ALL VALUATIONS ON K(X)

by

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This work is a natural continuation of our previous works [1], [2], [3]. We intend here to describe all types of valuations on K(X). This possibility is given by our main result in [2] which give a description of so-called residual transcendental extension of a valuation on K to K(X). Following an ideea of MacLane (see [7]) we define the notion of "ordered system of valuations on K(X)" (see 2) and the limit of such a system. The main result given in section 5 shows that every r.a.t. extension w to K(X) of a valution v on K may be defined as a limit of a suitable ordered system of r.t. extensions of v to K(X).

In the last sections we are concerned with the existence of r.a.t. extensions of v to K(X) with a given residue field, or with a given value group, or both.

Sometimes there exist some similarity between a lot of our results and results of MacLane [7] (and even with some results of Ostrowski [9]). However, we remark that all our considerations and methods of proof are based on our notion of "minimal pair of definition of a r.t. extension of a valuation v on K to K(X)" and on the results we obtained in [1], [2] and [3].

1. NOTATION AND DEFINITIONS

1. Let K be a field and v a valuation on K. We emphasize sometimes this situation saying that (K,v) is a valuation pair. Denote by k_v the residue field, by G_v the value group and by O_v the valuation ring of v. If $x \in O_v$, denote by x^* the image of x into k_v. We refer the reader to [5], [6] or [10] for general notions and definitions.

Let K'/K be an extension of fields. A valuation v' on K' will be called an extension of v if $v'(x) = v(x)$ for all $x \in K$. If v' is an extension of v, we shall identify canonically k_v with a subfield of $k_{v'}$ and G_v with a subgroup of $G_{v'}$.

In what follows we shall consider a fixed valuation pair (K,v). Let us denote by $\overline{K}$ a fixed algebraic closure of K and by $\overline{v}$ a fixed extension of v to K. It is easy to see that $G_{\overline{v}}$ is a divisible group, i.e. for every $\delta \in G_{\overline{v}}$ and $n \in N$, there exists an element $\gamma \in G_{\overline{v}}$ such that $n\gamma = \delta$. Moreover, $G_{\overline{v}} = QG_v$, i.e. $G_{\overline{v}}$ is the smallest divisible group which contains G_v.

As usual, by $K(X)$ we shall denote the field of rational functions of an indeterminate X over K.

2. Let w be an extension of v to $K(X)$. Denote by $\overline{w}$ a common extension of w and $\overline{v}$ to $\overline{K}(X)$, i.e. $\overline{w}$ is a valuation of $\overline{K}(X)$ which extends simultaneously w and $\overline{v}$. In [3, Proposition 3.1] it is proved that always exists a such common extension. Let us denote

(1) $$M_{\overline{w}} = \left\{ \overline{w}(X - a) / \ a \in \overline{K} \right\} \subseteq G_{\overline{w}} .$$

According to [8] (see also [1], [2]) w is called a residual transcendental (r.t.) extension of v if k_w/k_v is a transcendental extension. According to [2, Proposition 1.1] w is an r.t. extension of v if and only if: $G_{\overline{v}} = G_{\overline{w}}$, the set (1) is upper bounded in $G_{\overline{w}}$ and contains its upper bound. Let δ be the upper bound of the set (1). Then there exists $a \in \overline{K}$ such that $\delta = \overline{w}(X - a)$, and thus (see [2]) $\overline{w}$ is an r.t. extension of $\overline{v}$ defined by $\overline{v}$, inf, a and δ (see [2]). Since $\overline{w}$ is defined by a and δ, we shall say that (a, δ) is a pair of definition of $\overline{w}$. Generally w has many pairs of definitions. In [1] it is proved that two pairs (a, δ), (a', δ') of $\overline{K} \times G_{\overline{v}}$ define the same r.t. extension of $\overline{v}$ to $\overline{K}(X)$ if and only if $\delta = \delta'$ and $\overline{v}(a - a') \geq \delta$. According to [2], a pair of definition (a, δ) of $\overline{w}$ is called minimal relative to K if the number $[K(a) : K]$ is the smallest possible one, i.e. if (b, δ) is another pair of definition of w, then $[K(b) : K] \geq [K(a) : K]$. A (minimal) pair of definition of $\overline{w}$ (with respect to K) is also called a (minimal) pair of definition of w. In [2, Theorem 2.1] it is proved that an r.t. extension w is perfectly defined by v and a minimal pair of definition (a, δ). Later, we shall see that minimal pairs of definition are also useful to define other extensions of v to K(X).

3. Let w_1, w_2 be two r.t. extensions of v to K(X). According to [7] one says that w_2 dominate w_2 (written $w_1 \leq w_2$) if $w_1(f(x)) \leq w_2(f(x))$ for all polynomials $f \in K[X]$. This inequality may be understood in $QG_v = G_{\bar{v}}$, since G_{w_1} and G_{w_2} are of finite index over G_v (see [1], [2] or [3]), and so they are canonically imbedded in QG_v. If $w_1 \leq w_2$ and there exists $f \in K[X]$ such that $w_1(f) < w_2(f)$, then one write $w_1 < w_2$.

PROPOSITION 1.1. Let K be algebraically closed and let w_1, w_2 be two r.t. extensions of v to K(X). Let (a_i, δ_i) be a pair of definition of w_i, $i = 1,2$. The following statements are equivalent:

1) $w_1 \leq w_2$

2) $\delta_1 \leq \delta_2$ and $v(a_1 - a_2) \geq \delta_1$

Moreover, $w_1 < w_2$ if and only if $\delta_1 < \delta_2$ and $v(a_1 - a_2) \geq \delta_1$.

Proof. 1)$\Rightarrow$2) Since (a_i, δ_i) is a pair of definition of w_i, then $w_i(X - a_i) = \delta_i$, $i = 1,2$. If $w_1 \leq w_2$, then $w_1(X - a_1) = \delta_1 \leq w_2(X - a_1) = \inf(\delta_2, v(a_1 - a_2))$ and so $\delta_1 \leq \delta_2$ and $\delta_1 \leq v(a_1 - a_2)$.

2)$\Rightarrow$1) If $v(a_2 - a_1) \geq \delta_1$, then (see [1]) (a_2, δ_1) is also a pair of definition of w_2. Let $f(X) \in K[X]$ of the form $f(X) = \sum b_i(X - a_2)^i$. Then we have

$$w_1(f) = \inf_i (v(b_i) + i\,\delta_1)$$

$$w_2(f) = \inf_i (v(b_i) + i\,\delta_2)$$

Now since $\delta_1 \leq \delta_2$, one has $v(b_i) + i\delta_1 \leq v(b_i) + i\delta_2$, for all i, and so $w_1(f) \leq w_2(f)$, as claimed.

Furthermore, let us assume that $w_1 < w_2$. Then there exists an element $a \in K$ such that

$$w_1(X - a) = \inf(\delta_1, v(a_1 - a)) < w_2(X - a) = \inf(\delta_2, v(a_2 - a) \tag{2}$$

According to the above equivalence, this inequality is possible only if $\delta_1 < \delta_2$.

Conversely, if $w_1 \leq w_2$, and $\delta_1 < \delta_2$, then $w_1(X - a_2) = \delta_1 < \delta_2 = w_2(X - a_2)$, i.e. $w_1 < w_2$.

