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SINGULARITY OF RADIAL SUBALGEBRAS IN II_1 FACTORS ASSOCIATED WITH FREE PRODUCTS OF GROUPS

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A possible way to analyse the structure of type II_1 factors (from the point of view of their ergodic properties) is provided by the study of their maximal abelian subalgebras (briefly M.A.S.A.'s). This approach was initiated in the later 50's by J.Dixmier [7], L.Pukanszky [10] and M.Takesaki [16], who introduced a number of invariants related to such subalgebras.
An important result in this field is the Connes-Feldman-Weiss theorem [6], which asserts that regulars M.A.S.A.'s in the hyperfinite factors are also conjugate (see also [12] for an operator algebraic approach of the proof).

Recall that, following Dixmier's classification, a M.A.S.A. A in a von Neumann algebra M is regular (or Cartan) if the von Neumann subalgebra generated in M by the normaliser $N_M(A)=\{u\in M : u \text{ unitary in } M, uAu^*=A\}$ is M itself and A is singular if $N_M(A)''=A$. While examples of regular M.A.S.A.'s are rather easy to be obtained by the classical group - measure space construction of Murray and von Neumann, detecting singular M.A.S.A.'s is a more difficult task [7, 10, 12, 13].

The Pukanszky invariant [10] (also considered in an unpublished work of Ambrose and Singer) for a M.A.S.A. A in a type II_1 factor M with canonical trace τ gives a finer classification. The description of this invariant is briefly as follows: let M acting in standard way, by multiplication to the left on $L^2(M,\tau)$ (the Hilbert space completion of M with respect to the norm $\|x\|_{2,\tau}=\tau(x^*x)^{1/2}$, $x\in M$), let $J:L^2(M,\tau)$ $L^2(M,\tau)$, $Jx=x^*$ the canonical conjugation ($JMJ=M'$), $\mathcal{A}=(A\vee JAJ)''$ the abelian von Neumann algebra generated by A and JAJ in $\mathcal{B}(L^2(M,\tau))$ and for any ξ in $L^2(M,\tau)$, denote by p_ξ the corresponding cyclic projection onto $\overline{\mathrm{Span}\,\mathcal{A}\xi}^{\|\ \|_2}=\overline{\mathrm{Span}\,A\xi A}^{\|\ \|_2}$. Denote also by $\mathcal{A}'$ the commutant of $\mathcal{A}$ in $\mathcal{B}(L^2(M,\tau))$. Then $\mathcal{A}p_1\subseteq\mathcal{B}(p_1L^2(M,\tau))$ is maximal abelian (the unit 1 of M is regarded as an element in $L^2(M,\tau)$) and the Pukanszky invariant is the von Neumann algebra type of the discrete von Neumann algebra $\mathcal{A}'(1-p_1)$. Pukanszky showed that in the hyperfinite factor $\mathcal{R}$ there are M.A.S.A.'s A_n such that the corresponding $\mathcal{A}_n$ have the property that $\mathcal{A}_n'(1-p_1)$ are of homogeneous type I_n.

By the work of S. Popa [12], if A is a Cartan M.A.S.A. then $\mathcal{A}$ is maximal abelian and if $\mathcal{A}'(1-p_1)$ is of the homogeneous type I_n, $n\geq 2$, then A is singular.

We will be concerned with the von Neumann algebra $M=\mathcal{L}(G)$ of a group G which is the free product $G=G_1*\ldots*G_N$ of groups G_i, all of finite order k (but not necessarily isomorphic) or all isomorphic to $\mathbb{Z}$ and with their radial subalgebras A generated by $\chi_1\in\mathcal{L}(G)$, where χ_1 is the left multiplication with the characteristic function of the words of length one in G. These algebras where considered by Figà-Talamanca, Picardello, Cohen, Pytlik in connection with their work on harmonic analysis and representation theory on this groups. In particular it was proved by Pytlik [11] that the radial algebra is a M.A.S.A. in $\mathcal{L}(G)$ for $G=\mathbb{F}_N$ (moreover, it is singular [14] in this case) and also for $G=G_1*\ldots*G_N$ with $N\geq k$ [17].

