JOINT SPECTRAL PROPERTIES FOR PERMUTABLE LINEAR TRANSFORMATIONS

by

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1. INTRODUCTION

Let X be a complex Banach space and let $T_j:D(T_j)\subset X\longrightarrow X$ $(j=1,2)$ be linear transformations in X. Following [12], we say that T_1, T_2 are <u>permutable</u> (or <u>permute</u>) if

$$T_1T_2x = T_2T_1x \quad , \quad x\in D(T_1T_2)\cap D(T_2T_1) \quad . \tag{1.1}$$

Multiplication (unbounded) operators by independent variables or partial differential operators with constant coefficients on various function spaces provide examples of permutable transformations.

A motivation to introduce a joint spectrum for a permutable pair of linear transformations is given in [12]. In particular, it is shown that such a joint spectrum can be used to characterize the commutativity of the spectral measures attached to a pair of permutable selfadjoint operators (see Thm.2.5 from [12]).

The aim of the present paper is to define a joint spectrum for an arbitrary finite family of permutable paraclosed transformations (see the definition below), which is different and, perhaps, more natural than the corresponding definitions from [10] or [5]. The actual definition extends that given by J.L.Taylor in the case of commuting linear bounded operators [9] (another type of extension can be found in [14]). Unlike in [12], where only genuine complex numbers are used, the present joint spectrum is a (closed) subset of a Cartesian product of copies of the Riemann sphere $\hat{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$. Among other results, we prove that the projection property of the joint spectrum (see Thm.3.2 from [9]) still holds in this context. As an application, we shall characterize the commutativity of the strongly continuous semigroups of linear operators in terms of the joint spectrum of their infinitesimal generators. Some elements of Fredholm theory for permutable families of paraclosed transformations will also be mentioned.

Let Lat(X) denote the family of all <u>Banach subspaces</u> [13] of X, i.e. those linear subspaces Z of X which have a Banach space structure of their own that makes the inclusion $Z \subset X$ continuous. If $Z_1, Z_2 \in \mathrm{Lat}(X)$, then both $Z_1 + Z_2$ and $Z_1 \cap Z_2$ belong to Lat(X). In other words, Lat(X) is a lattice, as the notation suggests. A linear subspace $Z \subset X$ is in Lat(X) iff it is the range of a linear and continuous operator defined on a certain Banach space. A member $Z \in \mathrm{Lat}(X)$ has a uniquely determined Banach space topology and, moreover, $\mathrm{Lat}(Z) = \{W \in \mathrm{Lat}(X) ; W \subset Z\}$. All these assertions, which are simple consequences of the closed graph theorem, can be obtained as in Lemma 2.1 from [11, Part I].

It was G.Julia who firstly pointed out the importance of the class Lat(X), at least in the case of Hilbert spaces (see [2] for a complete list of references). The members of Lat(X) bear various names (for instance, they are called <u>paracomplete subspaces</u> in [6]) but we prefer the terminology of [13].

Now, let Y be another fixed Banach space. A linear transformation (or operator) $T: D(T) \subset X \longrightarrow Y$ is said to be <u>paraclosed</u> if its graph G(T) is a member of $\mathrm{Lat}(X \times Y)$. (Note that a paraclosed operator is called in [6] <u>paracomplete</u>).

Every closed operator is paraclosed but the converse is not true. (Indeed, if $Z \in \mathrm{Lat}(X)$ is not closed, then the inclusion $Z \subset X$ is paraclosed but not closed.)

The family of all paraclosed operators, defined on linear subspaces of X with values in Y, will be denoted by $\mathcal{D}(X,Y)$. The set $\mathcal{D}(X,X)$ will be briefly designated by $\mathcal{D}(X)$. If $T_1, T_2 \in \mathcal{D}(X,Y)$, then $T_1 + T_2$ (defined, of course, on $D(T_1) \cap D(T_2)$) is also in $\mathcal{D}(X,Y)$. Moreover, if $T \in \mathcal{D}(X,Y)$ and $S \in \mathcal{D}(Y,Z)$, then $ST \in \mathcal{D}(X,Z)$. Finally, if $T \in \mathcal{D}(X,Y)$ is injective, then $T^{-1} \in \mathcal{D}(Y,X)$ (these assertions follow as in Section 2 from [11, Part I]).

Let $T \in \mathcal{D}(X,Y)$. Since $D(T)$ is the projection of $G(T)$ on the first coordinate, we have $D(T) \in Lat(X)$. In particular, $D(T)$ has a (uniquely determined) Banach space topology and the operator $T: D(T) \longrightarrow Y$ becomes continuous. In particular, if $T \in \mathcal{D}(X,Y)$ and $D(T) = X$, then T is bounded (see also Prop. 2.1.5 from [6]). This simple remark will be often used in the sequel. The above discussion also shows that if $T \in \mathcal{D}(X,Y)$, then $\ker(T) \in Lat(X)$ and $im(T) \in Lat(Y)$.

The isolation of the class $\mathcal{D}(X,Y)$ also goes back to G. Julia (see [2] or [6]; see also [13], [11], [15] etc. for some extensions).

Let $T \in \mathcal{D}(X,Y)$ be bijective. The above arguments then show that T^{-1} is bounded. Therefore in this case T must be closed.

For an arbitrary $T \in \mathcal{D}(X)$, we denote by $\sigma_{\mathbb{C}}(T)$ the set of those $z \in \mathbb{C}$ such that the operator $z - T : D(T) \longrightarrow X$ is not bijective. We define the spectrum $\sigma(T)$ of T by the equality $\sigma(T) = \sigma_{\mathbb{C}}(T)$ if $D(T) = X$ and $\sigma(T) = \sigma_{\mathbb{C}}(T) \cup \{\infty\}$ if $D(T) \neq X$. The set $\rho(T) = \hat{\mathbb{C}} \setminus \sigma(T)$ is the resolvent set of T. It is easily seen that $\rho(T)$ is open in $\hat{\mathbb{C}}$. In fact, if $\rho(T) \neq \emptyset$, then $\rho(T) \cap \mathbb{C} \neq \emptyset$ and, as noticed above, in this case T is closed.

Let us specify what we mean by commutativity in the class $\mathcal{D}(X)$.

Let $T_1, T_2 \in \mathcal{D}(X)$. We say that T_1, T_2 commute if $\rho(T_j) \neq \emptyset$

and for some $z_j \in \rho(T_j) \cap \mathbb{C}$ $(j=1,2)$ the bounded linear operators $(z_1-T_1)^{-1}$ and $(z_2-T_2)^{-1}$ commute.

It is known (and easily seen) that this property does not depend on the particular choice of the points z_1, z_2. In other words, if T_1, T_2 commute, then $(w_1-T_1)^{-1}$ and $(w_2-T_2)^{-1}$ commute for all $w_j \in \rho(T_j) \cap \mathbb{C}$ $(j=1,2)$.

If T_1, T_2 commute, then T_1, T_2 permute. Indeed, if

$$x \in D(T_1T_2) \cap D(T_2T_1) = D((z_1-T_1)(z_2-T_2)) \cap D((z_2-T_2)(z_1-T_1)),$$

$x_1 = (z_1-T_1)(z_2-T_2)x$ and $x_2 = (z_2-T_2)(z_1-T_1)x$, then an obvious calculation with inverses shows that $x_1 = x_2$.