4. Let K be algebraically closed and w_1, w_2 two r.t. extensions of v to K(X). Let (a_i, δ_i) be a pair of definition of w_i, i = 1,2. We shall say that w_2 well dominates w_1 if $w_1 < w_2$ and $v(a_1 - a_2) = \delta_1$.

2. ORDERED SYSTEMS OF VALUATIONS

1. By an ordered system of r.t. extensions of v to K(X) we mean a family $(w_i)_{i\in I}$ of r.t. extensions of v to K(X), where I is a well ordered set without last element and such that w_j dominates w_i when i < j.

Let $(w_i)_{i\in I}$ be an ordered system of r.t. extensions of v to K(X). For every $f \in K[X]$ let us define:

$$w(f) = \sup_i w_i(f) \tag{3}$$

We remark that since w_i is an r.t. extension of v, then G_{w_i}/G_v is a finite group and so $G_v \subsetneq G_{w_i} \subsetneq G_{\bar{v}}$. Hence (3) must be understood in $G_{\bar{v}}$. However, the element in (3) may or may not be an element of $G_{\bar{v}}$. Therefore we say that the system $(w_i)_{i\in I}$ of r.t. extensions of v to K(X) has a limit if for every $f \in K[X]$, w(f) defined by (3) is an element of G_v. Then one easily sees that the assignement:

$$f \rightsquigarrow w(f)$$

defines a valuation w on K[X] which may be canonically extended to K(X). This valuation w is an extension of v to K(X), and will be called the limit of the given system $(w_i)_{i\in I}$. We write: $w = \sup_i w_i$.

Let K be algebraically closed and let $(w_i)_{i\in I}$ be an ordered system of r.t. extensions of v to K(X). For every $i\in I$ denote by (a_i,δ_i) a pair of definition of w_i. Then according to Proposition 1.1 the set $(\delta_i)_i$ is a well ordering subset of G_v. Moreover. if for every $i,j \in I$, i < j, w_j well dominates w_i, then $(a_i)_i$ is a pseudo-convergent sequence on K (see [10, p. 39]). Generally, $(a_i)_i$ contains a subset which is a pseudo-convergent sequence. However, we do not deal with this situation, since in our further consideration all dominate valuations will be well dominate. One has the following result:

PROPOSITION 2.1. Let K be algebraically closed and let $(w_i)_{i\in I}$ be an ordered system of r.t. extensions of v to K(X). The following statements are equivalent.

1) The ordered system $(w_i)_i$ has a limit w which is an r.t. extension of v to K(X).

2) There exists an element $a \in K$ such that $v(a - a_i) \geq \delta_i$ for all $i \in I$. (A sequence $(a_i)_i$ is pseudo-convergent in K if it has a pseudo-limit in K). Also $\sup_i \delta_i$ is defined in G_v.

Proof. 1)$\Rightarrow$2) Let (a, δ) be a pair of definition of w. According to (3) one sees that $w \geq w_i$ for all i. Hence, by Proposition 1.1 one has:

(4) $\qquad \delta \geq \delta_i$ and $v(a_i - a) \geq \delta_i$, $i \in I$.

Therefore, according to (4) it follows that:

$$\delta = w(X - a) = \sup_i w_i(X - a) = \sup_i (\inf(\delta_i, v(a - a_i)) = \sup_i \delta_i .$$

(Also by (4) it follows that a is a pseudo-limit of $(a_i)_i$.

2)$\Rightarrow$1) Let (a, δ) be such that $\delta = \sup_i \delta_i$ and let a be such that $v(a - a_i) \geq \delta_i$ for all $i \in I$. Let w be a valuation on K(X) defined by inf, v, a and δ. Then it is plain that $w = \sup_i w_i$.

The following result (somewhat complementary to Proposition 2.1) is valid.

PROPOSITION 2.2. Let K be algebraically closed and let $(w_i)_i$ be an ordered system of r.t. extensions of v to K(X). The following statements are equivalent:

1) The ordered system $(w_i)_i$ has a limit w which is not a r.t. extension of v to K(X).

2) For every $a \in K$ there exists $i \in I$ such that $w_i(X - a) < \delta_i$.

Proof. 1)$\Rightarrow$2) Let $w = \sup_i w_i$. Since by the hypothesis w is not an r.t. extension of v to $K(X)$, then according to [2, Proposition 1.1], the set (see (2)):

$$M_w = \{ w(X - b) \mid b \in K \} \subseteq G_w$$

is unbounded in G_w or it is bounded but does not contain its upper bound. Let $a \in K$. In both cases there exists $b \in K$ such that $w(X - a) < w(X - b)$. But $w(X - b) = \sup_i w_i(X - b)$ and so there exists $i \in I$ such that $w(X - a) < w_i(X - b) \leq \delta_i$. As $w(X - a) = \sup_i w_i(X - a)$, we have $w_i(X - a) \leq w(X - a) < w_i(X - b) \leq \delta_i$, as claimed.

2)$\Rightarrow$ 1) Let $a \in K$. Then since $w_j(X - a) < \delta_j$ for a suitable j, it results that $\sup_i w_i(X - a) = w_j(X - a)$. Since K is algebraically closed, it follows that for every $f \in K[X]$, $\sup_i w_i(f)$ exists and is in G_v, and so $w = \sup_i w_i$ is defined. Now we must prove that w is not an r.t. extension of v. Indeed, let us assume that w is an r.t. extension of v and let (a, δ) be a pair of definition of w. Then by the hypothesis there exists $j \in I$ such that $w_j(X - a) < \delta_j$. According to (3), it follows that $w(X - a) < \delta_j$. Also by (3) one has that $w \geq w_i$ for all $i \in I$. In particular, one has $w_j(X - a_j) = \delta_j \leq w(X - a_j) =$ $= w(X - a + a - a_j) = \inf(w(X - a), v(a - a_j)) \leq w(X - a)$, a contradiction. Hence w is not an r.t. extension of v, as claimed.

THEOREM 2.3. Let K be a (not necessarily algebraically closed) field, and let $(\overline{w}_i)_{i \in I}$ be an ordered system of r.t. extensions of $\overline{v}$ to $\overline{K}(X)$. For every $i \in I$ denote by (a_i, δ_i) a fixed minimal pair of definition of $\overline{w}_i$ with respect to K. Denote by w_i the restriction of $\overline{w}_i$ to $K(X)$ and by v_i the restriction of $\overline{v}$ to $K(a_i)$, $i \in I$. Then

a) For all $i, j \in I$, $i < j$ one has $w_i < w_j$, i.e. $(w_i)_{i \in I}$ is an ordered system of r.t. extensions of v to $K(X)$.

b) For all $i, j \in I$, $i < j$, one has $k_{v_i} \subseteq k_{v_j}$ and $G_{v_i} \subseteq G_{v_j}$.

c) Assume that $\overline{w} = \sup_i \overline{w}_i$ and $\overline{w}$ is not an r.t. extension of $\overline{v}$ to $\overline{K}(X)$. Let w be the restriction of $\overline{w}$ to $K(X)$. Then $w = \sup_i w_i$. Moreover one has:

$$k_w = \bigcup_i k_{v_i} \quad \text{and} \quad G_w = \bigcup_i G_{w_i}$$

Proof. a) Let us denote by f_i the monic minimal polynomial of a_i relative to K, and let $n_i = \deg f_i = [K(a_i) : K]$, $i \in I$. Since $\overline{w_i} < \overline{w_j}$, it follows that $w_i \leq w_j$ whereas $i < j$. We note that in fact $w_i < w_j$. Indeed, if $w_i = w_j$ then since (a_i, δ_i) is a minimal pair of w_i by [3, Theorem 2.2] it follows that $\delta_i = \delta_j$, contrary to the assumption $\overline{w_i} < \overline{w_j}$, i.e. $\delta_i < \delta_j$. (A short computation shows that $w_i(f_j) < w_j(f_j)$, if $i < j$.)

