MINIMAL PAIRS OF A RESIDUAL TRANSCENDENTAL EXTENSION OF A VALUATION

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In a previous paper (see [2]) we have given a characterization of so-called residual transcendental (r.t.) extensions of a valuation v on K to K(X). In that characterization the notion of "minimal pair of definition of r.t. extension" (see [2] or definition in § 1) was essential. In fact. describing all r.t. extension of v to K(X) is equivalent to describing all pairs $(a. \delta) \in \overline{K} \times G_{\overline{v}}$ which are minimal pairs with respect to K.

The aim of this work is to give an answer to the following problem: Let K be a field. v a valuation on K. $\overline{K}$ an algebraic closure of K. $\overline{v}$ an extension of v to $\overline{K}$ and $G_{\overline{v}}$ the value group of $\overline{v}$. Describe all pairs $(a. \delta) \in \overline{K} \times G_{\overline{v}}$ which are minimal with respect to K. This general problem seems to be very dificult. but in the present work we give some seemingly important results and applications (Sections 3 and 4). The results given in this work. are essential in the study of <u>all</u> extensions of v to K(X). This study will be given in a forthcoming paper.

The present work has four sections. The first section is concerned to notations and definitions.

Theorem 2.2 proved in Section 2 permit us to describe all common extensions to $\overline{K}(X)$ of $\overline{v}$ and w. where w is an r.t. extension of v to K(X). Also, in this section there are given considerations of some numbers (like as index of ramification and inertial degree) associated with an r.t. extension w of v to K(X). Since these numbers are defined using $\overline{v}$ and a minimal pair of definition of w. we shoe that in fact they are dependent only to w and v (Remark 2.4).

In Section 3 it is considered the question of constructing minimal pairs. It is shown that for every r.t. extension of v to K(X) there exists a minimal pair (a, δ) where a is separable over K (Theorem 3.1). Also, it is shown that the notion of minimal pair is closely related to completion in the sense of [5, Ch. VI] of K relative to v (Theorems 3.8 and 3.9).

In the last section some applications are given. Namely, Theorem 4.4 proves the existence of some r.t. extensions of v to K(X) with prescribed residue field and value group. Also, Theorem 4.5 gives a general frame-work under which the "fundamental inequality" of [7] becomes an equality. By this theorem easily result all conjectures states in [7].

1. NOTATION AND DEFINITIONS

Let K be a field and let v be a valuation on K. We emphasize this situation by calling (K,v) a <u>valuation pair</u>. If (K,v) is a valuation pair, denote by O_v the valuation ring of v; by G_v the value group of v, and by k_v the residue field of v. If $x \in O_v$, then x^* will denote the image of x into k_v. We refer the reader to [5, Ch. VI], [6] and [8] for general notation and definitions.

We say that a valuation pair (K',v') is an <u>extension</u> of the valuation pair (K,v) (or simply v' is an extension of v) if $K \subseteq K'$ and v' is an extension of v to K', i.e. $v'(x) = v(x)$ for all $x \in K$. If (K',v') is an extension of (K,v) we shall canonically identify k_v with a subfield of $k_{v'}$, and G_v with a subgroup of $G_{v'}$.

For the rest of this paper we consider a fixed valuation pair (K,v). Take $\bar{K}$ a fixed algebraic closure of K and $\bar{v}$ a fixed extension of v to $\bar{K}$. If G_v is the value group of v then the value group of $\bar{v}$, is in fact $G_{\bar{v}} = Q \otimes_Z G_v$ $= \mathbb{Q}G_v$, i.e. the smallest divisible group which contains G_v.

Let K(X) be the field of rational functions of an indeterminate X. An extension w of v to K(X) will be called an r.t. <u>(residual transcendental) extension</u> if k_w/k_v is a transcendental extension.

Let w be an extension of v to K(X). By a <u>common extension</u> of w and $\bar{v}$ to $\bar{K}(X)$ we shall mean a valuation $\bar{w}$ on $\bar{K}(X)$ which induces w on K(X) and $\bar{v}$ on $\bar{K}$. We shall prove later (Proposition 2.1) that such a common extension always exists.

If $(a, \delta) \in \bar{K} \times G_{\bar{v}}$, denote by $w_{(a, \delta)}$ the valuation on $\bar{K}(X)$ defined as follows. For a polynomial $f(X) \in \bar{K}[X]$ with the Taylor expansion:

$$f(x) = a_e + a_1(X - a) + \ldots + a_n(X-a)^n$$

we put

(1) $$w_{(a, \delta)}(f(X)) = \inf_i (v(a_i) + i\delta).$$

Now $w_{(a, \delta)}$ can be extended in a canonical way to $\bar{K}(X)$ (see [5. Ch. VI. § 10]). It is easy to see that $w_{(a, \delta)}$ is in fact an r.t. extension of $\bar{v}$. Sometimes we shall say that $w_{(a, \delta)}$ is the <u>valuation defined by inf. $\bar{v}$. $a \in K$ and $\delta \in G_{\bar{v}}$</u>.

In [1] it is shown that if w is an r.t. extension of $\bar{v}$ to $\bar{K}(X)$. then there exists a pair $(a, \delta) \in \bar{K} \times G_{\bar{v}}$ such that $w = w_{(a, \delta)}$. Such a pair is called in [2] a <u>pair of definition</u> of w. Moreover, in [1, Proposition 3] it is proved that two pairs (a, δ), (a', δ') of $\bar{K} \times G_{\bar{v}}$ define the same valuation w if and only if

(2) $$\delta = \delta' \text{ and } \bar{v}(a - a') \geq \delta .$$

By a <u>minimal pair with respect to K</u> we mean a pair $(a, \delta) \in \bar{K} \times G_{\bar{v}}$ such that for every $b \in K$ such that $\bar{v}(b - a) \geq \delta$, one has $[K(a) : K] \leq [K(b) : K]$.

Let $\bar{w}$ be a r.t. extension of $\bar{v}$ to $\bar{K}(X)$. By a <u>minimal pair of definition</u> of $\bar{w}$ <u>with respect to</u> K we mean a pair of definition (a, δ) of $\bar{w}$, which is a minimal pair with respect to K. If (a, δ) and (a', δ) are two minimal pairs of definition of $\bar{w}$, then $[K(a) : K] = [K(a') : K]$, and so this number denoted by

$$[K : \bar{w}]$$

<u>depends only on $\bar{w}$ and K.</u>

Since the field K is fixed, we shall usually write "a minimal pair of definition" instead of "a minimal pair of definition with respect to K".

Now let w be an r.t. extension of v to $K(X)$, and $\bar{w}$ a common extension of $\bar{v}$ and w to $\bar{K}(X)$. A pair of definition (respectively a minimal pair of definition) of $\bar{w}$ will be usually called a <u>pair of definition</u> (respectively a <u>minimal pair of definition with</u>

respect to K) of w. We define the number $[K : w]$ by the equality

(3) $\quad [K : w] = [K : \bar{w}]$

We shall see later [Theorem 2.2 and Remark 2.4] that the number $[K : w]$ depends only on w and K, and not on $\bar{w}$. Also, in Theorem 2.2 we shall prove that if w_1, w_2 are two common extensions of $\bar{v}$ and w to $\bar{K}(X)$, then w_1 is closely related to w_2. Moreover, there exists only a finite number of common extensions of $\bar{v}$ and w to $\bar{K}(X)$ (Corollary 2.3).

The following result will be useful later:

PROPOSITION 1.1. Let w be an r.t. extension of $\bar{v}$ to $\bar{K}(X)$ and let (a, δ) be a pair of definition of w. Let $r = (f(X))/(g(X))$ be an element of $\bar{K}(X)$ such that $w(r) = 0$ and such that r^* is transcendental over $k_{\bar{v}}$. Then there exists a root b of $f(X)g(X) = 0$ such that (b, δ) is a pair of definition of w.

Proof. Let $f(X) = c\prod_{i=1}^{n}(X - a_i)$, $g(X) = d\prod_{i=1}^{n}(X - b_i)$. Assume that

$\bar{v}(a - a_i) < \delta$ for all $i = 1, \ldots, n$

$\bar{v}(a - b_j) < \delta$ for all $j = 1, \ldots, m$.

i.e., according to (2), for every $b \in \{a_1, \ldots, a_n, b_1, \ldots, b_m\}$, (b, δ) is not a pair of definition of w. Now, if $a'_i \in \bar{K}$ is such that $w(X - a_i) = \bar{v}(a'_i)$, then $((X - a_i)/a'_i)^*$ is algebraic over $k_{\bar{v}}$. Also, if $b'_j \in \bar{K}$ is such that $w(X - b_j) = \bar{v}(b'_j)$, then $((X - b_j)/b'_j)^*$ is algebraic over $k_{\bar{v}}$. But then it follows that

$$r^* = \left(\frac{ca'_1 \cdots a'_n}{db'_1 \cdots b'_m} \cdot \frac{\prod_i ((X-a_i)/a'_i)}{\prod_j ((X-b_j)/b'_j)}\right)^* = \left(\frac{ca'_1 \cdots a'_n}{db'_1 \cdots b'_m}\right)^* \cdot \frac{\prod_i ((X-a_i)/a'_i)^*}{\prod_j ((X-b_j)/b'_j)^*}$$

is also algebraic over $k_{\bar{v}}$, and this contradicts our hypothesis.