On the other hand, it is well known that even for selfadjoint operators there are permutable pairs which do not commute (see, for instance, [8]).

To define a joint spectrum for several permutable linear transformations, the most appropriate class seems to be that of paraclosed operators. One reason is that this class leads to chain-complexes of Banach spaces and continuous linear operators for which a suitable perturbation theory is available (see [1]). To get a better understanding of the general case, we shall first perform our construction in the case of two operators.

Let $T=(T_1,T_2) \in \mathcal{D}(X)^2$ be a permutable pair. Let $X_T^0 = D(T_1T_2) \cap D(T_2T_1)$, let $X_T^1 = D(T_2) \oplus D(T_1)$ and let $X_T^2 = X$. For every $z=(z_1,z_2) \in \hat{\mathbb{C}}^2$ we define the mapping $\delta^0(T(z))$ from X_T^0 into X_T^1 by the formula

$$\delta^0(T(z))x = T_1(z)x \oplus T_2(z)x \quad , \quad x \in X_T^0 \quad ,$$

with $T_j(z)x = (z_j-T_j)x$ if $z_j \neq \infty$ and $T_j(z)x = x$ if $z_j = \infty$, $j=1,2$.

Let also $\delta^1(T(z)) : X_T^1 \to X_T^2$ be given by

$$\delta^1(T(z))x_2\oplus x_1 = T_1(z)x_1 - T_2(z)x_2 \quad , \quad x_2\oplus x_1\in X_T^1 \quad ,$$

with $T_j(z)$ defined as above. Since T_1,T_2 permute, it is easily seen that $\delta^1(T(z))\delta^0(T(z))=0$ for every $z\in\hat{\mathbb{C}}^2$. This shows that the sequence

$$0 \longrightarrow X_T^0 \xrightarrow{\delta^0(T(z))} X_T^1 \xrightarrow{\delta^1(T(z))} X_T^2 \longrightarrow 0 \qquad (1.2)$$

is a complex of linear spaces. As a matter of fact, by the preceding discussion the spaces X_T^k $(k=0,1,2)$ have a Banach space topology and the mappings $\delta^k(T(z))(k=0,1)$ become continuous. In other words, (1.2) is a complex of Banach spaces.

By definition, a point $z=(z_1,z_2)\in\hat{\mathbb{C}}^2$ is in the <u>joint spectrum</u> $\sigma(T)$ of T if the complex (1.2) is not exact.

In spite of some combinatorial (and graphical) difficulties, we shall see in the next section that the general case is well reflected by the above situation.

2. A JOINT SPECTRUM FOR PERMUTABLE OPERATORS

Let $n\geqslant 2$ be an integer and let $\mathcal{S}(n)$ be the group of permutations of the set $\{1,\dots,n\}$. Let also $\{j_1,\dots,j_p\}\subset\{1,\dots,n\}$ be such that $1\leqslant j_1<\dots<j_p\leqslant n$, where $1\leqslant p\leqslant n-1$. We shall designate by $\mathcal{S}_{j_1\dots j_p}(n)$ the set of all bijective mappings

$$\pi \; : \; \{1,\dots,n-p\} \longrightarrow \{1,\dots,n\}\setminus\{j_1,\dots,j_p\} \; .$$

If $p=0$, we define $\mathcal{S}_{\emptyset}(n)=\mathcal{S}(n)$. Note also that $\mathcal{S}_{1\dots n}(n)=\emptyset$.

When no confusion is possible, the set $\mathcal{S}_{j_1\dots j_p}(n)$ will be denoted by $\mathcal{S}_{j_1\dots j_p}$.

Let X be a Banach space. In this section we shall work with finite families of paraclosed linear transformations

$T=(T_1,\dots,T_n)\in\mathcal{D}(X)^n$,which will be designated,following [11,Part II],as <u>multioperators</u>.When every pair (T_j,T_k) permute, then T is said to be a <u>permutable multioperator</u> (briefly,a p.m.).

Let $T=(T_1,\dots,T_n)\in\mathcal{D}(X)^n$ .We associate T with the family of indeterminates $\sigma=(\sigma_1,\dots,\sigma_n)$ .If $0\leqslant p\leqslant n-1$ and $\pi\in\mathcal{S}_{j_1\dots j_p}$, we set $T_\pi=T_{\pi(1)}\cdots T_{\pi(n-p)}$.

Let $\Lambda^p[\sigma,X]$ be the space of all exterior forms in $\sigma_1,\dots,\sigma_n$, with coefficients in X ,of degree p (see [9] or [10] for details). We denote by X_T^p the linear subspace of $\Lambda^p[\sigma,X]$ which consists of vectors of the form

$$\begin{aligned} \xi &= \sum_{j_1<\dots<j_p} x_{j_1\dots j_p}\,\sigma_{j_1}\wedge\dots\wedge\sigma_{j_p} , \\ x_{j_1\dots j_p} &\in \bigcap_{\pi\in\mathcal{S}_{j_1\dots j_p}} D(T_\pi) . \end{aligned} \tag{2.1}$$

With the above convention, $X_T^0=\bigcap\{D(T_\pi)\ ;\ \pi\in\mathcal{S}(n)\}$ .We also define $X_T^n=\Lambda^n[\sigma,X]$ and $X_T^p=\{0\}$ if $p\leqslant -1$ or $p\geqslant n+1$.

2.1. LEMMA. Let $T=(T_1,\dots,T_n)\in\mathcal{D}(X)^n$ and let $z',z''\in\mathbb{C}^n$. Then for every $p\geqslant 0$ one has $X^p_{T-z'}=X^p_{T-z''}$.

Proof. It suffices to assume that $0\leqslant p\leqslant n-1$ .It is also sufficient to show that

$$\bigcap_{\pi\in\mathcal{S}_{j_1\dots j_p}} D((T-z')_\pi)=\bigcap_{\pi\in\mathcal{S}_{j_1\dots j_p}} D((T-z'')_\pi) , \tag{2.2}$$

for every family of indices $j_1<\dots<j_p$ .This can be easily obtained from the formal equality

$$(T_{k_1}-z''_{k_1})\dots(T_{k_q}-z''_{k_q}) = (T_{k_1}-z'_{k_1})\dots(T_{k_q}-z'_{k_q}) + R , \tag{2.3}$$

where $q=n-p$, R is a sum of monomials in $T_{k_1}-z'_{k_1},\dots,T_{k_q}-z'_{k_q}$ of degree less than q and $k_1,\dots,k_q$ is a family of distinct indices.Then (2.2) can be inferred via (2.3) .

Note that the translation invariance expressed by (2.2) does not require any condition of the type (1.1).

2.2. LEMMA. The multioperator $T=(T_1,\dots,T_n)\in\mathcal{D}(X)^n$ is permutable iff $T-z$ is permutable for a certain $z\in\mathbb{C}^n$.

Proof. Indeed, from (2.2) we have, in particular, for all pairs j,k

$$D(T_jT_k)\cap D(T_kT_j)=D((T_j-z_j)(T_k-z_k))\cap D((T_k-z_k)(T_j-z_j)),$$

from which we derive easily the assertion.