Since (a_i, δ_i) is a minimal pair of definition of w_i, $i \in I$, by Proposition 1.1 we have:

(5) $\qquad n_i \leq n_j, \quad v(a_i - a_j) \geq \delta_i, \quad$ if $i < j$, $i,j \in I$.

Therefore $(w_i)_{i \in I}$ is really an ordered system of r.t. extensions of v to K(X).

b) Let $c \in K(a_i)$. Then $c = f(a_i)$, where $f(X) \in K[X]$ and $n = \deg f < n_i$. Since (a_i, δ_i) is a minimal pair of definition of w_i, then for every root b of f one has $\overline{v}(a_i - b) < \delta_i$. Thus by (5) it follows:

(6) $\qquad \overline{v}(f(a_j)) = v_j(f(a_j)) = \overline{v}(f(a_i)) = v_i(f(a_i)) = v_i(c).$

Now let us assume that $v_i(c) = 0$. Then $v_j(c) = 0$ and c^*, the image of c in k_{v_i}, coincides with the image of c into k_{v_j}. Indeed, let $b_1, \ldots, b_n$ be all roots of $f(X)$ in $\overline{K}$. For any t, $1 \leq t \leq n$, let $d_t \in K$ be such that:

$$\overline{v}(a_i - b_t) = \overline{v}(a_j - b_t) = \overline{v}(d_t) \qquad 1 \leq t \leq n.$$

Then one has $\overline{v}((a_i - b_t)/d_t) = \overline{v}((a_j - b_t)/d_t) = 0$ and so

$$\overline{v}\left(\frac{(a_j - b_t)/d_t}{(a_i - b_t)/d_t} - 1\right) = \overline{v}\left(\frac{a_j - b_t}{a_i - b_t} - 1\right) = \overline{v}\left(\frac{a_j - a_i}{a_i - b_t}\right) > 0.$$

Hence

$$((a_j - b_t)/d_t)^* = ((a_j - b_t)/d_t)^*, \quad 1 \leq t \leq n.$$

By these equalities it follows that:

$$\frac{f(a_j)^*}{f(a_i)^*} = \left(\frac{f(a_j)}{f(a_i)}\right)^* = \left(\prod_{t=1}^{n} \frac{(a_j - b_t)/d_t}{(a_i - b_t)/d_t}\right)^* = \prod_{t=1}^{n} \frac{((a_j - b_t)/d_t)^*}{((a_i - b_t)/d_t)^*} = 1,$$

i.e., $f(a_i)^* = f(a_j)^* \in k_{v_j}$ as claimed.

The inclusion $G_{v_i} \subseteq G_{v_j}$ follows easily by (6).

c) Since $\overline{w} = \sup_i \overline{w}_i$, then it is easy to see that $w = \sup_i w_i$. Moreover, it is clear that w is not an r.t. extension of v.

Now we shall prove that $k_{v_i} \subseteq k_w$ and $G_{v_i} \subseteq G_w$ for all $i \in I$. For that, let $f(X) \in K[X]$ be such that $n = \deg f < n_i$, and let $b_1, \ldots, b_n$ be all roots of f in $\overline{K}$. Since (a_i, δ_i) is a minimal pair of definition of w_i, one has $\overline{v}(a_i - b_t) < \delta_i$, $1 \leq t \leq n$, and so $\overline{w}(X - b_t) = \overline{w}(X - a_i + a_i - b_t) = \overline{v}(a_i - b_t)$, $1 \leq t \leq n$. But then we have:

$$\overline{w}(f(X)) = w(f(X)) = \overline{v}(f(a_i)). \tag{7}$$

If $\overline{v}(f(a_i)) = v_i(f(a_i)) = 0$, then $w(f(x)) = 0$ and as above one obtains that $f(X)^* = f(a_i)^*$, i.e. $k_{v_i} \subseteq k_w$. Relation (7) implies that $G_{v_i} \subseteq G_w$. Hence one has

$$\bigcup_i k_{v_i} \subseteq k_w \quad \text{and} \quad \bigcup_i G_{v_i} \subseteq G_w. \tag{8}$$

For proving that these inclusions are in fact equalities, let $r(X) = f(X)/g(X) \in K(X)$. Let $b_1, \ldots, b_n$ and $c_1, \ldots, c_m$ be all roots (not necessarily distinct) of f, respectively of g, in $\overline{K}$. Since $\overline{w}$ is not an r.t. extension of $\overline{v}$ to $\overline{K}(X)$, then by Proposition 2.2, 2) there exists an $i \in I$ such that:

$$\begin{aligned} &w(X - b_t) < \delta_i \qquad 1 \leq t \leq n \\ &w(X - c_s) < \delta_i \qquad 1 \leq s \leq m. \end{aligned} \tag{9}$$

According to (9) one has: $\overline{v}(a_i - b_t) = \overline{w}(a_i - X + X - b_t) = \overline{w}(X - b_t)$, $1 \leq t \leq n$, and analogously $\overline{v}(a_i - c_s) = \overline{w}(X - c_s)$, $1 \leq s \leq m$. Therefore we have: $\overline{v}(f(a_i)) = w(f(X))$, $\overline{v}(g(a_i)) = w(g(X))$, and so:

$$\overline{v}(r(a_i)) = v_i(r(a_i)) = w(r(X)). \tag{10}$$

Now if $w(r) = 0$, then by (9), $v_i(r(a_i)) = 0$ and as above we can easily prove that $(r(a_i))^* = (r(X))^*$, i.e.

$$r(X)^* \in k_{v_i}. \tag{11}$$

Therefore by (8), (10) and (11) it follows that:

$$\bigcup_{i \in I} k_{v_i} = k_w, \quad \bigcup_{i \in I} G_{v_i} = G_w, \tag{12}$$

as claimed.

3. TYPES OF VALUATIONS OF K(X)

It is natural to ask for the description of all valuations on K(X). In this work we try to give an answer to this question. In this section we describe all types of valuations on K(X).

A) Valuations on K(X) which are trivial on K. These valuations are well known (see [10]): they are defined by the irreducible polynomials of K[X] and also by the valuation at "infinity", defined by 1/X. All these are of rank one and discrete. These valuations play a prominent part in algebraic theory of functions of one variable and elsewhere.

B) Valuations on K(X) which extend valuations on K. Since distinct valuations on K have distinct extensions to K(X) we deal only with extensions of a fixed valuation v on K. We classify these extensions as follows:

(RT) Residual transcendent extensions w of v to K(X). There are defined by the condition:

$$\deg \mathrm{tr}(k_w/k_v) = 1 .$$

R.t. extensions of v to K(X) had been described in [2, Theorem 2.1]. According to this result, to describe an r.t. extension w of v to K(X) we have to know an algebraic closure $\overline{K}$ of K, an extension $\overline{v}$ of v to $\overline{K}$, and a minimal pair of definition of w. Now, a minimal pair of definition (a, δ) of w is in fact a minimal pair of definition of a common extension $\overline{w}$ of w and $\overline{v}$ to $\overline{K}(X)$. Furthermore, one has $\overline{w} = w_{(a,\delta)}$, i.e. $\overline{w}$ is defined by inf, $\overline{v}$, a and δ. Finally, to know all r.t. extensions of v to K(X), we have to know all pairs $(a,\delta) \in \overline{K} \times G_{\overline{v}}$ such that (a,δ) is a minimal pair of definition of $w_{(a,\delta)}$ with respect to K. This question is discussed in [3]. Although a complete solution is not given in [3], the answer is given in some important cases.