According to [5] (see also [6]) a valuation v on K is said to be Henselian, if for every algebraic extension L/K, v has a unique extension to L.

2. COMMON EXTENSIONS OF $\bar{v}$ AND w TO $\bar{K}(X)$

In this section we show that the set of all common extensions of $\bar{v}$ and w to

$\overline{K}(X)$ is non-empty. and this set is finite. We show that the number $[K : w]$ defined in (3) is dependent only to v and w, and this number is closely related to the "fundamental inequality" defined in [7].

Let $(K.v)$ be a valuation pair. Let us remind that by $\overline{K}$ we denote a fixed algebraic closure of K and by $\overline{v}$ a fixed extension of v to $\overline{K}$.

PROPOSITION 2.1. If w is an extension of v to $K(X)$ there always exists a common extension $\overline{w}$ of w and $\overline{v}$ to $\overline{K}(X)$. Moreover, w is an r.t. extension of v if and only if $\overline{w}$ is an r.t. extension of $\overline{v}$.

Proof. First we notice that $\overline{K}/K$ and $\overline{K}(X)/K(X)$ are normal extensions. If σ is an automorphism of $\overline{K}/K$, denote by $\overline{\sigma}$ the automorphism of $\overline{K}(X)/K(X)$ defined in the following canonical way: if $f(X) = a_o + a_1X + ... + a_nX^n$ is an element of $\overline{K}[X]$. then set $\overline{\sigma}(f(X)) = \sigma(a_o) + \sigma(a_1)X + ... + \sigma(a_n)X^n$. and extend it canonically to $\overline{K}(X)$. The assignment

$$\sigma \rightsquigarrow \overline{\sigma}$$

is an isomorphism of Aut $(\overline{K}/K)$ onto $\mathrm{Aut}(\overline{K}(X)/K(X))$.

Now let w' be an extension of w to the algebraic extension $\overline{K}(X)$ of $K(X)$, and let v' be the restriction of w' to $\overline{K}$. It is clear that v' is an extension of v to $\overline{K}$. Since $\overline{K}/K$ is a normal extension, there exists (see [5, Ch. VI, §8]) an element $\sigma \in \mathrm{Aut}(\overline{K}/K)$ such that $v' = \sigma\overline{v}$, i.e. $v'(a) = \overline{v}(\sigma^{-1}(a))$, for all $a \in \overline{K}$. Let us define $\overline{w}$ by $\overline{w} = \overline{\sigma}^{-1}w'$. Then the restriction of $\overline{w}$ to $\overline{K}$ is exactly $\overline{v}$, as claimed.

The last part of the proposition is obvious.

THEOREM 2.2. Let w_1, w_2 be two r.t. extensions of $\overline{v}$ to $\overline{K}(X)$. The following statements are equivalent:

1) w_1 and w_2 coincide on $K(X)$.

2) There exists a minimal pair of definition (a_i, δ_i) of w_i, $i = 1,2$ (with respect to K) such that the following conditions are fulfilled:

a) $\delta_1 = \delta_2$, and a_1, a_2 are conjugate over K.

b) If $g(X) \in K[X]$ is such that $\deg g(x) < [K(a_1) : K]$, then $\bar{v}(g(a_1)) = \bar{v}(g(a_2))$.

Proof. The proof uses [2, Theorem 2.1]. We shall use notation, definitions and considerations related to [2, Theorem 2.1], which are briefly recalled in the following:

1) $\Rightarrow$ 2). Let w be the common restriction of w_1 and w_2 to $K(X)$. Let (a_i, δ_i) be a minimal pair of definition of w_i, $n_i = [K(a_i) : K]$, $i = 1,2$. Also, denote by $f_i(X)$ the (monic) minimal polynomial of a_i with respect to K, and let $\gamma_i = w_i(f_i(X))$, $i = 1,2$. Denote by e_i the smallest non-zero positive integer e_i such that $e_i \gamma_i \in G_{v_i}$ (here v_i is the restriction of $\bar{v}$ to $K(a_i)$, $i = 1,2$). According to [2, Theorem 2.1], there exists a polynomial $h_i(X) \in K[X]$, such that

$$
(4) \qquad \begin{aligned} & w_i(h_i(X)) = \bar{v}(h_i(a_i)) = e_i \gamma_i \\ & \deg h_i(X) < n_i \end{aligned}
$$

$i = 1,2$. Then, according to [2, Theorem 2.1],

$$r_i(X) = (f_i^{e_i}(X))/h_i(X)$$

$i = 1,2$, is the rational fraction in $K(X)$ of smallest degree such that $w_i(r_i) = 0$, and r_i^* is transcendental over k_v. As w_1 and w_2 coincide on $K(X)$, we have:

$$e_1 n_1 = \deg r_1 = \deg r_2 = e_2 n_2 \ .$$

First we shall prove that $n_1 = n_2$. Indeed, let us assume that $n_1 < n_2$. Thus since $w_1(r_1) = w_2(r_1) = 0$ and since r_1^* is transcendental over k_v by Proposition 1.1, it follows that there exists a root a of $f_1(X)$ or of $h_1(X)$ such that (a, δ_2) is a pair of definition of w_2. But this is false since $\deg h_1(X) < n_1 < n_2$. Therefore one necessarily has $n_1 \geq n_2$ and by symmetry $n_1 = n_2$, as claimed. We have implicitly proved that we may assume that w_2 has a minimal pair of definition (a, δ_2) such that a and a_1 are conjugate, i.e. we may assume that a_2 and a_1 are conjugate, and so $f_1 = f_2 = f$.

Furthermore, condition b) of 2) follows by [2, Theorem 2.1], since for every polynomial $g(X) \in K(X)$ such that $\deg g(X) < n_1 = \deg f_1$ one has

$$
(5) \qquad w(g(X)) = \bar{v}(g(a_1)) = \bar{v}(g(a_2)) \ .
$$

Finally, we shall prove that $\delta_1 = \delta_2$. If a_1 is purely inseparable over K, let e be the smalest positive integer such that $a_1^{p^e} = b$, belongs to K. Then the minimal polynomial of a is

$$f(x) = X^{p^e} - b,$$

Therefore one has $a_1 = a_2$, and so

$$w_1(f(X)) = w_1(X^{p^e} - a_1^{p^e}) = p^e w_1(X - a_1) = w_2(f(X)) = p^e w_2(X - a_1),$$

i.e.

$$w_1(X - a_1) = \delta_1 = w_2(X - a_1) = \delta_2 .$$

Let us assume that a_1 is not purely inseparable over K. Then, we may write

$$f(X) = \sum_{i=1}^{n} g_i(a_1)(X - a_1)^i = \sum_{i=1}^{n} g_i(a_2)(X - a_2)^i$$

where $g_i(X)$ $(i = 1,\dots,n)$ are polynomials, with coefficients in $\bar{K}$, of degree at most deg f - 1. Then, according to [2, Theorem 2.1] one has:

$$\text{(6)} \qquad \begin{aligned} w_1(f(X)) &= \inf_i (\bar{v}(g_i(a_1)) + i\delta_1) = \gamma_1 \\ w_2(f(X)) &= \inf_i (v(g_i(a_2)) + i\delta_2) = \gamma_2 . \end{aligned}$$

But since w_1 and w_2 coincide on K(X), then one has $\gamma_1 = \gamma_2$. Finally, the equality $\delta_1 = \delta_2$ follows by (5) and (6). Indeed, assume that $\gamma_1 = \bar{v}(g_{i_1}(a_1)) + i_1\delta_1$ and $\gamma_2 = \bar{v}(g_{i_2}(a_2)) + i_2\delta_2 = \gamma_1$. Hence $\bar{v}(g_{i_1}(a_1)) + i_1\delta_1 = \bar{v}(g_{i_2}(a_2)) + i_2\delta_2$. If $i_1 \neq i_2$, then by (5) one sees that $\bar{v}(g_{i_2}(a_1)) + i_2\delta_1 \geq \bar{v}(g_{i_1}(a_1)) + i_1\delta_1 = \bar{v}(g_{i_2}(a_2)) + i_2\delta_2$. Now since $\bar{v}(g_{i_2}(a_1)) = \bar{v}(g_{i_2}(a_2))$, (see [2, Theorem 2.1]), it follows that $\delta_1 \geq \delta_2$. By symmetry, we have $\delta_1 \leq \delta_2$, hence $\delta_1 = \delta_2$.

The implication 2) $\Rightarrow$1) follows immediately by [2, Theorem 2.1].

COROLLARY 2.3. Let w be an r.t. extension of v to K(X). Then there exists only a finite number of common extensions of $\bar{v}$ and w to $\bar{K}(X)$.