We shall define in what follows a joint spectrum for every p.m., following the ideas of J.L.Taylor in the bounded case [9]. Let $T=(T_1,\dots,T_n)\in\mathcal{D}(X)^n$ be a p.m. associated with the family of indeterminates $\sigma=(\sigma_1,\dots,\sigma_n)$. Let $0\leqslant p\leqslant n-1$, let $1\leqslant j_1<\dots<j_p\leqslant n$ and let

$$x\in\bigcap_{\pi\in\mathcal{S}_{j_1\dots j_p}} D(T_\pi)$$

(where $\mathcal{S}_{j_1\dots j_p}=\mathcal{S}_\emptyset=\mathcal{S}(n)$ if $p=0$). Let also $1\leqslant k\leqslant n$. Then for every $z\in\hat{\mathbb{C}}^n$ we set

$$\begin{aligned} T^k_{j_1\dots j_p}(z)x &= 0 && \text{if } k\in\{j_1,\dots,j_p\}, \\ &=(z_k-T_k)x && \text{if } k\notin\{j_1,\dots,j_p\} \text{ and } z_k\neq\infty, \qquad (2.4)\\ &=x && \text{if } k\notin\{j_1,\dots,j_p\} \text{ and } z_k=\infty. \end{aligned}$$

Note that if $k\notin\{j_1,\dots,j_p\}$, then

$$T^k_{j_1\dots j_p}(z)x\in\bigcap_{\pi\in\mathcal{S}_{j'_1\dots j'_{p+1}}} D(T_\pi), \qquad (2.5)$$

where $j'_1,\dots,j'_{p+1}$ is obtained by ordering $\{k,j_1,\dots,j_p\}$.

Let $\xi\in X^p_T$ be a vector represented by (2.1). We set

$$T^{k,p}(z)\xi=\sum_{j_1<\dots<j_p} T^k_{j_1\dots j_p}(z)x_{j_1\dots j_p}\,\sigma_{j_1}\wedge\dots\wedge\sigma_{j_p}. \qquad (2.6)$$

Then we define a linear mapping from X^p_T into $\Lambda^{p+1}[\sigma;X]$ by the equality

$$\delta^p(T(z))\xi = \sum_{k=1}^{n} \sigma_k \wedge T^{k,p}(z)\xi \quad . \tag{2.7}$$

It follows from (2.5) that $\delta^p(T(z))$ takes values actually in X_T^{p+1} .

2.3. LEMMA. The sequence

$$0 \longrightarrow X_T^0 \xrightarrow{\delta^0(T(z))} X_T^1 \xrightarrow{\delta^1(T(z))} \cdots \xrightarrow{\delta^{n-1}(T(z))} X_T^n \longrightarrow 0 \tag{2.8}$$

is a complex of Banach spaces and continuous mappings.

Proof. From (2.6) we obtain

$$\delta^{p+1}(T(z))\delta^p(T(z))\xi = \sum_{k<m} \sigma_k \wedge \sigma_m (T^{k,p+1}(z)T^{m,p}(z) - T^{m,p+1}(z)T^{k,p}(z))\xi$$

for each $\xi \in X_T^p$ .According to (1.1) and (2.4),we may assert that

$$T^{k,p+1}(z)T^{m,p}(z)\xi - T^{m,p+1}(z)T^{k,p}(z)\xi = 0$$

for all indices k,m .Therefore $\delta^{p+1}(T(z))\delta^p(T(z)) = 0$,that is (2.8) is a complex of linear spaces.

Since $X_T^p \in \mathrm{Lat}(\Lambda^p[\sigma, X])$ and $\delta^p(T(z))$ is paraclosed,it follows that X_T^p has a Banach space topology and the operator $\delta^p(T(z))$ becomes continuous (see the Introduction). Therefore (2.7) is actually a complex of Banach spaces.

2.4. DEFINITION. Let $T = (T_1,\dots,T_n) \in \mathcal{D}(X)^n$ be a p.m.The <u>joint spectrum</u> $\sigma(T)$ of T is defined as the set of those $z \in \hat{\mathbb{C}}^n$ such that the complex (2.8) is not exact.The set $\rho(T) = \hat{\mathbb{C}}^n \setminus \sigma(T)$ is called the <u>resolvent set</u> of T .We also define $\sigma_{\mathbb{C}}(T) = \sigma(T) \cap \mathbb{C}^n$.

If $T = (T_1)$ is singleton,then $X_T^0 = D(T_1)$, X_T^1 is isomorphic to X and $\delta^0(T(z))$ acts as $x \longrightarrow T_\emptyset^1(z)x$, $x \in D(T_1)$.Therefore a point $z_1 \in \mathbb{C}$ is in $\rho(T)$ iff $\mathrm{im}(z_1 - T_1) = X$ and $\ker(z_1 - T_1) = \{0\}$; in other words,iff $z_1 - T_1$ is bijective.The point $\infty \in \rho(T)$ iff $D(T_1) = X$ .This shows that $\sigma(T) = \sigma(T_1)$ and $\sigma_{\mathbb{C}}(T) = \sigma_{\mathbb{C}}(T_1)$ (see the Introduction).

One of the most important and useful properties of the joint spectrum of a commuting (bounded) multioperator is the projection property (see [9],Thm.3.2).We shall try to give in the following a version of this result.Unfortunately,Definition 2.4 does not suffice to our purpose.A more sophisticated definition is needed to present such an assertion (see also [12]).

Let $T'=(T_1,\dots,T_n)\in\mathcal{D}(X)^n$ and $T''=(T_{n+1},\dots,T_{n+m})\in\mathcal{D}(X)^m$ be such that $T=(T',T'')\in\mathcal{D}(X)^{n+m}$ is a p.m.We associate T with the family of indeterminates $\sigma=(\sigma_1,\dots,\sigma_{n+m})$.With the notation which precedes Definition 2.4,if $\xi\in X_T^p$,where $0\leqslant p\leqslant n+m$,there exists a uniquely determined decomposition $\xi=\xi'+\xi''$,where ξ' does not contain $\sigma_{n+1},\dots,\sigma_{n+m}$ .We denote by $X^p_{T';T''}$ the linear subspace of X_T^p which consists of such ξ' .Note that $X^p_{T';T''}=\{0\}$ if $n<p\leqslant n-m$ and that for $0\leqslant p\leqslant n$ we have $\xi'\in X^p_{T';T''}$ iff ξ' has the representation

$$\xi'=\sum_{1\leqslant j_1<\dots<j_p\leqslant n} x_{j_1\dots j_p}\,\sigma_{j_1}\wedge\dots\wedge\sigma_{j_p}\ ,$$

$$x_{j_1\dots j_p}\in\bigcap_{\pi\in\mathcal{S}_{j_1\dots j_p}(n+m)} D(T_\pi)\ . \tag{2.9}$$

In particular,for $p=n$, $X^n_{T',T''}=\{x\,\sigma_1\wedge\dots\wedge\sigma_n\ ;\ x\in X^0_{T''}\}$.

For every $z'\in\hat{\mathbb{C}}^n$ we define

$$\delta^p(T'(z');T'')=\delta^p(T(z))\,|\,X^p_{T';T''}\ , \tag{2.10}$$

where $z=(z',z'')\in\mathbb{C}^{n+m}$ (z'' may be arbitrary) and $\delta^p(T(z))$ is given by (2.7) .It follows from (2.5) that $\delta^p(T'(z');T'')$ maps $X^p_{T';T''}$ into $X^{p+1}_{T';T''}$.