(RA) Residual algebraic (r.a) extensions w of v to K(X). These are defined by the condition:

k_w/k_v is an algebraic extension.

Furthermore, r.a.-extensions are divided into two distinct classes according to the nature of the valued group G_w relative to G_v:

(RAT) Residual algebraic of torsion (r.a.t) extensions w of v to K(X). These are defined by the condition that the quotient group:

$$G_w/G_v$$

is a torsion group (i.e. every element is of finite order). It is plain to see that w is an r.a.t. extension of v to K(X) if and only if $G_v \subseteq G_w \subseteq G_{\bar{v}}$.

(RAF) Residual algebraic extension w of v to K(X) which are not of torsion (r.a.f). These are defined by condition that the quotient group G_w/G_v is not a torsion group. Later, (see §4) we shall see that G_w/G_v is in fact a free abelian group; more precisely, it is isomorphic to Z, the additive group of integral numbers.

4. RESIDUAL ALGEBRAIC EXTENSIONS. THE CASE K IS ALGEBRAICALLY CLOSED

Let K be an algebraically closed field, v a valuation on K, and w an r.a. extension of v to K(X).

1. First, we consider the case when w is an residual algebraic torsion extension of v to K(X). According to the above definition this means that k_w/k_v is an algebraic extension, and G_w/G_v is a torsion group. Now since K is algebraically closed then k_v is also algebraically closed and so $k_w = k_v$. Moreover, $G_v = G_w$ since G_v is a divisible group. But then, according to [16, Ch. II], (K(X),w) is an immediate extension of (K,v). Let us consider the set M_w defined in (1). Since w is not an r.t. extension of v, then according to [2, Proposition 1.1] it follows that M_w has not an upper bound, or it does not contain its upper bound. Furthermore, since M_w is a totally ordered set, then according to [4, § 2, Exercise 4] it follows that it contains a cofinal well ordered subset $\{\delta_i\}_{i \in I}$. Since M_w does not contain an upper bound, then I has not a last element. For every $i \in I$, we choose an element $a_i \in K$ such that:

(13) $\qquad w(X - a_i) = \delta_i, \quad i \in I$.

Consider $w_i = w_{(a_i, \delta_i)}$, i.e. w_i is the r.t. extension of v to $K(X)$ defined by inf, v, a_i and δ_i.

THEOREM 4.1. With above notation one has:

a) $w_i < w_j$ if $i < j$, i.e. $\{w_i\}_{i \in I}$ is an ordered system of r.t. extensions of v $K(X)$. Moreover, for every $i < j$, w_j well dominates w_i.

b) $w_i \leq w$ for all $i \in I$ and $w = \sup_{i \in I} w_i$.

Proof. a) Let $i < j$. We shall prove that for every $b \in K$ one has:

(14) $\qquad w_i(X - b) \leq w_j(X - b)$.

First, we note that, according to (13) and the inequality $\delta_i < \delta_j$, one has:

(15) $\qquad v(a_i - a_j) = w(a_i - a_j) = w(a_i - X + X - a_j) = w(a_i - X) = \delta_i$.

But then, for every $b \in K$, one has:

$$w_i(X - b) = \inf(\delta_i, v(a_i - b))$$

$$w_j(X - b) = \inf(\delta_j, v(a_j - b)) .$$

According to (15) we have that: $v(a_j - b) = v(a_j - a_i + a_i - b) \geq \inf(\delta_i, v(a_i - b)) = w_i(X - b)$. Hence:

$$w_j(X - b) = \inf(\delta_j, v(a_j - b)) \geq \inf(\delta_i, v(a_i - b)) = w_i(X - b),$$

i.e., $w_i \leq w_j$. In particular:

$$w_j(X - a_j) = \delta_j > \inf(\delta_i, v(a_i - a_j)) = w_i(X - a_j)$$

and so one has $w_i < w_j$. Moreover, by (15) it follows that w_j well dominates w_i.

b) Let $b \in K$. Then one has: $w(X - b) = w(X - a_i + a_i - b) \geq \inf(w(X - a_i), v(a_i - b)) = \inf(\delta_i, v(a_i - b)) = w_i(X - b)$. Hence $w_i \leq w$ for all $i \in I$. In proving that $w = \sup_i w_i$ it is enough to show that for every $b \in K$ one has

(16) $\qquad w(X - b) = \sup_i w_i(X - b)$

Indeed, since $w(X - b) \in M_w$, then there exists $i \in I$ such that $w(X - b) < \delta_i$. Hence

$w(X - b) = w(X - a_i + a_i - b) \geq \inf(\delta_i, v(a_i - b))$, and so $w(X - b) = v(a_i - b) < \delta_i$. Thus $w_i(X - b) = v(a_i - b)$. If $j > i$, then

(17) $\quad w_j(X - b) = w_i(X - b) = v(a_i - b) = w(X - b)$.

This shows that (16) is valid and so $w = \sup_i w_i$.

REMARK 4.2. 1. According to (15), it follows that $\{a_i\}_{i \in I}$ is a pseudo--convergent sequence (see [11, Ch. II]). By (13), it follows that X is a pseudo-limit of $\{a_i\}_{i \in I}$ in K(X). Moreover, since X is transcendental over K, then $\{a_i\}_{i \in I}$ is a transcendental pseudo-sequence. According to (17) it follows that for every $f(X) \in K[X]$, one has:

$$w(f(X)) = \sup_i w_i(f(X)) = \sup_i v(f(a_i)) .$$

These remarks permit to reobtain (using our considerations) the classical results of Ostrowski (see [9, Teil III] and to give a new proof of [10, Ch. II, Lemma 11].

2. We consider now the (r.a.f.)-extensions w of to K(X). Thus the quotient group G_w/G_v contains at least a free element (i.e. an element $\bar{\delta}$ such that $n\bar{\delta} \neq 0$ for all $n \in \mathbb{Z}$, $n \neq 0$). Hence in the group G_w there exists at least an element δ such that $\mathbb{Z}\delta \cap G_v = 0$. It is clear that we may assume that there exists a K such that:

$$\delta = w(X - a) .$$

We assert that:

(18) $\quad G_w = G_v + \mathbb{Z}\delta$.

Indeed, assume that there exists $\delta' \in G_w$ such that $\delta' \notin G_v + \mathbb{Z}$. Let $r \in K(X)$ be such that $w(r) = \delta'$. Write $r = f/g$, $f, g \in K[X]$, and $f = a\prod_i (X - a_i)$, $g = b\prod_j (X - b_j)$, one sees that $\delta' = w(r) = v(a) - v(b) + \sum_i w(X - a_i) - \sum_j w(X - b_j)$. Since $\delta' \notin G_v + \mathbb{Z}$, then for at least one i or one j, we have $w(X - a_i) \notin G_v + \mathbb{Z}$ or $w(X - b_j) \notin G_v + \mathbb{Z}$. Suppose that $\delta_1 = w(X - a_1) \notin G_v + \mathbb{Z}$. Then $v(a - a_1) = w(a - X + X - a_1) = \inf(\delta, \delta_1)$, a contradiction. Hence the equality (18) is valid.