Proof. Indeed, let $\bar{w}$ be a common extension of w and $\bar{v}$ to $\bar{K}(X)$, and let (a, δ) be a minimal pair of definition of $\bar{w}$. According to Theorem 2.2, there exist at most

$[K(a) : K]$ common extensions of $\overline{v}$ and w to $\overline{K}(X)$.

REMARK 2.4. At this point we show that the number $[K : w]$ defined in (3) depends only to v and w.

For that we prove:

A) The number $[K : w]$ does not depend of a common extension of $\overline{v}$ and w to $\overline{K}(X)$. Indeed, let w be an r.t. extension of v to $K(X)$ and let $\overline{w}$ be a common extension of $\overline{v}$ and w to $\overline{K}(X)$. Let also (a_1, δ), (a_2, δ) be two minimal pairs of definition of $\overline{w}$. According to (3) and to the definition of $[K : w]$ one has:

$$(7) \qquad [K : w] = [K : \overline{w}] = [K(a_1) : K] = [K(a_2) : K] .$$

It is clear that the number $[K : w]$ depends only on $\overline{w}$ and not on the choice of a minimal pair of $\overline{w}$. Now, according to Theorem 2.2, it follows that if w_1, w_2 are common extensions of $\overline{v}$ and w to $\overline{K}(X)$, so $[K : w_1] = [K : w_2]$. Hence the number $[K : w]$ in (7) does not depend on the choice of a common extension of $\overline{v}$ and w to $\overline{K}(X)$.

B) The number $[K : w]$ in (7) does not depends on the extension of v to $\overline{K}$. Indeed, let v' be another extension of v to $\overline{K}$. Then, since $\overline{K}/K$ is a normal extension, by [5, Ch. VI, § 8] there exists $\sigma \in \mathrm{Aut}(\overline{K}/K)$ such that $v' = \sigma\,\overline{v}$. Let w' be a common extension of v' and w to $\overline{K}(X)$. Then $\overline{w} = \sigma^{-1}w'$ is a common extension of $\overline{v}$ and w to $\overline{K}(X)$. Let (a', δ') be a minimal pair of definition of w'. and similarly let (a, δ) be a minimal pair of definition of $\overline{w}$. Consider also (see [2]):

$$M_{\overline{w}} = \left\{ \overline{w}(X - b) \,\middle|\, b \in \overline{K} \right\} \subseteq G_{\overline{v}} = G_{\overline{w}}$$

$$M_{w'} = \left\{ w'(X - b) \,\middle|\, b \in \overline{K} \right\} \subseteq G_{v'} = G_{w'} .$$

Now, since $v' = \sigma\overline{v}$, we may assume that $G_{\overline{v}} = G_{v'}$. Furthermore, one has that $w'(X - a') = \delta' = (\sigma^{-1}\overline{w})(X - a') = \overline{w}(X - \sigma^{-1}(a')) \leq \delta$ (because, according to [2], δ is an upper bound of $M_{\overline{w}}$). Similarly one sees that $\delta \leq \delta'$, and so $\delta = \delta'$. Hence the equality $\overline{w}(X - \sigma(a')) = \delta$ shows that $(\sigma(a'), \delta)$ is a pair of definition of $\overline{w}$. Therefore, $[K(\sigma(a') : K] = [K(a') : K] \geq [K(a) : K]$ because (a, δ) is a minimal pair of definition of $\overline{w}$. By symmetry, it follows that $[K(a') : K] \leq [K(a) : K]$, and hence

$[K(a):K] = [K(a'):K]$,

By this equality it follows that the number $[K:w]$ defined in (3) depends only on v and w, and not on $\overline{v}$.

REMARK 2.5. At this point we consider the relation between the number $[K:w]$ defined in (3) and the numbers deg (w/v), f(w/v) and e(w/v) defined in [7]. This relation will be used in the proof of Theorem 4.5.

If $r \in K(X)$, $r \notin K$, set $\deg r = [K(X):K(r)]$. As usual, if w is an r.t. extension of v to K(X), then by deg (w/v) we denote the least integer n such that there exists $r \in O_w$ of degree n such that r^* is transcendental over k_v.

Let $\overline{w}$ be a common extension of $\overline{v}$ and w to $\overline{K}(X)$ and let (a, δ) be a minimal pair of definition of $\overline{w}$. In [2, Theorem 2.1] it is proved that

(8) $\deg(w/v) = [K:w]e$,

where e is the smallest positive integer such that $ew(f) \in G_{v_1}$ (here f is the monic minimal polynomial of a ~~relative~~ with respect to K and v_1 is the restriction of $\overline{v}$ to K(a)). Since the numbers deg(w/v) and $[K:w]$ (see Remark 2.4), depend only on v and w, by (8) it follows that e depends also only on v and w. Moreover, it is easy to show that $\gamma = w(f(X))$ also depends only on w and v.

Furthermore, it is clear that $G_v \subseteq G_w$ and $[G_w : G_v] < \infty$. Set:

$e(w/v) = [G_w : G_v]$.

In [2, Theorem 2.1] it is proved that:

(9) $e(w/v) = e \cdot e(v_1/v)$

where e is defined as above, and $e(v_1/v) = [G_{v_1} : G_v]$. By (8) and (9) it follows that $e(v_1/v)$ depends only on w and v and not on $\overline{v}$.

Finally, in [2, Theorem 2.1] it is shown that k (the algebraic closure of k_v in k_w) can be canonically identified with k_{v_1}. Hence, if we denote $f(w/v) = [k : k_v]$, then one has:

(10) $f(w/v) = [k_{v_1} : k_v] = f(v_1/v)$

and obviously, this number also depends only to w and v, and not on $\bar{v}$. By [2, Theorem 2.1] and by the above considerations it follows that w is determined by v, $\bar{v}$ and a minimal pair of definition (a, δ) of $\bar{w}$. (We remind that such a pair (a, δ) has been also called a minimal pair of definition of w.) The set of minimal pairs of definition of w with respect to K is dependent only on w and v, and not on $\bar{v}$.

In [7] it is proved (see also [2]) that

(11) $\deg(w/v) \geq e(w/v)f(w/v)$

holds and the question of finding conditions under which (11) becomes an equality is raised. Since, by (8), (9), (10) and (11) we have

(12) $[K(a) : K] = [K : w] \geq e(v_1/v)f(v_1/v)$,

the equality in (11) is equivalent to the equality in (12).

In Theorem 4.5 we give a general condition under which the inequality in (12) becomes an equality.

3. MINIMAL PAIRS OF DEFINITION OF AN r.t. EXTENSION

Let v be a valuation on a field K, let $\bar{K}$ be a fixed algebraic closure of K and let $\bar{v}$ be a fixed extension of v to $\bar{K}$.

In this section we are concerned with the following questions:

I) Which pairs $(a, \delta) \in \bar{K} \times G_{\bar{v}}$ are minimal pairs for some r.t.-extensions of v to K(X)?

II) Let $(a, \delta) \in \bar{K} \times G_{\bar{v}}$ and let $w_{(a,\delta)}$ be the valuation on $\bar{K}(X)$ defined by inf: $\bar{v}$, a and δ. Find a minimal pair of $w_{(a,\delta)}$ with respect to K.

Since both questions I) and II) seem to be difficult in the general setting, we give a bunch of results in some particular but important cases.

Denote by $(\tilde{K}, \tilde{v})$ the <u>completion</u> of (K,v) in the sense of [5, Ch. VI, § 5]. By [5, Ch. VI, § 5], it follows that $(\tilde{K}, \tilde{v})$ is an immediate extension of (K,v) (see [8, Ch. II]), i.e.

$$k_v = k_{\tilde{v}}, \quad G_v = G_{\tilde{v}} .$$

As usual, we shall say that (K,v) is <u>complete</u> if $(K,v) = (\tilde{K},\tilde{v})$.

By a <u>(generalized) Cauchy sequence</u> in (K,v) we mean a pair $(I, \{a_\rho\})$, where I is a cofinal subset of G_v, and $\{a_\rho\}$ is a subset of K indexed by I, such that the following condition is fulfilled: for all $\delta \in G_v$, there exists $\rho(\delta) \in I$ so that $v(a_{\rho_1} - a_{\rho_2}) > \delta$ for any $\rho_1 \geq \rho(\delta)$, $\rho_2 \geq \rho(\delta)$. An element $a \in K$ is said to be a <u>limit</u> of the Cauchy sequence $(I, \{a_\rho\})$ (in writing $a = \lim a_\rho$) if for every $\delta \in G_v$, there exists $\rho(\delta) \in I$ such that $v(a_\rho - a) > \delta$, for any $\rho \geq \rho(\delta)$. According to the construction of $(\tilde{K},\tilde{v})$ given in [5, Ch. VI, § 5], it follows that every element of $\tilde{K}$ is the limit of a Cauchy sequence in (K,v). Moreover (K,v) is complete if and only if every Cauchy sequence in (K,v) has a limit in K. In particular, every Cauchy sequence in $(\tilde{K},\tilde{v})$ has a limit in $\tilde{K}$.