2.5. LEMMA. The sequence

$$0\longrightarrow X^0_{T';T''}\xrightarrow{\delta^0(T'(z');T'')}X^1_{T';T''}\xrightarrow{\delta^1(T'(z');T'')}\dots \tag{2.11}$$

is a complex of Banach spaces and continuous mappings.

The proof is similar to that of Lemma 2.3 and is omitted.

2.6. DEFINITION. Let $T' = (T_1,\dots,T_n) \in \mathcal{D}(X)^n$ and let $T'' = (T_{n+1},\dots,T_{n+m}) \in \mathcal{D}(X)^m$ be such that $T=(T',T'')$ is a p.m. The <u>relative joint spectrum</u> of T' with respect to T'' is the set $\sigma(T';T'')$ consisting of those points $z' \in \hat{\mathbb{C}}^n$ such that the complex (2.11) is not exact. If $T'' = \emptyset$, we set $\sigma(T';\emptyset) = \sigma(T')$.

Note that if $T' = (T_1)$ and $T'' = (T_2)$ is such that $T = (T_1,T_2)$ is a permutable pair, then $\sigma(T';T'') = \sigma(T_1;T_2)$ is the spectrum of the operator $T_1 : D(T_1T_2) \cap D(T_2T_1) \longrightarrow D(T_2)$, where $D(T_2)$ is endowed with its natural Banach space topology.

Let T be a multioperator. We use the symbol $S \prec T$ to specify that S is an ordered subset of T (and hence a multioperator "contained" in T) or the empty set.

We are now in the position to prove the <u>projection property</u> of the joint spectrum.

2.7. THEOREM. Let $T' = (T_1,\dots,T_n) \in \mathcal{D}(X)^n$ and let $T'' = (T_{n+1},\dots,T_{n+m}) \in \mathcal{D}(X)^m$ be such that $T = (T',T'')$ is a p.m. Then we have the equality

$$pr^n_{n+m}\, \sigma(T) = \bigcup_{S \prec T''} \sigma(T';S) , \tag{2.12}$$

where $pr^n_{n+m} : \hat{\mathbb{C}}^{n+m} \longrightarrow \hat{\mathbb{C}}^n$ is the projection on the first n coordinates.

Proof. Let $T'_n = (T',T_{n+1})$ and let $T''_n = (T_{n+2},\dots,T_{n+m})$ (if $m=1$ we put $T''_n = \emptyset$). We shall prove that

$$pr^n_{n+1}\, \sigma(T'_n;T''_n) = \sigma(T';T''_n) \cup \sigma(T';T'') . \tag{2.13}$$

Since n and m are arbitrary, by succesive projections we derive easily (2.12) from (2.13).

Let us prove (2.13). First of all, let us designate by $\{H^p(T'(z');T'')\}_{p\geq 0}$ the cohomology of the complex (2.11), which will be used in various instances. We need the following.

2.8. LEMMA. With the above notation, there are linear mappings $u^p : X^p_{T';T''_n} \longrightarrow X^{p+1}_{T'_n;T''_n}$ and $v^p : X^p_{T'_n;T''_n} \longrightarrow X^p_{T';T''}$ such that the sequence

$$0 \to X^p_{T';T''_n} \xrightarrow{u^p} X^{p+1}_{T'_n;T''_n} \xrightarrow{v^p} X^{p+1}_{T';T''} \longrightarrow 0 \qquad (2.14)$$

is an exact complex of Banach spaces. In addition, there exists a long exact cohomology sequence

$$\begin{aligned} \dots \longrightarrow H^p(T'(z');T''_n) &\xrightarrow{\hat{u}^p} H^{p+1}(T'_n(z'_n);T''_n) \xrightarrow{\hat{v}^{p+1}} \\ \xrightarrow{v^{p+1}} H^{p+1}(T'(z');T'') &\xrightarrow{\theta^{p+1}} H^{p+1}(T'(z');T''_n) \xrightarrow{u^{p+1}} \dots , \end{aligned} \qquad (2.15)$$

where $z'=(z_1,\dots,z_n)$, $z'_n=(z',z_{n+1})$, $\hat{u}^p$, $\hat{v}^p$ are induced by by u^p , v^p , respectively, and θ^p is induced by $(-1)^p$ times the action of (2.4) , computed for T_{n+1} and z_{n+1} .

Proof of Lemma 2.8. The mapping u^p is given by $u^p\xi'=(-1)^p\sigma_{n+1}\wedge\xi'$, which is well defined. Indeed if $\xi'\in X^p_{T';T''_n}$ is represented as in (2.9), we have

$$x_{j_1\dots j_p} \in \bigcap_{\pi\in\mathcal{S}_{j_1\dots j_p}(n+m-1)} D((T',T''_n)_{\pi}) =$$

$$= \bigcap_{\pi\in\mathcal{S}_{j_1\dots j_p,n+1}(n+m)} D((T'_n,T''_n)_{\pi}) .$$

To define v^p we first observe that for $\xi\in X^p_{T'_n;T''_n}$ there exists the following unique decomposition $\xi=\xi'+\sigma_{n+1}\wedge\xi''$, where both ξ' , ξ'' do not contain σ_{n+1} . It is clear that $\xi'\in X^p_{T';T''}$, and therefore we may put $v^p\xi=\xi'$.

It is easy to see that (2.14) becomes an exact complex of Banach spaces. Moreover, if $z'\in\hat{\mathbb{C}}^n$ and $z'_n=(z',z_{n+1})\in\hat{\mathbb{C}}^{n+1}$, it can be easily shown that

$$u^{p+1}\,\delta^{p}(T'(z');T''_n) = \delta^{p+1}(T'_n(z'_n);T''_n)\,u^p \; .$$

Therefore the family $\{u^p\}_{p\geq 0}$ induces a morphism of degree one of the complex $\{X^p_{T';T''_n}, \delta^p(T'(z');T''_n)\}_{p\geq 0}$ into the complex $\{X^p_{T'_n;T''_n}, \delta^p(T'_n(z'_n);T''_n)\}_{p\geq 0}$.

We also have

$$v^{p+1}\,\delta^{p}(T'_n(z'_n);T''_n) = \delta^{p}(T'(z');T'')\,v^p \; .$$

Hence the family $\{v^p\}_{p\geq 0}$ induces a morphism of degree zero of the complex $\{X^p_{T'_n;T''_n}, \delta^p(T'_n(z'_n);T''_n)\}_{p\geq 0}$ into the complex $\{X^p_{T';T''}, \delta^p(T'(z');T'')\}_{p\geq 0}$

The existence of the long exact cohomology sequence (2.15) is now a consequence of a well-known result (see,for instance, [7]). Nevertheless,we need a more explicit expression of the connecting morphism θ^p .From the general theory (see [7]),it follows that θ^p is induced by an additive relation $\xi \to \eta$,where $\xi \in X^p_{T';T''}$, $\delta^p(T'(z');T'')\xi = 0$, $\xi = v^p\zeta$, $\zeta \in X^p_{T'_n;T''_n}$ and $\delta^p(T'_n(z'_n);T'')\zeta = u^p\eta$. Since $X^p_{T';T''} \subset X^p_{T'_n;T''_n}$,we may take $\zeta = \xi$,and the relation $\delta^p(T'_n(z'_n);T'')\xi = u^p\eta$ shows that $T_{n+1}(z_{n+1})\xi = (-1)^p\, u^p\eta$,where $T_{n+1}(z_{n+1})\xi = (z_{n+1} - T_{n+1})\xi$ when $z_{n+1} \neq \infty$ and $T_{n+1}(z_{n+1})\xi = \xi$ if $z_{n+1} = \infty$ (this is precisely an action induced by (2.4) for T_{n+1} and z_{n+1}).This completes the proof of the lemma