The aim of the present paper is to give a precise description of $\mathcal{A}, \mathcal{A}'$ and of the inclusion $A, JAJ \subseteq \mathcal{A}$, when A is the radial algebra as before. We obtain that $\mathcal{A}'(1-p_1)$ is of the homogeneous type I_∞ if $N \geq 3$ and maximal abelian if G is the amenable group $G = \mathbb{Z}_2 * \mathbb{Z}_2$. Consequently, A is singular in the first case and Cartan in the second case. Moreover, if G is before with $N \geq 3$ or if $G = \mathbb{F}_N$, then $\mathcal{A}$ is isomorphic to $A \oplus (A \otimes A)$ (corresponding to the decomposition $1 = p_1 + 1 - p_1$) with A, JAJ sitting inside as direct tensor factors in the second summand and collapsing into the first. If $G = \mathbb{Z}_2 * \mathbb{Z}_2$ (so $\mathcal{L}(G)$ is the hyperfinite factor [5]), then $\mathcal{A}$ is isomorphic to $A \oplus A$, with A, JAJ collapsing (modulo a certain automorphism) onto both factors.

The idea is as in [14] to show an orthonormal infinite family $\{\xi_n\}_{n \geq 0}$ in $L^2(M, \tau)$, $\xi_0 = 1 \in L^2(M, \tau)$ such that $\overline{A \xi_n A}^{\|\ \|_2}$ are orthogonal and the corresponding cyclic projections p_{ξ_n} have all the same central support $1-p_1$ in $\mathcal{A}'$, for $n \geq 1$. In order to do this, we prove that the intertwining operators $a \xi_n \longrightarrow a \xi_m$, $a \in \mathcal{A}$ are invertible.

The last part of the paper contains a precise description of the spatial action of χ_1 and $J\chi_1 J$, which is shown to be related to the unilateral shift S on $l^2(\mathbb{N})$ in all these cases. In particular, the spectral measure of χ_1 and $J\chi_1 J$ on each ξ_n is computed.

1. THE PUKANSZKY INVARIANT FOR RADIAL SUBALGEBRAS

Let $G_1, G_2, \ldots, G_N$ be finite groups with the same order $k \geq 2$ and let $G = G_1 * G_2 * \ldots * G_N$ their free product. Denote $G_i^* = G_i - \{1_{G_i}\}$. Each element g in $G, g \neq 1_G$ may be written uniquely in the reduced form as $g = g_1 g_2 \ldots g_m$, where $g_j \in G_{i_j}^*$, $i_1 \neq \ldots \neq i_m$. Define the length of such a word g to be m and denote $|g| = m$, $|1_G| = 0$, $o(g) = g_1$, $t(g) = g_m$. This length function corresponds to the action of G on his associated tree. Denote $E_m = \{w \in G : |w| = m\}$ the set of words of length m, with cardinality $N(N-1)^{m-1}(k-1)^m$ and $E_o = \{1_G\}$. Denote also $\beta = \sqrt{(N-1)(k-1)}$. The radial function χ_m on G is the characteristic function of the set E_m.

Let $M = \mathcal{L}(G)$ be the associated von Neumann algebra of the group G. Clearly G has infinite conjugacy classes hence $\mathcal{L}(G)$ is a type II_1 factor acting standartly on $l^2(G)$, identified with the space $L^2(M, \tau)$ of the GNS representation associated to the trace τ on M and $\| \|_\tau$ coincides with the usual norm $\| \|_2$ on $l^2(G)$. Denote by $\langle , \rangle$ the scalar product on $l^2(G)$.

Denote the group ring $\mathbb{C}[G]$ of G over $\mathbb{C}$ by M_o and identify $M_o = \{x : x = \sum_{w \in G} \lambda_w . w$ finite sum, $\lambda_w \in \mathbb{C}\}$ with a subalgebra of $\mathcal{L}(G)$ which acts by left translation on $l^2(G)$. It is known [4] that

$$\chi_2 = \chi_1^2 - (k-2)\chi_1 - N(k-1)\chi_o \tag{1.1}$$

and

$$\chi_{m+1} = \chi_m \chi_1 - (k-2)\chi_m - \beta^2 \chi_{m-1}, \quad m \geq 2, \tag{1.2}$$

hence the von Neumann subalgebra A of $\mathcal{L}(G)$ generated by the χ_m's is abelian. It has been shown in [17] that A is maximal abelian if and only if $N \geq k$.