1. THEOREM 3.1. Every r.t. extension w of v to $K(X)$ has a minimal pair of definition $(b, \delta) \in \bar{K} \times G_{\bar{v}}$ such that b is separable over K.

Proof. Let, as usual, $\bar{w}$ be a common extension of $\bar{v}$ and w to $\bar{K}(X)$, and let (a, δ) be a minimal pair of definition of $\bar{w}$ with respect to K. Obviously we assume that char $K = p > 0$ and that a is not separable. Let e be the smallest positive integer such that a^{p^e} is separable over K, and let $K' = K(a^{p^e})$. It is clear that a is purely inseparable over K' and its minimal polynomial over K' is just $X^{p^e} - a^{p^e}$. Let $c \in K$ be a suitable non-zero element. Then the polynomial

$$f(x) = X^{p^e} - cX - a^{p^e}$$

is separable (i.e., its formal derivative is non-zero), and so it has at least a separable irreducible factor, or equivalently, there exists at least an element $b \in \bar{K}$ such that b is separable over K' and

(13) $$f(b) = b^{p^e} - cb - a^{p^e} = 0.$$

Then one has:

$$b^{p^e} - a^{p^e} = (b - a)^{p^e} = cb$$

and hence

(14) $\qquad p^e\bar{v}(b - a) = v(c) + \bar{v}(b).$

Let us assume that

(15) $\qquad v(c) \geq \sup(p^e\bar{v}(a),\ p^e\delta - \bar{v}(a)).$

By (13) it results that $\bar{v}(a) = \bar{v}(b)$. Indeed, let us assume that $\bar{v}(a) > \bar{v}(b)$. Then by (13) we infer that $p^e\bar{v}(b) = \bar{v}(b) + v(c)$, or equivalently $\bar{v}(b) = (p^e - 1)^{-1}v(c)$. But then one has: $v(c) < (p^e - 1)\bar{v}(a)$ which contradicts (15). Hence $\bar{v}(a) \leq \bar{v}(b)$. If $\bar{v}(a) < \bar{v}(b)$, then also by (13) it results that $p^e\bar{v}(a) = \bar{v}(b) + v(c)$, i.e. $\bar{v}(b) = p^e\bar{v}(a) - v(c)$. But then $\bar{v}(a) < p^e\bar{v}(a) - v(c)$, and so $v(c) < (p^e - 1)\bar{v}(a)$, which contradicts (15). Therefore $\bar{v}(a) = \bar{v}(b)$, and by (14) and (15) one has:

$$p^e\bar{v}(b - a) = v(c) + \bar{v}(a) \geq p^e\delta ,$$

i.e., $\bar{v}(b - a) \geq \delta$, and so (b, δ) is also a pair of definition of $\overline{w}$ and of w. It is clear that b is also separable over K and one has:

$$[K(b) : K] = [K' : K][K'(b) : K'] \leq [K' : K][K'(a);K'] = [K(a) : K],$$

Finally, since (a, δ) is a minimal pair of definition of $\overline{w}$, it follows that ~~(b, δ)~~ (b, δ) is also a minimal pair of definition of $\overline{w}$ (and w).

2. Now let us consider the question II). Let $\overline{w}$ be a common extension of w and $\bar{v}$ to $\bar{K}(X)$ and let $(a, \delta) \in \bar{K} \times G_{\bar{v}}$ be a pair of definition of $\overline{w}$ (and w). How can we find a minimal pair of definition of $\overline{w}$ (and w) with respect to K?

We shall consider this question in the special case when v is a Henselian valuation. Hence, for the rest of this subsection we assume that v is Henselian. Consider an element $a \in \bar{K}$, separable over K. Let us denote

$$\omega(a) = \sup\ \bar{v}(a - a')$$

where the supremum is taken over all $a' \in \bar{K}$ conjugate to a over K and $a' \neq a$. Now since v is Henselian, we have that $\bar{v}(a) = \bar{v}(a')$ if a' and a are conjugate over K, and so one has:

$$\omega(a) \geq \bar{v}(a).$$

PROPOSITION 3.2. Let δ be an element of $G_{\bar{v}}$ and let $a \in \bar{K}$ be an element which is separable over K. Denote by $\bar{w}$ the valuation on $\bar{K}(X)$ defined by inf, $\bar{v}$, a and δ. Then:

a) If $\delta \leq \bar{v}(a)$, $(0, \delta)$ is a minimal pair of definition of $\bar{w}$ with respect to K.

b) If $\delta > \omega(a)$, (a, δ) is a minimal pair of definition of $\bar{w}$ with respect to K.

Proof. The assertion a) is obvious, and the assertion b) follows by Krasner's lemma (see [6. pag. 122]).

What happens when $\omega(a) \geq \delta > \bar{v}(a)$?

Let $a \in \bar{K}$ be separable over K and let $C = \{a = a_1, a_2, \ldots, a_n\}$ be the set of all elements of $\bar{K}$ conjugated to a over K. If $I = \{a, a_{i_2}, \ldots, a_{i_r}\}$ is a subset of C, denote by $\{s_j(I)\}_{1 \leq j \leq r}$ the symmetric fundamental polynomials defined by the elements of I. Denote also $K_I := K(s_1(I), \ldots, s_r(I))$.

PROPOSITION 3.3. Let $a \in \bar{K}$ be separable over K and let $\delta \in G_{\bar{v}}$ be such that $\omega(a) \geq \delta > \bar{v}(a)$. Denote by $\bar{w}$ the valuation on $\bar{K}(X)$ defined by inf, $\bar{v}$, a and δ. Assume that one of the following conditions is fulfilled:

a) char $k_v = 0$.

b) K is perfect and v is of rank one.

Then there exists a subset $I = \{a, a_{i_2}, \ldots, a_{i_r}\}$ of C and an element $b \in K_I$ such that $K(b) = K_I$ and that (b, δ) is a minimal pair of definition of $\bar{w}$ with respect to K.

Proof. Let (b, δ) be a minimal pair of definition of $\bar{w}$. Denote by $I = \{a, a_{i_2}, \ldots, a_{i_r}\}$ the subset of C consisting of all conjugates of a over K(b). With the above notation one has $K_I \subseteq K(b)$. Also if $a' \in I$, then, since v is Henselian, one has $\bar{v}(a' - b) = \bar{v}(a - b) \geq \delta$. Therefore if $a \neq a'$ it follows that:

$$\bar{v}(a - a') = \bar{v}(a - b + b - a') \geq \delta .$$

According to [3, Proposition 2, pag. 425], there exists $b' \in K_I$ such that

$$\bar{v}(a - b') \geq \delta .$$

Therefore (b', δ) is also a pair of definition of $\bar{w}$. But (b, δ) is a minimal pair of definition of $\overline{w}$ and since $K \subseteq K_I \subseteq K(b)$, it follows that $K(b) = K_I = K(b')$. The proof is complete.

REMARK 3.4. If char $k_v = 0$, we may determine the element b as follows: Since

$$\overline{v}(a - a') \geq \delta \qquad a' \in \{a, a_{i_2}, \ldots, a_{i_r}\}$$

then one has that: $\overline{v}(a + a_{i_1} + \ldots + a_{i_r} - ra) \geq \delta$ and so

$$\overline{v}\left(\frac{a + a_{i_2} + \ldots + a_{i_r}}{r} - a\right) \geq \delta$$

since $\overline{v}(r) = 0$. Hence, we can take

$$b = \frac{a + a_{i_2} + \ldots + a_{i_r}}{r} \in K_I .$$

3. Now we consider the question I). We reformulate I) with respect to an element $a \in K$ in the following somewhat equivalent form:

CONDITION 3.5. There exists an element $\delta \in QG_v = G_{\overline{v}}$ such that (a, δ) is a minimal pair of definition of $w_{(a,\delta)}$ with respect to K.

As usual $w_{(a,\delta)}$ is the valuation of $\overline{K}(X)$ defined by inf. $\overline{v}$, a and δ.

One has the following obvious result:

REMARK 3.6. Let $(a, \delta) \in \overline{K} \times G_{\overline{v}}$.

1) The following statements are equivalent:

a) (a, δ) is a minimal pair of definition of $w_{(a,\delta)}$ with respect to K.

b) If $b \in \overline{K}$ is such that $[K(b) : K] < [K(a) : K]$ then $\overline{v}(a - b) < \delta$.

2) If (a, δ) is a minimal pair of definition of $w_{(a,\delta)}$, then for every $\delta' \in G_{\overline{v}}$, $\delta' \geq \delta$, (a, δ') is a minimal pair of definition of $w_{(a,\delta')}$.