Finally, the valuation w can be easily described. Let $f(X) \in K[X]$. Write:

$$f(X) = a_0 + a_1(X - a) + \ldots + a_n(X - a)^n .$$

Then according to (10), we have that:

(19) $$w(f(X)) = \inf_i (v(a_i) + i\delta).$$

THEOREM 4.3. Let w be an (r.t.f.)-extension of v to K(X). Then there exists a pair $(a, \delta) \in K \times G_w$ such that $w(X - a) = \delta$. Moreover $G_w = G_v + Z\delta$ and w is defined by (19).

Conversely, let G be an ordered group which contains G_v as a subgroup, and $\delta \in G$ be such that $Z\delta \cap G_v = 0$. Let $a \in K$ and let $w : K(X) \to G$ be defined by the equality (19). Then w is an (r.a.f.) extension of v to K(X). Moreover, $G_w = G_v + Z\delta$, and $k_w = k_v$.

The first part of the theorem results by the above considerations. The proof of the last part is obvious.

Let w be a (r.a.f.) extension of v to K(X). A pair $(a, \delta) \in K \times G_w$ as in the above theorem is also called a <u>pair of definition</u> of w. How many pairs of definition has w? One has the following result

REMARK 4.4. Let w be a r.a.f. extension of v to K(X) and (a_1, δ_1), (a_2, δ_2) be two pairs of definition of w. Then

(20) $$\delta_1 = \delta_2 \quad \text{and} \quad v(a_1 - a_2) \geq \delta_1 .$$

Proof. Indeed, we have that: $w(X - a_1) = \delta_1$, $w(X - a_2) = \delta_2$. According to (19), we infer: $w(X - a_2) = w(X - a_1 + a_1 - a_2) = \inf(\delta_1, v(a_1 - a_2)) = \delta_2$. Hence $\delta_1 \geq \delta_2$, $v(a_1 - a_2) \geq \delta_2$. By symmetry, it follows that $\delta_1 \leq \delta_2$ and $v(a_1 - a_2) \geq \delta_1$. Finally, $\delta_1 = \delta_2$ and $v(a_1 - a_2) \geq \delta_1$, as claimed.

By the above considerations one sees that r.a.f. extensions of v to K(X) are similar to r.t.-extensions. They are defined by inf, v and a suitable pair $(a, \delta) \in K \times G_w$. Moreover, (20) shows that the relation between various pairs of definition of an r.a.f. valuation is the same as the relation between various pairs of definition of an r.t.-extension (see [1]). The only (but essential) difference is relative to the nature of

δ. For r.t.-extensions, $\delta \in G_v = QG_v$ while for r.a.f. extensions δ belongs to an ordered group G which strictly contains G_v and $\delta \notin G_v$.

3. We define now a family of r.a.f. extensions of v to K(X), namely those extensions w whose rank (see [5, Ch. VI] or [10, Ch. I]) is different from the rank of v (of course, we assume that the rank of v is finite).

Let us consider the group $G = G_v \times Z$ ordered lexicographically. Then one has $rg(G) = rg(G_v) + 1$. Let $\delta = (0,1) \in G$, and $a \in K$. Denote by w the valuation on K(X) defined by inf, v, a and δ (see (19)). Since $\delta \notin G_v$, then w is an r.a.f. extension of v. Denote by w_1 the r.t.-extension of v to K(X) defined by the pair $(a,0) \in K \times G_v$ (i.e. w_1 is defined by inf, v, a and 0). It is easy to see that $O_w \subset O_{w_1}$, $G_w = G$, $G_{w_1} = G_v$. Let M_w and M_{w_1} be the maximal ideal of O_w and O_{w_1}, respectively. Then one has $M_{w_1} \subset M_w$ and O_{w_1} is the ring of quotients of O_w relative to the complement of M_{w_1}.

Conversely, let a be an element of K and let w_1 be the r.t.-extension of v to K(X) defined by the pair $(a,0) \in K \times G_v$. Let O_{w_1} the value ring of w_1 and M_{w_1} the maximal ideal of O_{w_1}. Denote $t = (X - a)^*$; then t is transcendental over k_v and $k_{w_1} = k_v(t)$, i.e. k_{w_1} is the field of rational functions of t over k_v. Denote v the valuation on $k_v(t)$ (trivial on k_v) defined by the irreducible polynomial t. One has $k_{v'} = k_v$, $G_{v'} = \mathbb{Z}$. Let $\varphi : O_{w_1} \longrightarrow k(t)$ be the canonical homomorphism. Denote $O_w = \varphi^{-1}(O_{v'})$ $M_w = \varphi^{-1}(M_{v'})$. Then one has $M_{w_1} \subset M_w \subset O_w \subset O_{w_1}$. It is easy to see that O_w is in fact the value ring of the valuation w on K(X) defined by the pair (a, δ), where $= (0,1) \in G_v \times \mathbb{Z}$ (ordered lexicographically).

5. THE r.a.-EXTENSIONS. THE GENERAL CASE

Now let K be a (not necessarily algebraically closed field) and v a valuation on K. We consider the r.a. extensions w of v to K(X). As usual we denote by $\overline{K}$ a fixed algebraic closure of K and by $\overline{v}$ a fixed extension of v to K. Let $\overline{w}$ be a fixed common extension of v and w to $\overline{K}(X)$.

1. First, we assume that w is a r.a.t.-extension of v. Then it is easy to see that

$\overline{w}$ is also a r.a.t.-extension of v. Consider the set $M_{\overline{w}}$ defined in (1). As in § 4, 1., let $\{\delta_i\}_{i\in I}$ be a cofinal well ordered subset of M_w. Since by the hypothesis w is not a r.t.-extension, then I has not a least element. For every $i\in I$ we choose an element $a_i\in\overline{K}$ such that

(21) $\quad \overline{w}(X - a_i) = \delta_i$ and $[K(a_i) : K]$ is the smallest possible one

(this means that if $w(X - b) = \delta_i$ then $[K(b) : K] \geq [K(a_i) : K]$). Denote by $\overline{w}_i$ the r.t. extension of $\overline{v}$ to $\overline{K}(X)$ defined by the pair (a_i, δ_i). By (21) it follows that (a_i, δ_i) is a minimal pair of definition of $\overline{w}_i$ relative to K. According to Theorem 4.1 we infer that:

$$(22)\qquad \begin{aligned} &\overline{w}_i < \overline{w}_j \quad \text{if} \quad i < j \\ &\overline{w}_i < \overline{w} \quad \text{for all } i\in I \text{ and } \overline{w} = \sup_i \overline{w}_i . \end{aligned}$$

For all $i\in I$ denote by w_i the restriction of $\overline{w}_i$ to K(X), and by v_i the restriction of $\overline{v}$ to $K(a_i)$. It is easy to see that (a_i, δ_i) is in fact a minimal pair of definition of w_i. Since $(w_i)_{i\in I}$ is an ordered system of r.t.-extensions of v to K(X) and $\overline{w} = \sup_i \overline{w}_i$, then, according to Theorem 2.3, one has the following result:

THEOREM 5.1. Let w be a r.a.t. extension of v to K(X). Then with above notation, we have that::

1) $w_i < w_j$, $k_{v_i} \subset k_{v_j}$ and $G_{v_i} \subset G_{v_j}$ whereas $i < j$.