We go back to the proof of Theorem 2.7,more precisely to the proof of (2.13).Let $z' \in pr^n_{n+1}\,\sigma(T'_n;T''_n)$ and assume that $z' \notin \sigma(T';T''_n) \cup \sigma(T';T'')$ .Then the complexes $\{X^p_{T';T''_n}, \delta^p(z');T''_n\}_{p\geq 0}$ and $\{X^p_{T';T''}, \delta^p(T'(z');T'')\}_{p\geq 0}$ are exact and so the complex $\{X^p_{T'_n;T''_n}, \delta^p(T'_n(z'_n);T''_n)\}_{p\geq 0}$ is also exact,by (2.15),for every $z_{n+1} \in \hat{\mathbb{C}}$ .This is equivalent to saying that $z' \notin pr^n_{n+1}\,\sigma(T';T''_n)$, which contradicts our assumption.

Conversely, let $z' \in \hat{\mathbb{C}}^n$ and assume that $z' \notin pr^n_{n+1}\sigma(T'_n;T''_n)$. This means that for every $z_{n+1} \in \hat{\mathbb{C}}$ one has $z'_n = (z', z_{n+1}) \notin \sigma(T'_n;T''_n)$. Using again (2.15), the exactness of the complex $\{X^p_{T'_n;T''_n}, \delta^p(T'_n(z'_n);T''_n)\}$ shows that the linear spaces

$$\begin{aligned} &\ker \delta^p(T'(z');T'')/\mathrm{im}\,\delta^{p-1}(T'(z');T'') , \\ &\ker \delta^p(T'(z');T''_n)/\mathrm{im}\,\delta^{p-1}(T'(z');T''_n) \end{aligned} \qquad (2.16)$$

are isomorphic, and the isomorphism is induced by the action of $T_{n+1}(z_{n+1})$. Looking at the formula of $T_{n+1}(z_{n+1})$ (Lemma 2.8), we deduce that the canonical mapping from the first space in (2.16) into the second one is just an isomorphism. With the terminology of [13], the spaces (2.16) are quotient Banach spaces, and T_{n+1} induces in each of them a morphism whose spectrum is empty. This may happen only if the spaces (2.16) are null (see [13]) for all p, which is impossible by our assumption. This contradiction completes the proof of the theorem.

To make Theorem 2.7 really useful, we should observe that it remains true even if the projection has no privileged form.

2.9. THEOREM. Let $T = (T_1,\dots,T_n) \in \mathcal{D}(X)^n$ be a p.m. and let $M = \{m_1,\dots,m_p\} \subset \{1,\dots,n\}$ be a family of distinct integers. Set $T_M = (T_{m_1},\dots,T_{m_p})$. If $pr_M : \mathbb{C}^n \to \mathbb{C}^p$ is the projection $z = (z_1,\dots,z_n) \to z_M = (z_{m_1},\dots,z_{m_p})$, then we have the formula

$$pr_M\,\sigma(T) = \bigcup_{S \prec R} \sigma(T_M;S) ,$$

where $R = (T_{k_1},\dots,T_{k_{n-p}})$ and $\{k_1,\dots,k_{n-p}\} = \{1,\dots,n\} \setminus M$.

Proof. We first observe that if $\pi \in \mathcal{S}(n)$, then

$$\sigma((T_{\pi(1)},\dots,T_{\pi(n)});S) = \{(z_{\pi(1)},\dots,z_{\pi(n)});(z_1,\dots,z_n) \in \sigma(T;S)\}$$

for every multioperator S such that (T,S) is permutable (see, for instance, Lemma III.9.6 from [10], which can be easily adapted to this case). This remark allows us to reduce the actual statement to that of Theorem 2.7, from which we derive readily our assertion.

2.10. THEOREM. Let $T=(T_1,\ldots,T_n)\in\mathcal{D}(X)^n$ be a p.m. Then $\sigma(T)$ is a closed and nonempty subset of $\hat{\mathbb{C}}^n$.

Proof. Let $1\leq p\leq n-1$ and set

$$X_T(j_1,\ldots,j_p)=\bigcap_{\pi\in\mathcal{S}_{j_1\ldots j_p}} D(T_\pi) .$$

Since $X_T(j_1,\ldots,j_p)\in \mathrm{Lat}(X)$, we may fix a norm $\|x\|_{j_1\ldots j_p}$ in $X_T(j_1,\ldots,j_p)$. Note that X_T^p is isomorphic to the direct sum

$$\bigoplus_{1\leq j_1<\ldots<j_p\leq n} X_T(j_1,\ldots,j_p) ,$$

and therefore X_T^p can be given the norm

$$\|\xi\|_p=\sum_{j_1<\ldots<j_p}\|x_{j_1\ldots j_p}\|_{j_1\ldots j_p} , \quad \xi\in X_T^p ,$$

where ξ has the representation (2.1). It is also clear that X_T^0 can be given a norm $\|x\|_0$.

Now, let $z^0=(z_1^0,\ldots,z_n^0)\notin\sigma(T)$ and let also $z=(z_1,\ldots,z_n)\in\hat{\mathbb{C}}^n$ be such that $z_k\neq\infty$ if $z_k^0\neq\infty$ and $z_k\neq 0$ if $z_k^0=\infty$. We define

$$\begin{aligned}\hat{T}^k_{j_1\ldots j_p}(z)x &= (z_k-T_k)x && \text{if } z_k^0\neq\infty,\ z_k\neq\infty,\\ &= x-z_k^{-1}T_kx && \text{if } z_k^0=\infty,\ 0\neq z_k\neq\infty,\\ &= x && \text{if } z_k^0=\infty,\ z_k=\infty\end{aligned}$$

for all $x\in X_T(j_1,\ldots,j_p)$ and $k\notin\{j_1,\ldots,j_p\}$. When $k\in\{j_1,\ldots,j_p\}$ we set $T^k_{j_1\ldots j_p}(z)x=0$. Clearly, this definition can also be adapted for $p=0$.