PROPOSITION 3.7. Let $a \in \overline{K}$. Consider the following statements:

a) The condition (3.5) is satisfied for a.

b) The set

$$M(a,K) = \left\{ \bar{v}(a-b) \middle/ b \in \bar{K}, \text{ such that } [K(a):K] > [K(b):K] \right\}$$

is upper bounded in $G_{\bar{v}}$.

c) The set

$P(a,K) = \left\{ \bar{v}(g(a)) \middle/ g \in K[X], \ g \text{ monic and } \deg g < [K(a):K] \right\}$ is upper bounded in $G_{\bar{v}}$.

d) Let $n = [K(a):K]$. There exists $\nu \in G_{\bar{v}}$, $\nu \leq \inf\limits_{1 \leq j \leq n-1} (j\bar{v}(a))$ such that the set: $P(a,K,\nu) = \left\{ \bar{v}(g(a)) \middle/ g(X) = X^m + a_1X^{m-1} + \ldots + a_m \in K[X], \ m < n \text{ and } v(a_i) \geq \nu, \ 1 \leq i \leq m \right\}$ is upper bounded in $G_{\bar{v}}$.

Then the implications: a)$\Leftrightarrow$b)$\Leftrightarrow$c)$\Rightarrow$d) are always valid. Moreover, if v is Henselian then also d) $\Rightarrow$b).

Proof. The equivalence a)$\Leftrightarrow$b) follows by Remark 3.6, 1).

b)$\Rightarrow$c) Let δ be a positive upper bound of $M(a,K)$ in $G_{\bar{v}}$ and let $g(X) \in K[X]$ be a monic polynomial of degree smaller than $n = [K(a):K]$. Let $g(X) = \prod\limits_i (X - b_i)$ be the decomposition of $g(X)$ in $\bar{K}(X)$. For each i, one has $[K(b_i):K] < n$ and so $\bar{v}(a - b_i) < \delta$. Hence $\bar{v}(g(a)) = \sum\limits_i \bar{v}(a - b_i) < n\delta$, and so $P(a,K)$ is upper bounded by $n\delta$.

The implication c) $\Rightarrow$d) is obvious.

Now let us assume v to be Henselian. If b and b' are conjugate over K (i.e., they have the same minimal polynomial over K), then $\bar{v}(a) = \bar{v}(a')$. We shall show that the implication d) $\Rightarrow$b) is also valid. Let us assume that d) is valid and b) is false, i.e. the set $M(a,K)$ is not upper bounded in $G_{\bar{v}}$. Let λ be a positive upper bound of $P(a,K,\nu)$ such that $\lambda > \bar{v}(a)$. Let $b \in \bar{K}$ be such that $[K(b):K] < [K(a):K] = n$, and that

(16) $\qquad \bar{v}(a-b) > n\lambda - (n-1)\bar{v}(a)$.

Then $\bar{v}(a-b) > \bar{v}(a)$ and so $\bar{v}(a) = \bar{v}(b)$. If b' and b are conjugate over K, and since v is Henselian, one has $\bar{v}(b) = \bar{v}(b')$ and hence $\bar{v}(a - b') \geq \bar{v}(a)$. Let g be the monic minimal polynomial of b over K. Then $g(X) = \prod\limits_{i=1}^{m} (X - b_i)$, $m < n$, $b = b_1$. If $g(X) = X^m +$

$+ a_1X^{m-1} + \dots + a_m$, then according to well known relations between roots and coefficients of a polynomial. one gets: $v(a_i) \geq \inf_{i \leq j \leq m} (j\bar{v}(b)) \geq \inf_{i \leq j \leq n} (j\bar{v}(a)) \geq \gamma$. Therefore $\bar{v}(g(a)) \in P(a.K.\gamma)$. Moreover one has: $\bar{v}(g(a)) = \bar{v}(\prod_i (a - b_i)) = \bar{v}(a - b) + \sum_{i=2}^{m} \bar{v}(a - b_i) \geq$ $\geq \bar{v}(a - b) + (m - 1)\bar{v}(a)$. By (16) it follows that $\bar{v}(g(a)) > \lambda$, which is a contradiction. Therefore d) $\Rightarrow$ b) as claimed.

THEOREM 3.8. Let $(\tilde{K}.\tilde{v})$ be the completion of $(K.v)$. Denote by $\hat{K}$ an algebraic closure of $\tilde{K}$ which contains $\bar{K}$ and by $\hat{v}$ a common extension of $\bar{v}$ and $\tilde{v}$ to $\hat{K}$. For an element $a \in \bar{K}$ the following statements are equivalent:

a) The set $P(a,K)$ is upper bounded in $G_{\bar{v}}$.

b) The set $P(a,\tilde{K})$ is upper bounded in $G_{\hat{v}} = G_{\bar{v}}$, and $[K(a) : K] = [\tilde{K}(a) : \tilde{K}]$.

Proof. Since the implication b) $\Rightarrow$ a) is obvious, we have only to show that a) $\Rightarrow$ b). First, we shall show that $[\tilde{K}(a) : \tilde{K}] = [K(a) : K] = n$. Let

$$(17) \qquad g(X) = X^m + a_{m-1}X^{m-1} + \dots + a_1X + a_o$$

be the minimal polynomial of a over $\tilde{K}$. Assume that $m < n$. Denote by I a well ordered cofinal subset of G_v (see [4, § 2, Exercise 4]). Since $a_i \in \tilde{K}$, $i = 0,\dots,m - 1$, for each $\rho \in I$ and each i, $0 \leq i \leq m - 1$, there exists an element $a_i^{(\rho)} \in K$ such that

$$(18) \qquad \tilde{v}(a_i - a_i^{(\rho)}) \geq \rho.$$

Set $g_\rho(X) = a_o^{(\rho)} + a_1^{(\rho)}X + \dots + a_{m-1}^{(\rho)}X^{m-1} + X^m \in K[X]$. By hypothesis one has: $\bar{v}(g_\rho(a)) \leq \lambda$, where λ is the upper bound of $P(a,K)$. But according to (18) the set

$$\bar{v}(g_\rho(a)) = \hat{v}(g_\rho(a) - g(a)) \quad , \rho \in I$$

is unbounded, and this contradicts a). Hence $[\tilde{K}(a) : \tilde{K}] = n$.

Now, let $g(X) \in \tilde{K}[X]$ be a monic polynomial whose degree is smaller than n (see also (17)). As above, we define the polynomials $g_\rho(X) \in K[X]$, such that the conditions (18) are fulfilled. If λ is an upper bound of $P(a,K)$, then $\bar{v}(g_\rho(a)) \leq \lambda$ for all ρ. But, according to (18) the set $\left\{\hat{v}(g_\rho(a) - g(a))\right\}_\rho$ is unbounded and so $\hat{v}(g_\rho(a)) = \bar{v}(g_\rho(a)) =$

$= \hat{v}(g(a)) \leq \lambda$ for suficiently large ρ. Therefore the set P(a.K) has also an upper bound, as claimed.

THEOREM 3.9. Let v be Henselian. Then:

a) Condition (3.5) is verified for every element $a \in \overline{K}$, separable over K.

b) If K is complete with respect to v (i.e. $(K.v) = (\tilde{K},\tilde{v})$) then condition 3.5 is valid for all elements of $\overline{K}$.

Proof. a) follows by Proposition 3.2, b).

b) Let $a \in \overline{K}$; it is clear that we may assume that $a \notin K$. Let $[K(a) : K] = n$. According to Proposition 3.7, it will be enough to show that the set $P(a,K,\nu)$ is upper bounded for some $\nu \leq \inf\limits_{i \leq j \leq n-1} (j\bar{v}(a))$.

Let us assume that for every ν, $\nu \leq \inf\limits_{j} (j\bar{v}(a))$, the set $P(a,K,\nu)$ is not upper bounded. This means that for every ν, $\nu \leq \inf (j\bar{v}(a))$, there exists a triplet

(19) $\quad T_{\nu} = (I, \{g_{\rho}(X)\}, m)$

where I is a well ordered and cofinal subset of G_v (see [4, §2, Ex. 4] and $\{g_{\rho}(X)\}$ is the set of polynomials over K, monic, and of the same degree $m < n$.

(20) $\quad g_{\rho}(X) = X^m + a_{1\rho}X^{m-1} + \dots + a_{m\rho} \qquad \rho \in I$

such that

(21) $\quad \begin{aligned} & v(a_{i\rho}) \geq \nu, \qquad i \leq i \leq m, \quad \rho \in I \\ & \bar{v}(g_{\rho}(a)) \longrightarrow \infty \end{aligned}$

(The notation $\bar{v}(g_{\rho}(a)) \to \infty$ means that for every $\gamma \in G_{\bar{v}}$ there exists $\rho(\gamma) \in I$ such that $\bar{v}(g_{\rho}(a)) > \gamma$, for all ρ such that $\rho \geq \rho(\delta)$.)

In the set of all triples (19) we choose a triplet T_{ν_0}, corresponding to a suitable $v_0 \leq \inf\limits_{j} (j\bar{v}(a))$, such that m is as small as possible. For this triplet we shall use also the notation of (19) and (20).

Since K is assumed to be complete, we note that $m > 1$. We claim that we can assume that not all sequences:

(22) $\quad \{a_{i\rho}\} \qquad 0 \leq i \leq m, \qquad a_{o\rho} = 1, \quad \rho \in I$

contain a (generalized) Cauchy subsequence.

