1'(x) = -\eta_o k^2 C_o \cdot f(x) + C_7 \quad , \quad (C_7 \text{ constant}) .$$

Next, the relation (3.39), together with (3.37)-(3.38), yield

(3.40) $$\eta_o = k^{-1}(\alpha + 1) \cdot E_k^{-1}$$

where $E_k = (1 - \alpha) - (T + S) k^2$. Finally we obtain:

(3.41) $$g_1(x) = \eta_o k^2 C_o (N (0)^{-1} N(x) - x - 1)$$

where $$N(x) = \int_{-1}^{x} f^{-1}(s)ds$$

We have to remark here that (3.40) agrees with the eigenvalue obtained in [1] and [2], which corresponds to the case without intermediate region.

We get at the next order, proceeding as before, the following result:

(3.42) $$\eta_1 = ((T+S)^2k^4-2(1-\alpha)(T+S)k^2-4\alpha)E_k^{-3}M(0) + (\alpha + 1)^2E_k^{-3}N(0)^{-1}$$

At the second -order approximation σ is given by

(3.43) $$\sigma = \eta_0^{-1} - \eta_0^{-2}\cdot\eta_1\cdot\varepsilon + O(\varepsilon^2)$$

and using (3.40) and (3.42) it follows

(3.44) $$\sigma = (\alpha+1)^{-1}\cdot kE_k + (\alpha+1)^{-2}\cdot k^2(E_k^{-1}(\alpha+1)^2(M(0)-N(0)^{-1})-E_kM(0))\cdot\varepsilon + O(\varepsilon^2)$$

If $M(0)N(0) > 1$ it is obvious that for k close to $\sqrt{(1-\alpha)/(T+S)}$ we have high positive values for σ. In order to avoid this situation and because $M(0)\cdot N(0) \geqslant 1$, it seems reasonable to impose the condi - tion $M(0)\cdot N(0) = 1$. This is equivalent to

(3.45) $f(x) = \text{const.}$ for $x \in (-1 , 0)$.

The above relation can be obtained in the following way:

$$\int_{-1}^{0} (\sqrt{f(x)} - M(0)/\sqrt{f(x)})^2 dx = \int_{-1}^{0} (f(x)-2M(0)+M(0)^2/f(x))dx =$$

$$= M(0)\cdot(M(0)\cdot N(0) - 1) = 0,$$

that is, $f(x) = M(0)$.

Hence, the final expression of σ is

$$\sigma = (\alpha + 1)^{-1}\cdot k\cdot E_k -(\alpha + 1)^{-2}k^2\cdot E_k\cdot M(0)\cdot\varepsilon + O(\varepsilon^2)$$

and the limiting effect of the intermediate region upon the range of unstable disturbances is clear.

KEY WORDS : SECONDARY OIL RECOVERY , STEADY DISPLACEMENT , INTERFACE STABILITY

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