Let $\hat{T}^{k,p}(z)$ and $\delta^p(\hat{T}(z))$ be given by (2.6) and (2.7), where $T^k_{j_1\ldots j_p}(z)$ is replaced by $\hat{T}^k_{j_1\ldots j_p}(z)$ and $T^{k,p}(z)$ is replaced by $\hat{T}^{k,p}(z)$, respectively. Then, by using the argument from the proof of Lemma 2.3, one can easily show that $\{X_T^p,\delta^p(\hat{T}(z))\}_{p\geq 0}$ is a complex of Banach spaces. Note also that

$$\|(\delta^p(\hat{T}(z)) - \delta^p(T(z^0)))\xi\|_{p+1} \leqslant$$

$$\leqslant \sum_{z_k^0 \neq \infty} \sum_{j_1<\dots<j_p} c^k_{j_1\dots j_p} |z_k - z_k^0| \, \|x_{j_1\dots j_p}\|_{j_1\dots j_p} +$$

$$+ \sum_{z_k^0 = \infty} \sum_{j_1<\dots<j_p} c^k_{j_1\dots j_p} |z_k|^{-1} \, \|x_{j_1\dots j_p}\|_{j_1\dots j_p} \leqslant \qquad (2.17)$$

$$\leqslant C_p\big(\max_{z_k^0 \neq \infty} |z_k - z_k^0| + \max_{z_k^0 = \infty} |z_k|^{-1}\big) \|\xi\|_p \ ,$$

where $c^p_{j_1\dots j_p}$, C_p are constants independent of z ,whose existence is insured by the continuity of the mappings

$$T_k : X_T(j_1,\dots,j_p) \longrightarrow X_T(j'_1,\dots,j'_{p+1}) \ ,$$

with $k,j'_1,\dots,j'_{p+1}$ as in (2.5) (note that the embedding $X_T(j_1,\dots,j_p)$ into $X_T(j'_1,\dots,j'_{p+1})$ is also continuous).

Since the complex $\{X_T^p \ ,\delta^p(T(z^0))\}_{p\geqslant 0}$ is exact by the hypothesis, if $\max\{|z_k - z_k^0| \ ; \ z_k^0 \neq \infty\}$ and $\max\{|z_k|^{-1} \ ; \ z_k^0 = \infty\}$ are sufficiently small,it follows from the stability of the exactness of the complexes of Banach spaces (see,for instance, [9] or [10]),via (2.17),that the complex $\{X_T^p \ ,\delta^p(\hat{T}(z))\}_{p\geqslant 0}$ is also exact.

We have only to show that the latter complex is exact iff the complex $\{X_T^p \ ,\delta^p(T(z))\}_{p\geqslant 0}$ is exact.Indeed,let $w_k = z_k$ if $z_k^0 = \infty$ and $0 \neq z_k \neq \infty$,and let $w_k = 1$ otherwise.Then we have $T^k_{j_1\dots j_p}(z) = w_k \hat{T}^k_{j_1\dots j_p}(z)$ for all indices.We define a transformation τ^p of X_T^p into itself by setting

$$\tau^p \xi = \sum_{j_1<\dots<j_p} w_{j_1}\dots w_{j_p} \, x_{j_1\dots j_p} \, \sigma_{j_1} \wedge \dots \wedge \sigma_{j_p} \ ,$$

with ξ given by (2.1).We also put $\tau^0 =$ identity .The mapping τ^p is clearly bijective for every $p \geqslant 0$ .We also have

$$\delta^p(T(z))\tau^p\xi = \tau^{p+1}\delta^p(\hat{T}(z))\xi \ .$$

Therefore, the complex $\{X_T^p, \delta^p(\hat{T}(z))\}_{p\geq 0}$ is exact iff the complex $\{X_T^p, \delta^p(T(z))\}_{p\geq 0}$ is exact. Using the first part of the proof, we obtain that the complex $\{X_T^p, \delta^p(T(z))\}_{p\geq 0}$ is exact for z in a neighbourhood of z^0, that is, the set $\rho(T)$ is open.

Let us prove that the set $\sigma(T)$ is nonempty. By virtue of Theorem 2.9, if we project the set $\sigma(T)$ on the j-th coordinate, we obtain a union of sets of the form $\sigma(T_j;R)$, where $R \prec (T_1,\ldots,T_{j-1},T_{j+1},\ldots,T_n)$. But the sets $\sigma(T_j;R)$ are nonempty for some j and R, and hence $\sigma(T)$ is nonempty.

2.11. REMARK. Theorem 2.10 can be stated for any relative spectrum of T, by using similar arguments. We omit the details.

3. AN APPLICATION TO SEMIGROUPS OF OPERATORS

In this section we shall characterize the commutativity of several strongly continuous semigroups of linear operators in terms of the joint spectrum of their infinitesimal generators.

Let $\{S(t)\}_{t\geq 0}$ be a strongly continuous semigroup of linear operators acting in the Banach space X. We denote by B the infinitesimal generator of $S=\{S(t)\}_{t\geq 0}$. We have

$$Bx = \lim_{t\to 0} t^{-1}(S(t)x - x) ,$$

provided the limit exists in X, and $B : D(B) \longrightarrow X$ is closed. If

$$\omega_0(S) = \inf\{t^{-1}\ln\|S(t)\| ;\ t>0\},$$

then we have the formula

$$(z - B)^{-1}x = \int_0^{\infty} e^{-zt}\, S(t)x\, dt , \quad x\in X , \tag{3.1}$$

whenever $\operatorname{Re} z > \omega_0(S)$ (see [3] for details). In other words

$$\sigma(B) \subset \{z\in\hat{\mathbb{C}} ;\ \operatorname{Re} z \leq \omega_0(S)\} ;$$

here and in the sequel we set $\operatorname{Re} z = -\infty$ if $z = \infty$.

3.1. THEOREM. Let $S_1=\{S_1(t)\}_{t\geq 0}, \dots, S_n=\{S_n(t)\}_{t\geq 0}$ be strongly continuous semigroups of operators acting in X and let $B_1,\dots,B_n$ be their infinitesimal generators, respectively. The following conditions are equivalent:

(1) For all $j,k\in\{1,\dots,n\}$ and $t',t''\geq 0$ we have $S_j(t')S_k(t'')=S_k(t'')S_j(t')$.

(2) $B=(B_1,\dots,B_n)$ is a p.m. and $\sigma(B)\subset\{z=(z_1,\dots,z_n)\in\hat{\mathbb{C}}^n ;\ \mathrm{Re}\ z_j\leq\omega_0(S_j),\ j=1,\dots,n\}$.

Proof. Assume that (1) holds and let the indices j,k be fixed $(j\neq k)$. Let also $x\in D(B_j)$. Since

$$t^{-1}(S_j(t)S_k(s)x - S_k(s)x) = S_k(s)(t^{-1}(S_j(t)x - x)),$$

it follows that $S_k(s)x\in D(S_j)$ and $B_jS_k(s)x=S_k(s)B_jx$ for each $s\geq 0$. Therefore, if $x\in D(B_k)\cap D(B_kB_j)$, we have

$$B_kB_jx = \lim_{t\to 0} t^{-1}(S_k(t)B_jx - B_jx) = \lim_{t\to 0} B_j(t^{-1}(S_k(t)x - x)).$$

Since B_j is closed, this implies that $B_kx\in D(B_j)$ and $B_jB_kx=B_kB_jx$. As a matter of fact we have proved that

$$D(B_k)\cap D(B_kB_j) = D(B_j)\cap D(B_jB_k) = D(B_jB_k)\cap D(B_kB_j),$$

and so $B=(B_1,\dots,B_n)$ is a p.m.

For the second statement of (2), let $z=(z_1,\dots,z_n)\in\hat{\mathbb{C}}^n$ be such that $\mathrm{Re}\ z_j>\omega_0(S_j)$ for some j. With no loss of generality we may assume $j=1$. We shall show that $z\notin\sigma(B)$.