Indeed, if this were not the case let $\{a_{1\rho_1}\}$ be a Cauchy subsequence of $\{a_{1\rho}\}$, where ρ_1 belongs to a cofinal subset I_1 of I. Replacing I by I_1 we may assume that the sequence $\{a_{1\rho}\}$ is Cauchy. Furthermore, we pass to the sequence $\{a_{2\rho}\}$ and so on. Finally, we may assume that all the sequences (22) are Cauchy and since K is complete let $a_i = \lim a_{i\rho}$. Set $g(X) = X^m + a_1X^{m-1} + \ldots + a_m$. Then one has:

$$\bar{v}(g(a) - g_\rho(a)) = \bar{v}(\sum_{i=1}^{m} (a_i - a_{i\rho})a^{m-i})$$

and since $v(g_\rho(a)) \to \infty$, then $g(a) = 0$. This is a contradiction since $m < n = [K(a) : K]$.

Let i_o be the first index such that all the sequences $\{a_{i\rho}\}$, $0 \leq i < i_o$ are Cauchy but $\{a_{i_o\rho}\}$ does not contain any Cauchy subsequence. This means that there exists an element $\gamma \in G_{\bar{v}}, \gamma > 0$, such that for all $\rho \in I$ there exists an element $\rho' \geq \rho$ such that

(23) $\quad v(a_{i_o\rho'+1} - a_{i_o\rho'}) \leq \gamma.$

Denote by I' the set of all elements ρ' constructed as above; it is easy to see that I' is well ordered, and a cofinal subset of I, hence of $G_{\bar{v}}$. Consider the set of all polynomials:

$$h_\rho(X) = g_{\rho+1}(X) - g_\rho(X) = b_{1\rho} X^{m-1} + \ldots + b_{i_o\rho} X^{m-i_o} + \ldots + b_m .$$

We write I instead of I' and ρ instead of ρ'. Since $\bar{v}(g_\rho(a)) \to \infty$, then also $\bar{v}(h_\rho(a)) \to \infty$. By hypothesis one has

(24) $\quad v(b_{1\rho}) \to \infty, \ldots . v(b_{(i_o-1)\rho}) \to \infty$.

According to (23) we have:

(25) $\quad v(b_{i_o\rho}) \leq \gamma \qquad$ for all ρ.

Denote

$$k'_\rho(X) = h_\rho(X) - \sum_{j=1}^{i_o-1} b_{j\rho} X^{m-j} = b_{i_o\rho} X^{m-i_o} + \ldots + b_{m\rho} .$$

Since $\tilde{v}(h_\rho(a)) \to \infty$, then by (24) one checks that $\bar{v}(k'_\rho(a)) \to \infty$ and by (25) one sees that $b_{i_o\rho} \neq 0$ for all ρ; also by (25) it follows that $v(b_{i_o\rho}^{-1}) \geq -\gamma$. Therefore one has

$$\tilde{v}(k_\rho(a)) \to \infty$$

where $k_\rho(X) = (b_{i_o\rho})^{-1} k'_\rho(X)$. It is clear that the polynomials $k_\rho(X)$ are monic of the same degree, and conditions (21) are fulfilled for the triplet $T_{\nu_o - \gamma} = (I, \{k_\rho(X)\}, m - i_o)$. Since $\nu_o - \gamma < \nu_o \leq \inf_j (j\bar{v}(a))$ we obtain a contradiction with the definition of the triplet (19) because the common degree $m - i_o$ of $k_\rho(X)$ is smaller than m. Hence the set $P(a,K.\nu)$ is upper bounded for some $\nu \leq \inf_{1\leq j\leq n-1} (j\bar{v}(a))$, as claimed.

COROLLARY 3.10. Let $(\tilde{K},\tilde{v})$ be the completion of (K,v). Assume that $\tilde{v}$ is Henselian. Let $a \in \bar{K}$. Then with the notation from Proposition 3.7, one has that $M(a,K)$ is upper bounded in $G_{\bar{v}}$ if and only if $[K(a) : K] = [\tilde{K}(a) : \tilde{K}]$.

Proof. If $M(a.K)$ is upper bounded in $G_{\bar{v}}$ then by Theorem 3.8 it results that $[K(a) : K] = [\tilde{K}(a) : \tilde{K}]$. The other implication results by Theorems 3.8, 3.9 and Proposition 3.7.

In particular. one has:

COROLLARY 3.11. Let v be of rank one and let $a \in \bar{K}$. There exists an element $\delta \in G_{\bar{v}}$ such that (a, δ) is a minimal pair for $w_{(a,\delta)}$ if and only if $[K(a) : K] = [\tilde{K}(a) : \tilde{K}]$.

The proof follows from Corollary 3.10, since $\tilde{v}$ is in this case Henselian (see [8, Ch. II]).

COROLLARY 3.12. Let v be Henselian and let the valuation pair (K,v) be complete, i.e. $(K,v) = (\tilde{K},\tilde{v})$. If $a_1 \in \bar{K}$, then there exists $a_2 \in \bar{K}$, separable over K, such $k_{v_1} = k_{v_2}$ where v_i is the unique extension of v to $K(a_i)$, $i = 1,2$.

Proof. According to Theorem 3.9 there exists $\delta \in G_{\bar{v}}$ such that (a_1, δ) is a minimal pair of definition of $\bar{w} = w_{(a_1,\delta)}$. According to Theorem 3.1 there exists $a_2 \in \bar{K}$

such that (a_2, δ) is also a minimal pair of definition of $\overline{w}$ with respect to K, and a_2 is separable over K. If w is the restriction of $\overline{w}$ to K(X), then according to [2, Corollary 2.3], $k_{v_1} = k_{v_2}$, and this is the algebraic closure of k_v in k_w.

Moreover, we can give a somewhat independent proof. Let $b \in K(a_1)$ be such that $v_1(b) = 0$. Then one has $b = g(a_1)$, where $g(X) \in K[X]$ and $r = \deg g(X) < [K(a_1) : K] = [K(a_2) : K] = n$. Let

$$g(X) = c\prod_{i=1}^{r}(X - d_i)$$

be the splitting of g(X) over $\overline{K}$ as a product of linear factors. Now, since (a_1, δ) and (a_2, δ) are both minimal pairs of definition of $\overline{w}$, it results that for suitable elements $h_i \in \overline{K}$ one has

$$\overline{v}(a_1 - d_i) = \overline{v}(a_2 - d_i) = \overline{v}(h_i) < \delta , \quad 1 \leq i \leq r,$$

and so

$$w(g(X)) = v(c + \sum_{i=1}^{r} \overline{w}(X - d_i) = v(c) + \sum_{i=1}^{r} \overline{v}(a_1 - d_i) =$$

$$= v(c) + \sum_{i=1}^{r} \overline{v}(a_2 - d_i) = \overline{v}(g(a_1)) = \overline{v}(g(a_2)) = \overline{v}(b) .$$

Finally, since $v(a_1 - a_2) \geq \delta$, then $\left(\frac{a_1 - d_i}{h_i}\right)^* = \left(\frac{a_1 - d_i}{h_i}\right)^*$, $1 \leq i \leq r$, and so $g(a_1)^* = b^* = (g(a_2))^*$. Therefore $k_{v_1} \subseteq k_{v_2}$, and by symmetry it follows that $k_{v_1} = k_{v_2}$, as claimed.

REMARK 3.13. By Theorem 3.8 it follows that, in general, not for all $a \in \overline{K}$ there exists $\delta \in G_{\overline{v}}$ such that (a, δ) is a minimal pair of definition of $w_{(a,\delta)}$. This is the case, for example, if $a \notin K$ but $a \in \widetilde{K}$.

4. SOME APPLICATIONS

In this section we give some applications of the results proved in previous sections. Theorem 4.4 gives the existence of some r.t. extensions of v to K(X) with prescribed residue field and valued group. Also in Theorem 4.5 we give a general framework in which the fundamental inequality of [7] becomes on equality.

1. In what follows we shall denote by (K_1, v_1) the Hensealization of (K, v) included in $(\bar{K}, \bar{v})$ (see [6, §7]). It is known that (K_1, v_1) is an immediate extension of (K, v), i.e. $k_v = k_{v_1}$ and $G_v = G_{v_1}$. One has the following result which will be useful later (compare with Theorem 3.8):

PROPOSITION 4.1. Let $a \in \bar{K}$ be separable over K and suppose that $[K(a) : K] = [K_1(a) : K_1]$. Then there exists $\delta \in G_{\bar{v}}$ such that (a, δ) is a minimal pair of definition of $w_{(a,\delta)}$ ~~relative~~ with respect to K (i.e. Conditions 3.5 is fulfilled for a).