Let $\mathcal{A}=(A \vee JAJ)''$ be the abelian von Neumann subalgebra of $\mathcal{B}(l^2(G))$ generated by A and JAJ ($J: l^2(G) \longrightarrow l^2(G)$, $J(v)=v^{-1}$, for $v \in G$, is the canonical conjugation). For each vector ξ in $l^2(G)$, denote by $p_\xi \in \mathcal{A}'$ the cyclic projection of $l^2(G)$ onto $\overline{\mathcal{A}\xi}^{\|\ \|_2} = \overline{\text{Span } A\xi A}^{\|\ \|_2}$ and by $z(p_\xi)$ the central support of p_ξ in $\mathcal{A}'$. Our aim is to show that $\mathcal{A}'$ is of the homogeneous type I_∞ on $I-p_1$ (where $1=\chi_{E_o} \in M$) and to give an explicitely description of the operators χ_1 and $J\chi_1 J$ on $l^2(G)$. In order to do this, we construct as in [14] a family of vectors $\{\xi_n\}_{n \in \mathbb{N}}$ in $l^2(G)$ such that the corresponding cyclic projections p_{ξ_n} are orthogonal, with the same central support $I-p_1$ in $\mathcal{A}'$ and $\sum_{n \in N} p_{\xi_n} = I-p_1$.

The linear span of words with length l is denoted by M_o^l and the projections from $l^2(G)$ onto M_o^l and respectively onto $\overline{\text{Span}\{AwA: w \in E_l\}}^{\|\ \|_2}$ by q_l and by p_l. An important step in our proof is to check that $p_{l-1}q_l = q_l p_{l-1}$ and the range of $p_{l-1} \wedge q_l$ is precisely $\mathcal{S}_l = \text{Span}\{q_l(\chi_1 w), q_l(w\chi_1): |w| \leq l-1\}$. For any vector ξ in M_o^l, $l \geq 1$, we denote

$$\xi_{r,s} = \begin{cases} q_{l+r+s}(\chi_r \xi \chi_s) & , \text{ for } r,s \geq 0 \\ 0 & , \text{ for } r<0 \text{ or } s<0. \end{cases}$$

Let us consider in $\mathcal{B}(M_o^l)$ the self-adjoint operators $q_l \chi_1 q_l$ and $Jq_l\chi_1 q_l J = q_l J\chi_1 J q_l$. For each $w \in E_{l-1}$, $l \geq 2$ we get

$$q_l \chi_1 q_l(\chi_1 w) = (k-2) q_l(\chi_1 w)$$

and

$$q_l \chi_1 q_l (w\chi_1) = \sum_{\substack{|a|=1 \\ |aw|=l-1}} q_l (aw\chi_1) \quad ,$$

so $q_l \chi_1 q_l (\mathcal{S}_l) \subset \mathcal{S}_l$ for $l \geq 2$. Clearly this inclusion is still true also for $l=1$ and it follows that for any $l \geq 1$, $B_l = q_l \chi_1 q_l \big|_{M_o^l \ominus \mathcal{S}_l}$ and $C_l = JB_lJ = q_l J\chi_1 J q_l \big|_{M_o^l \ominus \mathcal{S}_l}$ are self-adjoint operators in $\mathcal{B}(M_o^l \ominus \mathcal{S}_l)$ defined by

$$B_l \Big(\sum_{|v|=l} \lambda_v \cdot v \Big) = \sum_{|v|=l} \Big(\sum_{\substack{|a|=1 \\ |av|=l}} \lambda_{av} \Big) v$$

$$C_l \Big(\sum_{|v|=l} \lambda_v \cdot v \Big) = \sum_{|v|=l} \Big(\sum_{\substack{|a|=1 \\ |va|=l}} \lambda_{va} \Big) v.$$

Remark also that $B_1 = C_1$, $B_l C_l = C_l B_l$ for $l \geq 2$ and $B_l = C_l = 0$ whenever $k=2$.