Let $\sigma=(\sigma_1,\dots,\sigma_n)$ be a system of indeterminates associated with B. We have to prove that the complex $\{X_B^p, \delta^p(B(z))\}_{p\geq 0}$ is exact. To this end let us define a mapping τ^p on X_B^p by the relation $\tau^p\xi=(z_1-B_1)^{-1}\xi'$, where $\xi=\sigma_1\wedge\xi'+\xi''$ is uniquely represented with ξ', ξ'' not containing σ_1. Then τ^p takes values in X_B^{p-1}. Indeed, if

$$\xi' = \sum_{j_1<\cdots<j_{p-1}} x_{j_1\cdots j_{p-1}} \sigma_{j_1}\wedge\cdots\wedge\sigma_{j_{p-1}} ,$$

then $1 \notin \{j_1,\ldots,j_{p-1}\}$ and we must show that

$$(z_1-B_1)^{-1} x_{j_1\cdots j_{p-1}} \in \bigcap_{\pi\in\mathcal{S}_{j_1\cdots j_{p-1}}} D(B_\pi) .$$

This follows from the next computation, via (3.1):

$$(z_1-B_1)^{-1} B_j x = \int_0^\infty e^{-z_1 t} S_1(t)B_j x \, dt =$$

$$= \int_0^\infty B_j(e^{-z_1 t} S_1(t)x) \, dt = B_j(z_1-B_1)^{-1}x ,$$

valid for every $x\in D(B_j)$ and all j (B_j commutes with the integral by Theorem III.6.20 from [3]).

We now prove the equality

$$\tau^{p+1} \delta^p(B(z))\xi + \delta^{p-1}(B(z))\tau^p\xi = \xi , \quad \xi\in X_B^p . \qquad (3.2)$$

Indeed, if $\xi = \sigma_1\wedge\xi' + \xi''$ is as above, then, using (2.6) and (2.7) in the present context,

$$\tau^{p+1} \delta^p(B(z))\xi = \xi'' - \sum_{k=2}^{n} \sigma_k\wedge B^{k,p-1}(z)(z_1-B_1)^{-1}\xi'$$

and

$$\delta^{p-1}(B(z))\tau^p\xi = \sigma_1\wedge\xi' + \sum_{k=2}^{n} \sigma_k\wedge B^{k,p-1}(z)(z_1-B_1)^{-1}\xi' ,$$

whence we infer (3.2) .

Certainly, (3.2) implies the exactness of the complex $\{X_B^p, \delta^p(B(z))\}_{p\geq 0}$, and so $z\notin\sigma(B)$.

Conversely, assume that (2) holds and let us prove (1) .

It follows from Theorem 2.9 that

$$\sigma(B_j,B_k) \subset \mathrm{pr}\,\sigma(B) \subset \{(z_j,z_k)\in\hat{\mathbb{C}}^2 ; \mathrm{Re}\, z_q \leq \omega_0(S_q) , \ q=j,k\}$$

for every pair of indices $j,k=1,\ldots,n$, $j\neq k$, where $\mathrm{pr} : (z_1,\ldots,z_n) \longrightarrow (z_j,z_k)$. Therefore, with no loss of generality, we may suppose $n=2$.

Let $z=(z_1,z_2)\in\mathbb{C}^2$ be such that $\mathrm{Re}\, z_j>\omega_0(S_j)$ $(j=1,2)$. From the exactness of the complex $\{X_B^p, \delta^p(B(z))\}_{p\geq 0}$ it follows that

$$(z_1-B_1)^{-1}(z_2-B_2)^{-1}x = (z_2-B_2)^{-1}(z_1-B_1)^{-1}x \tag{3.3}$$

for every $x\in X$ .Indeed, the form $\eta=(z_2-B_2)^{-1}x\,\sigma_1 + (z_1-B_1)^{-1}x\,\sigma_2$ satisfies $\delta^1(B(z))\eta=0$ .Hence there exists $v\in X_B^0$ such that $(z_1-B_1)v=(z_2-B_2)^{-1}x$ and $(z_2-B_2)v = (z_1-B_1)^{-1}x$,which implies (3.3)

From (3.1) and (3.3) we derive the equality

$$\int_0^\infty\int_0^\infty e^{-(t_1z_1+t_2z_2)}(S_1(t_1)S_2(t_2)-S_2(t_2)S_1(t_1))x\,dt_1dt_2=0 \tag{3.4}$$

for all $x\in X$ and $(z_1,z_2)\in\mathbb{C}^2$ with $\mathrm{Re}\, z_j\geqslant\omega_0(S_j)$ $(j=1,2)$. Relation (3.4) implies the statement (1) via the following.

3.2. LEMMA. Let $f\in L_1((0,\infty)\times(0,\infty))$ be such that

$$\int_0^\infty\int_0^\infty e^{-(\lambda_1t_1+\lambda_2t_2)}f(t_1,t_2)\,dt_1dt_2=0$$

for $\mathrm{Re}\,\lambda_1$, $\mathrm{Re}\,\lambda_2$ sufficiently large.Then $f(t_1,t_2)=0$ almost everywhere.

Lemma 3.2 is (a version and) a consequence of Lemma VIII.1.15 from [3], by Fubini's theorem .

The proof of Theorem 3.1 is now complete.

3.3. REMARK. Condition (2) from Theorem 3.1 is obviously equivalent to the condition

(2') $B=(B_1,\dots,B_n)$ is a p.m. and

$$\sigma_{\mathbb{C}}(B)\cap\{z=(z_1,\dots,z_n)\in\mathbb{C}^n\ ;\ \mathrm{Re}\, z_j>\omega_0(S_j)\ ,\ j=1,\dots,n\}=\emptyset\ .$$

Theorem 3.1 yields the following extension of Theorem 2.5 from [12].

3.4. COROLLARY. Let $A_1,\dots,A_n$ be (unbounded) selfadjoint operators in a certain Hilbert space. The following conditions are equivalent:

(a) The spectral measures attached to $A_1,\dots,A_n$ mutually commute.

(b) $A=(A_1,\dots,A_n)$ is a p.m. and $\sigma_{\mathbb{C}}(A)\subset\mathbb{R}^n$.

Proof. Indeed, condition (b) is equivalent (by Theorem 3.1 and Remark 3.3) to the commutation of the groups of (unitary) operators $\{\exp(itA_1)\}_{t\geq 0},\dots,\{\exp(itA_n)\}_{t\geq 0}$, which in turn is well-known to be equivalent to the commutation of the spectral measures attached to $A_1,\dots,A_n$.

3.5. REMARK. If $A=(A_1,A_2)$ is a permutable pair of selfadjoint operators, then the following dichotomy holds: Either $\sigma_{\mathbb{C}}(A)=\mathbb{C}^2$ or $\sigma_{\mathbb{C}}(A)\subset\mathbb{R}^2$ (see [12]). The direct extension of this fact is no longer true for $n\geq 3$. A similar result for a permutable family of selfadjoint operators $A=(A_1,\dots,A_n)$ $(n\geq 3)$ involves not only the joint spectrum but some relative joint spectra as well. We omit the details.

4. ELEMENTS OF FREDHOLM THEORY

In this section we intend to define the essential joint spectrum of a p.m. and to give some of its properties. It is beyond our scope to present this subject in a more comprehensive way. A more detailed list of the properties of the essential joint spectrum will be published elsewhere.