Proof. Since a is also separable over K_1, Theorem 3.9 a) implies that there exists $\delta \in G_{\bar{v}}$ such that (a, δ) is a minimal pair of definition of $w_{(a,\delta)}$ with respect to K_1. Now we assert that (a, δ) is also a minimal pair of definition of $w_{(a,\delta)}$ with respect to K. Indeed, if $b \in \bar{K}$ is such that $[K(b) : K] < [K(a) : K]$, then by hypothesis one has:

$$[K_1(b) : K_1] \leq [K(b) : K] < [K(a) : K] = [K_1(a) : K_1].$$

Now since (a, δ) is a minimal pair of definition of $w_{(a,\delta)}$ with respect to K_1 it follows that $\bar{v}(a - b) < \delta$. But this means that (a, δ) is also a minimal pair of definition of $w_{(a,\delta)}$ with respect to K.

LEMMA 4.2. (see also [7, Theorem 4.6]). Let k/k_v be a finite extension. There exist (infinitely many) r.t. extensions w of v to K(X) such that k_w is isomorphic to k(t), with t transcendental over k. Moreover we can find w such that:

$$\deg [w/v] = f(w/v) = [k : k_v] .$$

Proof. Let $k = k_v(x_1, \ldots, x_n)$. We shall prove that there exists an element $a \in \bar{K}$, separable over K such that $[K(a) : K] = [k : k_v]$ and that $k_{v_2} = k$, where v_2 is the restriction of v to K(a). For that we shall proceed by induction over n. Now since $k_{\bar{v}}$ is an algebraic closure of k_v, we can assume that $k_v \subseteq k \subseteq k_{\bar{v}}$. Let us assume that $n \geq 1$, $[k_v(x_1) : k_v] = s > 1$ and let $\varphi_1(Y) = c_0 + c_1 Y + \ldots + c_s Y^s$ be the minimal polynomial of x_1 over k_v. Let $A_i \in O_v$ be such that $A_i^* = c_i$, $0 \leq i \leq s$, and such that $A_1 \neq 0$. Then the polynomial:

$$\varphi(Y) = A_0 + A_1 Y + \dots + A_s Y^s$$

is irreducible and separable over K. Let $b_1,\dots,b_s$ be all roots of $\varphi(Y)$ in $\overline{K}$. It is easy to see that $\overline{v}(b_i) \geq 0$, $1 \leq i \leq s$, and that there exists an i such that $b_i^* = x_1$. Denote $b_i = a_1$. Now, by induction hypothesis, there exists $a_2 \in \overline{K}$, separable over $K(a_1)$, such $[K(a_1)(a_2) : K(a_1)] = [k : k(x_1)]$. Let $a \in \overline{K}$ be such that $K(a) = K(a_1, a_2)$. This a is the desired element.

Let v_2 be restriction of v to K(a) and let v_2' be the restriction of v to $K_1(a)$. It is clear that $k = k_{v_2}$ and $[K(a) : K] = [k : k_v]$. Now since (K_1, v_1) is an immediate extension of (K,v), then one has: $[K_1(a) : K_1] \geq [k_{v_2'} : k_v] = [k_{v_2} : k_v] = [K(a) : K] \geq$ $\geq [K_1(a) : K_1]$. Therefore $[K(a) : K] = [K_1(a) : K]$, and since a is separable over K, according to Proposition 4.1 there exists an element $\delta \in G_{\overline{v}}$ such that (a, δ) is a minimal pair of definition of $w_{(a,\delta)}$ ~~relative~~ with respect to K. Now since G_v is cofinal in $G_{\overline{v}}$, we can choose $\delta \in G_v$ (see Remark 3.6. 2)). Finally, let w be the restriction of $w_{(a,\delta)}$ to K(X). By [2, Theorem 2.1] it follows that $k_w = k(t)$, where t is transcendental over k_v. Moreover, one has (see (8), (9), (10)):

(26) $\qquad \deg[w/v] = [K : w]e = [K(a) : K]e = f(v_2/v)e(v_2/v)e = f(w/v)e(w/v).$

Furthermore, since $[K(a) : K] = [k_{v_2} : k_v]$ it follows that $e(v_2/v) = 1$, or equivalently $G_v = G_{v_2}$. Let f be the monic minimal polynomial of a over K. Then one has:

$$f(x) = \sum_{i=1}^{n} f_i(X)(X - a)^i$$

where $f_i(X) \in K(a)[X]$. Thus:

$$\gamma = w(f) = \inf_i \left(v_2(f_i(a)) + i\delta\right).$$

By this equality and the condition $\delta \in G_v$, it follows that $\gamma = w(f) \in G_{v_2} = G_v$ and so in the equalities (26) one has e = 1. Therefore by (26) it follows that $\deg(w/v) = f(w/v) =$ $= [k : k_v]$, as claimed.

LEMMA 4.3. Let G be a subgroup of $G_{\overline{v}}$ such that $G_v \subseteq G$ and G/G_v is finite.

Then there exists an r.t. extension w of v to K(X) such that $G_w = G$. Moreover, we can choose w such that $\deg(w/v) = e(w/v) = [G : G_v]$. (As usual $[G : G_v]$ denotes the cardinal of the finite group G/G_v).

Proof. Let $n = [G : G_v]$. By induction over n we shall prove that there exists an element $a \in \overline{K}$, separable over K, such that $[K(a) : K] = [G : G_v] = e(v_2/v)$, and $G_{v_2} = G$ where v_2 is the restriction of $\overline{v}$ to K(a).

Let us assume the assertion valid for all $n' < n$. In fact, we assume that for every field K' such that $K \subseteq K' \subseteq \overline{K}$ and for every $G' \subseteq G_{\overline{v}}$ such that $G_{v'} \subseteq G'$, with $[G' : G_{v'}] < n$, there exists $a'' \in \overline{K}$, a'' separable over K', such that $[K'(a'') : K'] = [G' : G_{v'}] = e(v''/v')$ and $G_{v''} = G''$. (Here v' is the restriction of $\overline{v}$ to K' and v'' the restriction of $\overline{v}$ to K'(a'').)

Let G_1 be a subgroup of G such that G_1/G_v is cyclic and $[G_1 : G_v] = p$ is prime. Let λ be a positive element of G such that $\overline{\lambda}$. the coset of λ relative to G_v, generates G/G_v. It is clear that p is the smallest non-zero positive integer e such that $e\lambda \in G_v$.

Let b be an element of K such that $v(b) = p\lambda$. Let $c \in K$ be such that $v(c) > (p-1)\lambda$. Consider the polynomial $h(X) = X^p + cX + b$ of K[X]. Let a' be a root of h in $\overline{K}$. Then one has:

$$a'^p + ca' + b = 0 \tag{27}$$

According to general properties of a valuation it results that in (27) two terms has the same valutation and this valuation is smaller than the third. Since $v(c) > (p-1)\lambda$, the only posibility is $\overline{v}(a'^p) = v(b) = p\lambda$. Hence $\overline{v}(a') = \lambda$. This show that h is irreducible. Since h is a separable polynomial then a' is separable over K.

Let $K' = K(a')$ and let v' the restriction of $\overline{v}$ to K'. It is easy to see that $G_{v'} = G_1$ and $[K' : K] = [G_{v'} : G_v] = p$. If $p \neq n$ then we apply the induction hypothesis relative to K' and G, since $[G : G_{v'}] = n/p < n$. Hence there exists $a'' \in \overline{K}$, a'' separable over K', and such that $[K'(a'') : K'] = [G : G_{v'}] = e(v''/v')$; here v'' is restriction of $\overline{v}$ to K'(a''). Also, one has $G_{v''} = G$.

Let $a \in \overline{K}$ be such that $K(a) = K(a', a'')$. Then a is separable over K and by the above assumptions one has: $[K(a):K] = n = [G:G_v]$ and also $G_{v_2} = G$, where $v_2 = v''$ is the restriction of $\bar{v}$ to $K(a)$.

Furthermore, as above, let v_2 be the restriction of $\bar{v}$ to $K_1(a)$. Since (K_1,v_1) is an immediate extension of (K,v), then one has: $[G_{v_2} : G_{v_1}] = [G_{v_2} : G_v] = [K(a):K] =$ $= [K_1(a) : K_1]$. Hence $G_{v_2} = G$, and

(28) $\qquad e(v_2/v_1) = e(v_2/v) = n = [G:G_v] = [K(a):K].$

Now, according to Proposition 4.1, there exists $\delta \in G_{\bar{v}}$ such that (a, δ) is a minimal pair of definition of $w_{(a,\delta)}$ with respect to K. As usual, since G_v is cofinal in $G_{\bar{v}}$ we may choose $\delta \in G_v$ (see Remark 3.6, 2)). As in the proof of the previous lemma we may check that if f is the minimal (monic) polynomial of a, with respect to K, then $w(f) \in G$, where w is the restriction of $w_{(a,\delta)}$ to $K(X)$. Finally by (28) it follows that $f(w/v) = 1$; from (26) and since $w(f) \in G$ we have $\deg(w/v) = e(w/v) = n = [G:G_v]$. Hence $G_w = G$, as claimed.