2) $(w_i)_{i\in I}$ is an ordered system of r.t.-extensions of v to K(X) and $w = \sup_i w_i$.

Moreover, we have

$$k_w = \bigcup_i k_{v_i} \ ; \quad G_w = \bigcup_i G_{v_i}$$

CPOROLLARY 5.2. If w is a r.a.t. extension of v to K(X) then:

a) k_w/k_v is an algebraic extension countably generated (i.e. k_w is obtained by adjoining to k_v at most countably many algebraic elements).

b) The group G_w/G_v is countable.

The proof results by Theorem 5.1 since $\{k_{v_i}\}_i$ and $(G_{v_i})_i$ are totally ordered sets.

2. Now we consider the r.a.f. extensions of v to K(X).

Let w be a r.a.f. extension of v to K(X). Denote by $\overline{w}$ a common extension of w and $\overline{v}$ to $\overline{K}(X)$. It is easy to see that $\overline{w}$ is also a r.a.f. extension of $\overline{v}$ to $\overline{K}(X)$. According to Theorem 4.3, $\overline{w}$ is defined by a pair of definition (a, δ). We shall say that (a, δ) is a minimal pair of definition of w with respect to K if $[K(a) : K]$ is smallest possible one. Hence if $[K(b) : K] < [K(a) : K]$, then according to Remark 4.4 one has: $v(b - a) < \delta$.

THEOREM 5.3. Let w be a r.a.f. extension of v to K(X) and (a, δ) a minimal pair of definition of w relative to K. Denote by f the minimal monic polynomial of a over K and let $\gamma = w(f)$. If $g \in K[X]$, then $g = g_o + g_1 f + \ldots + g_n f^n$, where $\deg g_i < \deg f$, $0 \leq i \leq n$. One has:

$$w(g) = \inf_i (v(g_i(a)) + i\gamma).$$

Moreover if v_1 is the restictin of v to K(a) then

$$k_w = k_{v_1} \quad \text{and} \quad G_w = G_{v_1} \oplus Z\gamma \quad .$$

Proof. Let $a = a_1, \ldots, a_m$ be all roots of f in $\overline{K}$. Then: $\gamma = w(f(x)) = \overline{w}(\prod_{i=1}^{m}(X - a_i)) = \sum_i \overline{w}(X - a_i)$. But according to (11) we have: $w(X - a_1) = \delta_1$, $w(X - a_i)\ \inf(\delta, v(a - a_i))$ $i = 1, \ldots, m$. This means that $\gamma \notin G_v$ and so $Z\gamma \cap G_v = 0$. The proof follows now in a canonical manner.

6. EXISTENCE OF EXTENSIONS OF v TO K(X) WITH A GIVEN RESIDUE FIELD.

1. Let us assume that (K,v) is such that k_v is not algebraically closed. By Corollary 5.2 it follows that if w is a r.a.t. extension of v to K(X) then k_w/k_v is an extension countably generated. There exists a somewhat converse result:

THEOREM 6.1. Let k/k_v be an infinite algebraic extension countably generated. Then there exists a r.a.t. extension w of v to K(X) such that $k_w \simeq k$. Moreover w can be choosen such that $G_w = G_v$.

Proof. Since $k_{\overline{v}}$ is in fact an algebraic closure of k_v we can assume that $k_v \subset k \subset k_{\overline{v}}$.

Since k/k_v is countable generated, there exists a tower $k_v \subseteq k_1 \subseteq k_2 \subseteq \ldots$ of finite extensions of k such that $\bigcup_n k_n = k$. We shall prove that for every natural number n there exists an element $b_n \in \overline{K}$ such that:

1) b_n is separable over K and $[K(b_n) : K] = [k_n : k_v]$.

2) If v_n is the restriction of $\overline{v}$ to $K(b_n)$, then $k_{v_n} = k_n$.

3) $K(b_n) \subseteq K(b_{n+1})$, $n \geq 1$.

The proof results by induction over n. Indeed, according to [3, Lemma 4.2] there exists b_1 such that 1) and 2) are verified. Let us assume that $n \geq 1$ and $b_1, \ldots, b_n$ are defined such that all condition 1) - 3) are verified. Again according to [3, Lemma 4.2] there exists an element $c \in \overline{K}$ such that c is separable over $K(b_n)$, $[K(b_n,c) : K(b_n)] = [k_{n+1} : k_n]$ and $k_{v_{n+1}} = k_{n+1}$, where v_{n+1} is the restriction of v to $K(b_n,c)$. Since $K(b_n)/K$ and $K(b_n,c)/K(b_n)$ are separable extensions, then $K(b_n,c)/K$ is separable and so $K(b_n,c) = K(b_{n+1})$, for a suitable element b_{n+1} of $\overline{K}$.

Furthermore, let (K',v') be the Henselisation of (K,v) included in $(\overline{K},\overline{v})$ (see [6, pag. 131]). This means that $K \subseteq K' \subseteq \overline{K}$, v' is the restriction of v to K', v' is Henselian and (K',v') is an immediate extension of (K,v), i.e. $k_v = k_{v'}$ and $G_v = G_{v'}$ (see [10, Ch. II]).

We assert that $[K'(b_n) : K'] = [K(b_n) : K]$. Indeed, one has $K(b_n) \subseteq K'(b_n)$ and $k_{v_n} \subseteq k_{v'_n}$, where v'_n is the restriction of $\overline{v}$ to $K'(b_n)$. Since $k_v = k_{v'}$ thus $k_{v_n} = k_{v'_n}$ and so according to 1) it follows that: $[K(b_n) : K] = [K'(b_n) : K]$, as claimed. Moreover, by 3) it follows that for all n one has:

$$K'(b_n) \subseteq K'(b_{n+1}) \tag{23}$$

Now, for every positive integer n we shall define a pair $(a_n, \delta_n) \in \overline{K} \times G_{\overline{v}}$ such that:

α) If we denote by $\overline{w}_n$ the r.t. extension of $\overline{v}$ to $\overline{K}(X)$ defined by inf, $\overline{v}$, a_n and δ_n, then (a_n, δ_n) is a minimal pair of definition of $\overline{w}_n$ relative to K.

β) $\delta_n < \delta_{n+1}$ and $\overline{v}(a_{n+1} - a_n) \geq \delta_n$, or equivalently $\overline{w}_n < \overline{w}_{n+1}$ (see Proposition 1.1).

γ) $K(a_n) = K(b_n)$ for all n.

The defintion of the pair (a_n, δ_n) is done by induction over n. Let us denote $a_1 = b_1$. Since according to 1) a_1 is separable over K', by [3, Theorem 3.9] it follows that there exists $\delta_1 \in G_v$ such that (a_1, δ_1) is a minimal pair of defintion of $\overline{w}_1$ with respect to K.

Let us assume that $n \geq 1$ and that there are defined all pairs (a_i, δ_i), $I = 1,\ldots,n$, such that the conditions α) - γ) are accomplished. Since $[K'(b_n) : K'] = [K(b_n) : K]$ by η) it follows that $K'(a_n) = K'(b_n)$ and so by (23) we have

(24) $\quad K'(b_n) = K'(a_n) \subseteq K'(b_{n+1})$.