The first Lemma describes the spectral properties of B_l and C_l and gives nice formulas relating the actions of χ_1 and $J\chi_1 J$ on $\mathrm{Span}\{\xi_{m,n}\}_{m,n \geq 0}$.

LEMMA 1.1. i) For any $l \geq 1$ one has

$$B_l^2 = (k-3) B_l + (k-2) I;$$
$$C_l^2 = (k-3) C_l + (k-2) I$$

(In this Lemma, I denotes the identity operator on $M_o^l \ominus \mathcal{S}_l$).

ii) For any $l \geq 2, \xi \in M_o^l \ominus \mathcal{S}_l$, $n \geq 0$ one has

$$\chi_1 \xi_{o,n} = \xi_{1,n} + (B_l \xi)_{o,n};$$

$$\xi_{n,o}\chi_1=\xi_{n,1} + (C_1\xi)_{n,o}.$$

iii) For any $\xi\in M_o^1\ominus\mathcal{S}_1$, $n\geq 0$ one has

$$\chi_1\xi_{o,n}=\xi_{1,n} + (B_1\xi)_{o,n}-(JB_1\xi+J\xi)_{o,n-1};$$

$$\xi_{n,o}\chi_1=\xi_{n,1}+(C_1\xi)_{n,o}-(JC_1\xi+J\xi)_{n-1,o}.$$

Proof. i) Let $\xi=\sum_{|v|=1}\lambda_v\cdot v\in M_o^1\ominus\mathcal{S}_1$. Then

$$B_1^2\xi=\sum_{|v|=1}\Big(\sum_{\substack{|a|=1\\|av|=1}}\ \sum_{\substack{|b|=1\\|bav|=1}}\lambda_{bav}\Big)v$$

Letting $ba=c$, only two cases occur: $b=a^{-1}$ with $|a|=1$, $|av|=1$ or $|c|=1$, with $|cv|=1$, $|a|=1$, $|a^{-1}c|=1$, $|av|=1$. There are $k-2$ choices for a in the first case and $k-3$ in the second (since $c\neq o(v)^{-1}$). Thus

$$B_1^2\xi=(k-2)\sum_{|v|=1}\lambda_v\cdot v+(k-3)\sum_{|v|=1}\Big(\sum_{\substack{|c|=1\\|cv|=1}}\lambda_{cv}\Big)v=(k-2)\xi+(k-3)B_1\xi.$$

This finishes the proof of part i), since $C_1=JB_1J$.

ii) Note first that $\chi_1\xi_{o,n}=\xi_{1,n}+q_{1+n}(\chi_1\xi_{o,n})+q_{1+n-1}(\chi_1\xi_{o,n})$. For $1\geq 1$ we get

$$q_{1+n}(\chi_1\xi_{o,n})=\sum_{\substack{|v|=1\\|w|=n\\|vw|=1+n}}\ \sum_{\substack{|a|=1\\|avw|=1+n}}\lambda_v\cdot avw=\sum_{\substack{|v'|=1\\|w|=n\\|v'w|=1+n}}\Big(\sum_{\substack{|a|=1\\|av'|=1}}\lambda_{av'}\Big)v'w=$$

$$=(B_1\xi)_{o,n}.$$

Also, for $1\geq 2$ we get

$$q_{l+n-1}(\chi_1 \xi_{o,n}) = \sum_{\substack{|v|=l \\ |w|=n \\ |vw|=l+n}} \sum_{\substack{|a|=1 \\ |avw|=l+n-1}} \lambda_v avw = \sum_{\substack{|v|=l \\ |w|=n \\ |vw|=l+n}} \lambda_v o(v)^{-1} vw =$$

$$= \sum_{\substack{|v'|=l-1 \\ |b|=1 \\ |bv'|=l}} \sum_{\substack{|w|=n \\ |v'w|=l+n-1}} \lambda_{bv'} v'w = \sum_{\substack{|w|=n \\ |v'|=l-1 \\ |v'w|=l+n-1}} \Big(\sum_{\substack{|b|=1 \\ |bv'|=l}} \lambda_{bv'} \Big) v'w.$$