Let $T=(T_1,\dots,T_n)\in\mathcal{D}(X)^n$ be a p.m. and let $\sigma=(\sigma_1,\dots,\sigma_n)$ be a system of indeterminates associated with T . For every integer $p\geq 0$ we define the mapping $\delta^p(T): X_T^p\longrightarrow X_T^{p+1}$ by the equality $\delta^p(T)=\delta^p(-T(0))$ (i.e. $T^{k,p}(z)$ is replaced in (2.7) by $-T^{k,p}(0)$). As noticed in the proof of Theorem 2.10, we can always fix a norm in X_T^p which makes the mapping $\delta^p(T)$ continuous.

4.1. DEFINITION. A p.m. $T=(T_1,\dots,T_n)\in\mathcal{D}(X)^n$ is said to be (<u>semi-</u>) <u>Fredholm</u> if the complex $\{X_T^p,\delta^p(T)\}_{p\geq 0}$ is (semi-) Fredholm (in the sense of [1]). In this case we may define the <u>index</u> ind (T) of T to be equal to the index (i.e. the extended Euler characteristic see [1]) of the complex $\{X_T^p, \delta^p(T)\}_{p\geq 0}$.

The <u>essential joint spectrum</u> $\sigma_{ess}(T)$ of T consists, by definition, of those points $z\in\hat{\mathbb{C}}^n$ such that the complex $\{X_T^p, \delta^p(T(z))\}_{p\geq 0}$ is not Fredholm. Obviously $\sigma_{ess}(T)\subset\sigma(T)$.

4.2. REMARK. If $T=(T_1)$ is singleton, then T is (semi-) Fredholm iff $T_1 : D(T_1)\subset X \longrightarrow X$ is (semi-) Fredholm, that is, im (T_1) is closed in X and either ker (T_1) or $X/$im (T_1) is finite dimensional. Note that in this case T_1 is necessarily a closed operator in X. Indeed, let $\{x_k\}_k\subset D(T_1)$ be a sequence such that $x_k\longrightarrow x$ and $T_1x_k\longrightarrow y$ $(k\rightarrow\infty)$ in X. Since im (T_1) is closed, we may find a sequence $\{x'_k\}_k\subset D(T_1)$ such that $x'_k\longrightarrow x'$ $(k\rightarrow\infty)$ in the topology of $D(T_1)$ and $x'_k-x_k\in$ ker (T_1). Therefore $T_1x'=T_1x$ and so $x'-x\in$ ker (T_1), showing that $x\in D(T_1)$ and $T_1x=y$.

4.3. THEOREM. Let $T=(T_1,\dots,T_n)\in\mathcal{D}(X)^n$ be a p.m. Then the set $\sigma_{ess}(T)$ is closed in $\hat{\mathbb{C}}^n$.

Proof. We can give the same argument as in the first part of the proof of Theorem 2.10, using the stability of Fredholm complexes under small perturbations (see [10] or [1]).

4.4. THEOREM. Let $T=(T_1,\dots,T_n)\in\mathcal{D}(X)^n$ be a (semi-) Fredholm p.m. Assume that X_T^p is given a Banach space norm which makes the mapping $\delta^p(T) : X_T^p \longrightarrow X_T^{p+1}$ continuous. Then there exists an $\varepsilon_T>0$ such that if $\tilde{T}=(\tilde{T}_1,\dots,\tilde{T}_n)\in\mathcal{D}(X)^n$ is a p.m. with the properties $D(\tilde{T}_j)=D(T_j)$ for all j, $\delta^p(\tilde{T}) : X_T^p\longrightarrow X_T^{p+1}$ is continuous and $\|\delta^p(\tilde{T})-\delta^p(T)\|<\varepsilon_T$ for all $p\geq 0$, then $\tilde{T}$ is (semi-) Fredholm,

$$\dim\ker\delta^p(\tilde{T})/\mathrm{im}\,\delta^{p-1}(\tilde{T})\leq\dim\ker\delta^p(T)/\mathrm{im}\,\delta^{p-1}(T)$$

for all $p\geq 0$ and ind $(\tilde{T})=$ ind (T).

Proof. The assertion is a direct consequence of Theorem 1.4 from [1].

4.5. EXAMPLE. Let $\Omega\subset\mathbb{C}^2$ be a strongly pseudoconvex domain and let $L_2(\Omega)$ be the space of all square integrable functions on Ω. We denote by $L_2^p(\Omega)$ the space of (0,p)-forms on Ω, whose coefficients are functions from $L_2(\Omega)$. With our notation, $L_2^p(\Omega)$ equals the space $\Lambda^p[d\bar{z},L_2(\Omega)]$, where $d\bar{z}=(d\bar{z}_1,d\bar{z}_2)$ is regarded as a system of indeterminates.

Let T_j be the closed operator that is induced in $L_2(\Omega)$ by the differential operator $\partial/\partial\bar{z}_j$ $(j=1,2)$. It is easily seen that $T=(T_1,T_2)$ is a permutable pair in $L_2(\Omega)$ and that $D(T_1T_2)=D(T_2T_1)$. Moreover, the operator $\delta^p(T)$ is induced by $\bar{\partial}=(\partial/\partial\bar{z}_1)d\bar{z}_1+(\partial/\partial\bar{z}_2)d\bar{z}_2$. We want to prove that $T=(T_1,T_2)$ is semi-Fredholm.

If $N: L_2^2(\Omega)\longrightarrow L_2^2(\Omega)$ is the Neumann operator, it is known that $\eta=\bar{\partial}\vartheta N\eta$ for each $\eta\in L_2^2(\Omega)$, where ϑ is the formal adjoint of $\bar{\partial}$, extended in the sense of the theory of distributions (see [4] for details). Let $\xi=\vartheta N\eta\in L_2^1(\Omega)$. If $\|*\|_p$ designates the norm of the Hilbert space $L_2^p(\Omega)$, then we have the estimate

$$\sum_{j,k}^{2}\left\|\frac{\partial f_j}{\partial\bar{z}_k}\right\|_0^2\leqslant\frac{1}{4}(\|\vartheta\xi\|_0^2+\|\bar{\partial}\xi\|_2^2)=\frac{1}{4}\|\eta\|^2 , \tag{4.1}$$

since $\vartheta\xi=0$, where $\xi=f_1\,d\bar{z}_1+f_2\,d\bar{z}_2$. In fact, (4.1) is easily obtained for forms that are smooth in a neighbourhood of $\overline{\Omega}$ (see [4], Section II.1). The general assertion then follows by a standard procedure, using star-shaped neighbourhoods of the boundary points and a partition of unity. We omit the details.

It follows from (4.1) that $f_j\in D(T_1)\cap D(T_2)$ $(j=1,2)$, and so the mapping $\delta^1(T)$ is surjective.

Next, let $\zeta=g_1\,d\bar{z}_1+g_2\,d\bar{z}_2\in(L_2(\Omega))_T^1$ be such that $\delta^1(T)\zeta=\bar{\partial}\zeta=0$.

From the general theory it also results that there exists $h \in L_2(\Omega)$ such that $\bar{\partial} h = \zeta$; hence $g_1 = \partial h/\partial \bar{z}_1$ and $g_2 = \partial h/\partial \bar{z}_2$, showing that $h \in D(T_1 T_2) = (L_2(\Omega))^0_T$.

In this way the complex $\{(L_2(\Omega)^p_T, \delta^p(T)\}_{p \geq 0}$ is semi-Fredholm, that is, T is semi-Fredholm.

It would be interesting to prove a similar property for a strongly pseudoconvex domain of dimension ≥ 3 .

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