Let $a \in K$ be such that

(25) $\quad v(a) > \sup(\delta_n, \omega(a_n)) - v(b_{n+1})$

where $\omega(a_n) = \sup(v(a_n - a_n')/a_n'$ over all elements of K conjugated with a_n over K and distinct to a_n). Let us denote:

$$a_{n+1} = ab_{n+1} + a_n \ .$$

Obviously by (25) one has $v(a_{n+1} - a_n) > \omega(a_n)$, and so according to Krasner's Lemma (see [6, pag. 22]) it results that $K'(a_n) \subseteq K'(a_{n+1})$. According to (24) and the inductive hypothesis γ) it follows that $K'(b_{n+1}) = K'(a_{n+1})$ and $K(a_{n+1}) = K(b_{n+1})$.

Let $\delta_{n+1} \in G_{\overline{v}}$ be such that

(26) $\quad \delta_{n+1} > \sup(\delta_n, \omega(a_{n+1}))$.

Thus, by [3, Proposition 3.2], it follows that (a_{n+1}, δ_{n+1}) is a minimal pair of defintion of $\overline{w}_n$ with respect to K'. Moreover, since $[K'(a_{n+1}) : K'] = [K(a_{n+1}) : K]$, and a_{n+1} is separable over both K' and K, by [3, Proposition 4.1], it follows that (a_{n+1}, δ_{n+1}) is a minimal pair of definition of $\overline{w}_{n+1}$ with respect to both K' and K. Therefore it is plain that conditions α) - γ) are verified by all pairs (a_i, δ_i), $i = 1,\ldots,n + 1$.

Finally, let us denote by w_n the restriction of $\overline{w}_n$ to K(X). By β) it follows that $w_n \leq w_{n+1}$ for all n and so $\{w_n\}_n$ is an ordered system of r.t. extensions of v to K(X). We assert that this system has a limit. For that we show that the ordered system $(\overline{w}_n)_n$ has a limit. To do this we shall prove that the condition 2) of Proposition 2.2 is verified. Indeed, let $c \in K$ and assume that for every n one has $\overline{w}_n(X - c) \geq \delta_n$. This means that $\overline{v}(a_n - c) = \overline{w}_n(a_n - X + X - c) = \delta_n$. According to (26) it follows that $\overline{v}(a_n - c) > \omega(a_n)$ if $n \geq 2$. Hence by Krasner's Lemma, it follows that $K'(a_n) \subseteq K(c)$ for all $n \geq 2$. But this is a contradiction since the sequence $[K'(a_n) : K'] = [k_n : k_v]$ tends to infinity because $[k : k_v]$ is not finite by hypothesis. Therefore by Proposition 2.2 it results that $(\overline{w}_n)_n$ has a limit $\overline{w}$ which is not an r.t. extension of $\overline{v}$. Then, according to Theorem 2.4, it results that w, the restriction of $\overline{w}$ to K(X), is a limit of $(w_n)_n$ and $k_w = \bigcup_n k_{v_n} = \bigcup_n k_v = k$. Moreover, according to [3, Lemma 4.2], we can choose δ_n such that $G_{w_n} = G_v$ for all n. Then by Theorem 5.1 one has $G_w = G_v$, as claimed.

Now, let us consider a finite extension k/k_v (assume also that $k_v \subset k \subset k_{\overline{v}}$). The existence of a r.a.t. extension w of v to K(X) such that $k_w = k$ is proved under additional assumptions.

THEOREM 6.2. Let k/k_v be a finite extension. Let $(\widetilde{K},\widetilde{v})$ be the completion of (K,v) (see [5, Ch. VI, § 5]). Assume that tr.deg $\widetilde{K}/K > 0$. Then there exists an r.a.t. extension w of v to K(X) such that $k_w = k$. Moreover, we can choose w such that $G_v = G_w$.

Proof. Since k/k_v is finite according to [3, Lemma 4.2] there exists an element $a \in \overline{K}$ such that a is separable over K, $[K(a) : K] = [k : k_v]$ and $k_{v_1} = k$, where v_1 is the restriction of $\overline{v}$ to K(a). Moreover, if (K',v') is the Henselisation of (K,v) included in $(\overline{K},\overline{v})$ (see [6, pag. 131]) then as above

(27) $$[K'(a) : K'] = [K(a) : K] = [k : k_v].$$

Since there exists an element $\widetilde{a} \in \widetilde{K}$ transcendental over K, then there exists a well ordered set $\{\delta_i\}_{i \in I}$ of elements of G_v and a system $\{a_i\}_i$ of elements of K such that:

1) δ_i is a cofinal subset of G_v,

(28) 2) $v(a_i - a_j) = \delta_i$ whereas $i < j$, $i,j \in I$,

3) $v(a_i - \tilde{a}) = \delta_i$ for all $i \in I$.

Let $a = a^{(1)},\ldots,a^{(n)}$, be all conjugates of a over K.

Denote $\omega(a) = \sup(v(a - a^{(t)}), t = 2,\ldots,n)$. According to (28) 1) there exists $i_o \in I$ such that $\delta_{i_o} > \omega(a)$. By a suitable modification of the set I, we can assume that

(29) $\omega(a) < \delta_i$ for all $i \in I$.

Let $\overline{w}_i$ be the r.t. extension of $\overline{v}$ to $\overline{K}(X)$ defined by inf, $\overline{v}$, $a_i + a$ and δ_i. Since all conjugates of $a_i + a$ relative to K are obviously $a_i + a^{(1)},\ldots,a_i + a^{(n)}$, it results that $\omega(a_i + a) = \omega(a)$. Hence, according to (29) and [3, Proposition 3.2.], it follows that $(a_i + a, \delta_i)$ is a minimal pair of definition of $\overline{w}_i$ relative to K'. Now since $K(a) = K(a_i + a)$, by (27) and [3, Proposition 4.1], it follows that $(a_i + a, \delta_i)$ is also a minimal pair of definition of $\overline{w}_i$ with respect to K.

We assert that in fact $(\overline{w}_i)_i$ is an ordered system of r.t. extensions of $\overline{v}$ to $\overline{K}(X)$. Indeed, one has: $\overline{v}(a_i + a - (a_j + a)) = \overline{v}(a_i - a_j) = \delta_i$ if $i < j$ (see (28), 2)). Since $\delta_i < \delta_j$ whereas $i < j$, by Proposition 1.1 it follows that $\overline{w}_i < \overline{w}_j$.

Furthermore we assert that the ordered system $(\overline{w}_i)_{i \in I}$ has a limit. For that we shall prove that the condition 2) of Proposition 2.2 is verified. Indeed, let $b \in \overline{K}$. Assume that for any $i \in I$ one has $\overline{w}_i(X - b) \geq \delta_i$. Then $\overline{v}(b - (a_i + a)) = \overline{w}_i(b - X + X - (a_i + a)) = \delta_i$. Hence the element $b - a \in \overline{K}$ is also a limit of the Cauchy system $(a_i)_{i \in I}$, or equivalently $\tilde{a}$ is algebraic over K, a contradiction. Therefore the condition 2) of Proposition 2.2 is verified for all $b \in \overline{K}$, and so $(\overline{w}_i)_i$ has a limit $\overline{w}$.