THEOREM 4.4. Let (K,v) be a valued field and let $k_v \subset k \subseteq k_{\bar{v}}$ and $G_v \subseteq G \subseteq G_{\bar{v}}$ be such that $[k:k_v] < \infty$ and $[G:G_v] < \infty$. Then there exists an r.t. extension w of v to $K(X)$ such that k_w is isomorphic to $k(t)$, with t transcendental over k and $G_w = G$. Moreover we can choose w such that

$$f(w/v) = [k:k_v], \; e(w/v) = [G:G_v]$$

and that

$$\deg(w/v) = [k:k_v][G:G_v].$$

Proof. The proof is based mainly on the previous lemmas. According to Lemma 4.2 we can find a separable element $a' \in \bar{K}$ such that $[K(a'):K] = [k:k_v]$ and that, if v' is the restriction of $\bar{v}$ to $K(a')$, then $k_{v'} = k$. By Lemma 4.3 we can find a separable element $a'' \in \bar{K}$ such that $[K(a''):K] = [G:G_v]$ and that if v'' is the restriction of $\bar{v}$ to $K(a'')$ then $G_{v''} = G$. Let $a \in \bar{K}$ be such that $K(a) = K(a',a'')$ and let v_2 be the restriction of $\bar{v}$ to $K(a)$. Then, since $k_{v'} \subseteq k_{v_2}$, one sees that $[k_{v_2} : k_v] \geq [K(a'):K]$. Also, since $G_{v''} \subseteq G_{v_2}$, one has that $[G_{v_2} : G_v] \geq [G_{v''} : G_v] = [K(a''):K]$. Finally, one gets that

$[K(a):K] = [K(a'):K][K(a''):K]$ and

(29) $\qquad k_{v_2} = k_{v'} = k, \qquad G_{v_2} = G_{v''} = G.$

Furthermore, since (K_1, v_1) is an immediate extension of (K,v), one easily sees that $[K(a'):K] = [K_1(a'):K_1]$, $[K(a''):K] = [K_1(a''):K_1]$ and so $[K(a):K] = [K_1(a):K_1]$. Hence, by Proposition 4.1, there exists $\delta \in G_{\bar{v}}$ such that (a, δ) is a minimal pair of definition of $w_{(a,\delta)}$ with respect to K. Since G_v is cofinal in $G_{\bar{v}}$ we can choose $\delta \in G_v$ (see Remark 3.6, 2)). Let w be the restriction of $w_{(a,\delta)}$ to $K(X)$. Since $\delta \in G_v$, then by (29) it follows that the algebraic closure of k_v in k_w is just $k_{v_2} = k$ and so $k_w = k(t)$, with t transcendental over k. Moreover, $G_w = G$ and $e(w/v) = [G : G_v]$. Finally, one has (see (8), (9), (10)):

$$\deg(w/v) = f(w/v)e(w/v) = [k : k_v][G : G_v],$$

as claimed.

2. At this point we shall give a somewhat general framework in which the fundamental inequality

$$\deg(w/v) \geq e(w/v)f(w/v)$$

becomes an equality. The notation are as usual. Remind that $(\tilde{K}, \tilde{v})$ is the completion of (K,v) (see [5, Ch. VI, § 5]) and that $(\tilde{K}, \tilde{v})$ is an immediate extension of (K,v).

THEOREM 4.5. Let (K,v) be a valuation pair. The following statements are equivalent:

1) For every r.t. extension w of v to $K(X)$ one has:

$$\deg(w/v) = e(w/v)f(w/v)$$

2) The valuation $\tilde{v}$ is Henselian and for every finite simple extension $L/\tilde{K}$, $L = \tilde{K}(a)$, one has:

$$[L : \tilde{K}] = e(v_1/v)f(v_1/\tilde{v})$$

where v_1 is the only extension of $\tilde{v}$ to L.

Proof. 1) $\Longrightarrow$ 2) Let w be such that $e(w/v) = f(w/v) = 1$. Then by 1) it follows that w has a minimal pair of definition $(a, \delta) \in K \times G_v$, i.e. w is defined by inf, v, $a \in K$ and $\delta \in G_v$. But then, according to [2, Theorem 3.3] it follows that $\tilde{K}$ is algebraically closed in a maximally complete extension (K', v') of $(\tilde{K}, \tilde{v})$. First, we have that $(\tilde{K}, \tilde{v})$ is Henselian. Indeed, let $K_1/\tilde{K}$ be an algebraic extension. Then $K'K_1/K'$ is also an algebraic extension. Now, since (K', v') is Henselian (see [8, Ch. II, Theorems 6 and 7]), it follows that v' has a unique extension to $K'K_1$. But then it follows that necessarily $\tilde{v}$ has a unique extension to K_1, i.e. $(\tilde{K}, \tilde{v})$ is Henselian.

Furthermore, let $\hat{K}$ be an algebraic closure of $\tilde{K}$ which contains also $\bar{K}$. To complete the proof of the implication 1) $\Longrightarrow$ 2) it will be enough to show that for every $a \in \hat{K}$ one has: $[\tilde{K}(a) : \tilde{K}] = e(\hat{v}/\tilde{v})f(\hat{v}/\tilde{v})$, where $\hat{v}$ is the unique extension of $\tilde{v}$ to $\hat{K}$. For that, let X be an indeterminate over $\hat{K}$ and $a \in \hat{K}$. According to Theorem 3.9 a), there exists $\delta \in G_{\hat{v}}$ such that (a, δ) is the minimal pair of $w_{(a,\delta)}$ with respect to $\tilde{K}$ (here $w_{(a,\delta)}$ is the valuation on $\hat{K}(X)$ defined by inf, $\hat{v}$, a and δ). Let w' be the restriction of $w_{(a,\delta)}$ to $\tilde{K}(X)$ and w restriction of w' to $K(X)$. Since w' is an r.t. extension of $\tilde{v}$, and $(\tilde{K}, \tilde{v})$ is an immediate extension of (K, v), it follows that w is also an r.t. extension of v, and one has:

(30) $\quad \deg[w'/\tilde{v}] = \deg[w/v], \; e(w'/\tilde{v}) = e(w/v), \; f(w'/\tilde{v}) = f(w/\tilde{v}).$

Hence the equality

$$[\tilde{K}(a) : \tilde{K}] = e(v_1/\tilde{v})f(v_1/\tilde{v})$$

follows from (8), (9), (10), (30) and assumption 1).

2) $\Longrightarrow$ 1) Let w be an r.t. extension of v to $K(X)$ and $(a, \delta) \in \bar{K} \times G_{\bar{v}}$ a minimal pair of definition of w with respect to K. Since (a, δ) is a minimal pair of definition for w, then a verifies the condition 3.5 and so, according to Proposition 3.7, the set $P(a,K)$ is upper bounded in $G_{\bar{v}}$. Now, according to Theorem 3.8, one has

(31) $\quad [K(a) : K] = [\tilde{K}(a) : \tilde{K}]$.

Furthermore, since $(\tilde{K}, \tilde{v})$ is an immediate extension of (K, v), we have:

(32) $e(v'/v) = e(v_1/\tilde{v}),\ f(v'/v) = f(v_1/\tilde{v})$,

where v' is restriction of v to K(a) and v_1 the restriction of $\hat{v}$ to $\widetilde{K}(a)$. Finally, since by hypothesis one has

$$[\widetilde{K}(a) : \widetilde{K}] = e(v_1/\tilde{v})f(v_1/\tilde{v}),$$

relations (8), (9), (10), (31) and (32) imply

$$\deg[w/v] = e(w/v)f(w/v),$$

as claimed.

COROLLARY 4.6. Condition 2) of Theorem 4.5 is fulfilled if:

a) v is of rank one and discrete;

b) v is of rank one and $\mathrm{char}(k_v) = 0$;

c) v is Henselian and $\mathrm{char}(k_v) = 0$.

The proof is straightforward. The statements a), b), c) in Corollary 4.6 are respectively the conjectures (0.1), (0.3) and (0.4) of [7].

REFERENCES

[1] V. Alexandru et N. Popescu, Sur une classe de prolongements a K(x) d'une valuation sur une corp K, Rev. Roumaine Math. Pures Appl., 5 (1988), 393-400.

[2] V. Alexandru; N. Popescu; A. Zaharescu, A theorem of characterization of residual transcendental extensions of a valuation (to appear in J. Math. Kyoto Univ.).

[3] J. Ax, Zeros of polynomials over local fields. The Galois action, J. Algebra 15 (1970), 417-428.

[4] N. Bourbaki, Theorie des Ensembles, Ch. III: Ensembles ordonnes-Cardinaux--Nombres Entiers, Hermann, Paris, 1956.

[5] N. Bourbaki, Algebre Commutative, Ch. V: Entiers, Ch. VI: Valuations, Hermann. Paris. 1964.

[6] O. Endler, Valuation Theory, Springer, Berlin-Heidelberg-New York, 1972.

[7] J. Ohm, Simple transcendental extensions of valued fields, II: A fundamental inequality. J. Math. Kyoto Univ. 25 (1985), 583-596.

[8] O.F.G. Schilling, The theory of valuations, A.M.S. Math. Surveys, no.4, Providence, Rhode Island. 1950.

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