Let us denote by w_i the restriction of $\overline{w}_i$ to $K(X)$ for all $i \in I$, and let w be the restriction of $\overline{w}$ to $K(X)$. According to Theorem 2.3 we have that: $w = \sup_i w_i$ and $k_w = \bigcup_i k_{v_i} = k$. As usual v_i is the restriction of $\overline{v}$ to $K(a_i + a) = K(a) = K_1$. Finally $G_w = \bigcup_i G_{v_i} = G_v$, since by the equality $[K(a) : K] = [k : k_v]$ one has $G_{v_1} = G_v$, and since $\delta_i \in G_v$ one has $G_{v_1} = G_v = G_{v_i}$ for all i.

2. If w is an r.a.f. extension of v to K(X) then by Theorem 5.3 it follows that

k_w/k_v is a finite extension. Now a somewhat converse result is valid:

PROPOSITION 6.3. Let k/k_v be a finite extension. Then there exists an r.a.f. extension w of v to K(X) such that $k_w = k$.

Proof. Since k/k_v is finite, according to [3, Lemma 4.2], there exists an element $a \in \overline{K}$ such that a is separable over K, $[K(a) : K] = [k : k_v]$ and $k_{v_1} = k$, where v is the restiction of $\overline{v}$ to K(a).

Let $G = Z \times G_{\overline{v}}$ ordered lexicographically and $\delta = (1,0) \in G$. Let $\overline{w}$ be the extension of $\overline{v}$ to $\overline{K}(X)$ defined by inf, $\overline{v}$, a and δ. It is clear that $\overline{w}$ is an r.a.f. extension of $\overline{v}$ to K(X) and so w, the restriction of $\overline{w}$ to K(X), is also an r.a.f. extension of v. Furthermore since $\delta > \gamma$ for all $\gamma \in G_v$ (we remark that G_v is identified to $O \times G_v$) then (a, δ) is a minimal pair of definition of w with respcet to K. Therefore, according to Theorem 5.3, we have: $k_w = k_{v_1} = k$, as claimed.

7. EXISTENCE OF EXTENSIONS OF v TO K(X) WITH GIVEN VALUE GROUP

Let us assume that (K,v) is such that G_v is not divisible. By Corollary 5.2 it follows that w if r.a.t. extension of v to K(X) then the group G_w/G_v is countable. There exists a somewhat converse result:

THEOREM 7.1. Let (K,v) be a valued field. Let $G_v \subset G \subsetneq QG_v = G_{\overline{v}}$ be such that G/G_v is an infinite but countable group. Then there exists a r.a.t. extension of v to K(X) such that $G_w = G$. Moreover one can choose w such that $k_w = k_v$.

Proof. Since G/G_v is a countable torsion group, we may define a sequence of subgroups:

$$G_v \subset G_1 \subset G_2 \subset \dots \quad G_n \subset \dots \quad G$$

such that $G_n \neq G_{n+1}$, G_n/G_v is finite for all n, and that $\bigcup_n G_n = G$.

Now we shall define for all positive integer n an element $a_n \in \overline{K}$, separable over K, such that:

a) $[K(a_n) : K] = [G_n : G_v] = |G_n/G_v|$

b) $K(a_n) \subset K(a_{n+1})$

c) If we denote by v_n the restriction of $\overline{v}$ to $K(a_n)$ then $G_{v_n} = G_n$.

The element a_n will be defined by induction over n. Indeed, according to [3, Lemma 4.3], there exists an element a_1 such that a) and c) are verified. Let us assume that $n \geq 1$ and that the elements $a_1, \ldots, a_n$ are defined such that a), b) and c) are satisfied. Again, according to [3, Lemma 4.3], there exists an element $b_{n+1} \in \overline{K}$ separable over $K(a_n)$ such that $[K(a_n)(b_{n+1}) : K[a_n] = [G_{n+1} : G_n]$ and $G_{v_{n+1}} = G_{n+1}$, where v_{n+1} is the restriction of v to $K(a_n, b_{n+1})$. Now, since b_{n+1} is separable over $K(a_n)$ and a_n is separable over K by hypotheses, there exists an element $a_{n+1} \in K$ such that $K(a_n, b_{n+1}) = K(a_{n+1})$. It is clear that the elements $a_1, \ldots, a_n, a_{n+1}$ are such that the conditions a), b), c) are satisfied.

The rest the proof is made in the same way as the proof of Theorem 6.1 and it is left to the reader.

In the same manner as we have proved Theorem 6.2, we can prove the following result:

THEOREM 7.2. Let (K,v) be a valued pair and let G be an ordered group such that $G_v \subset G$ and G/G_v is finite. Assume that $\mathrm{tr.deg}(\widetilde{K}/K) > 0$, where $(\widetilde{K}, \widetilde{v})$ is the completion of (K,v) (see [4, Ch. V, §5]). Then there exists an r.a.t. extension w of v to K(X) such that $G_w = G$. Moreover we can choose w such that $k_w = k_v$.

By Theorem 6.1 and 7.1 one may derive in a canonical way the following result:

COROLLARY 7.3. Let (K,v) be a valued pair. Assume that there exists an infinite algebraic extension k/k_v countable generated and an ordered group G such that $G_v \subset G$ and G/G_v is an infinite countable torsion group. Then there exists an r.a.t. extension w of v to K(X) such that $k_w \simeq k$ and $G_w \simeq G$.

Also by Theorems 6.2 and 7.2 it follows:

COROLLARY 7.4. Let (K,v) be a valued pair. Let k/k_v be a finite extension and let G be an ordered group such that $G_v \subseteq G$ and G/G_v is finite. Assume that $\text{tr.deg}(\widetilde{K}/K) > 0$ where $(\widetilde{K},\widetilde{v})$ is the completion of (K,v) (see [4, Ch. V, § 5]). Then there exists an r.a.t. extension of w to $K(X)$ such that $k_w \simeq k$ and $G_w \simeq G$.

REFERENCES

[1] V. Alexandru, N. Popescu, Sur une classe de prolongements a K(X) d'une valuation sur un corps K (to appear in Rev. Roumaine Math. Pures Appl.).

[2] V. Alexandru, N. Popescu, A. Zaharescu; A theorem of characterization of residual transcendental extensions of a valuation (to appear in J. Math. Kyoto Univ.).

[3] V. Alexandru, N. Popescu, A. Zaharescu; Minimal pairs of definition of a residual transcendental extension of a valuation (to appear).

[4] N. Bourbaki, Theorie des Ensembles, Ch. III: Ensembles Ordonnes-Cardinaux--Nombres Entiers, Hermann, Paris, 1956.

[5] N. Bourbaki, Algebre Commutative, Ch. V: Entiers, Ch. Vi: Valuations, Hermann, Paris 1964.

[6] O. Endler, Valuation Theory, Springer, Berlin-Heidelberg-New York; 1972.

[7] S. MacLane, A construction for absolute values in polynomial rings, Trans. Amer. Math. Soc. 40 (1936) 363-395.

[8] M. Nagata, A theorem on valuation rings and its applications, Nagoya Math. J. 29 (1967), 85-91.

[9] A. Ostrovski, Untersuchungen zur aritmetischen Theorie der Korper, Teil I, II, III, Math. Zeit, 39 (1935), 269-404.

[10] O.F.G. Schilling, The theory of valuations, AMS Surveys, Nr.4, Providence, Rhode Island, 1950.

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