Since ξ is orthogonal on $\mathcal{S}_l$, one gets for any $v' \in E_{l-1}$, $\sum_{\substack{|b|=1 \\ |bv'|=l}} \lambda_{bv'} = \langle \xi, \chi_1 v' \rangle = 0$, so we obtain $q_{l+n-1}(\chi_1 \xi_{o,n}) = 0$.

iii) For $l=1$ the last calculation is different. More precisely

$$q_n(\chi_1 \xi_{o,n}) = \sum_{\substack{|v|=1 \\ |w|=n \\ |vw|=n+1}} \sum_{\substack{|a|=1 \\ |avw|=n}} \lambda_v avw = \sum_{\substack{|v|=1 \\ |w|=n \\ |vw|=n+1}} \lambda_v w = \sum_{|w|=n} \Big(\sum_{\substack{|v|=1 \\ |vw|=n+1}} \lambda_v \Big) w.$$

Since ξ is orthogonal on $\mathcal{S}_1$, $\sum_{|v|=1} \lambda_v = 0$ and we obtain

$$q_n(\chi_1 \xi_{o,n}) = -\sum_{|w|=n} \Big(\sum_{\substack{|v|=1 \\ |vw|=n}} \lambda_v \Big) w - \sum_{|w|=n} \lambda_{o(w)^{-1}} \cdot w . \tag{1.3}$$

The statement follows noticing that

$$JB_1 \xi = \sum_{|b|=1} \Big(\sum_{\substack{|a|=1 \\ |ab^{-1}|=1}} \lambda_{ab^{-1}} \Big) b = \sum_{|b|=1} \Big(\sum_{\substack{|v|=1 \\ |vb|=1}} \lambda_v \Big) b$$

and

$$J\xi = \sum_{|b|=1} \lambda_{b^{-1}} b \quad ,$$

hence the first term in the right side of (1.3) is equal to $-(JB_1\xi)_{o,n-1}$ and the second to $-(J\xi)_{o,n-1}$. ▨

By the previous Lemma, each B_l has two eigenvalues, namely -1 and $k-2$ and similarly C_l. Denote by P_{l1} and respectively by Q_{l1} the projections of $M_o^l \ominus \mathcal{S}_l$ onto $\{\xi \in M_o^l \ominus \mathcal{S}_l : B_l\xi = -\xi\}$ and respectively onto $\{\xi \in M_o^l \ominus \mathcal{S}_l : C_l\xi = -\xi\}$. Then $P_{l2} = I - P_{l1}$ and $Q_{l2} = I - Q_{l1}$ (I is the identity operator on $M_o^l \ominus \mathcal{S}_l$) are the projections of $M_o^l \ominus \mathcal{S}_l$ onto $\{\xi \in M_o^l \ominus \mathcal{S}_l : B_l\xi = (k-2)\xi\}$ and respectively onto $\{\xi \in M_o^l \ominus \mathcal{S}_l : C_l\xi = (k-2)\xi\}$. Since $B_lC_l = C_lB_l$, the projections P_{li} and Q_{lj} commute, $i,j = 1,2$. Denote by $\mathcal{E}_{ij}^l$ the range of $P_{li}Q_{lj}$, $i,j = 1,2$. Then, for any $l \geq 2$

$$M_o^l \ominus \mathcal{S}_l = \bigoplus_{i,j=1,2} \mathcal{E}_{ij}^l .$$

For $l=1$, $M_o^1 \ominus \mathcal{S}_1 = \mathcal{E}_1^1 \oplus \mathcal{E}_2^1$, where $\mathcal{E}_1^1 = \{\xi \in M_o^1 \ominus \mathcal{S}_1 : B_1\xi = -\xi\}$ and $\mathcal{E}_2^1 = \{\xi \in M_o^1 \ominus \mathcal{S}_1 : B_1\xi = (k-2)\xi\}$.

The six standard recurrence formulas listed below constitute the basic tools in the proof of the main results of the paper.

LEMMA 1.2. i) For any vector $\xi \in M_o^l$, $l \geq 1$

$$\chi_1\xi_{m,n} = \xi_{m+1,n} + (k-2)\xi_{m,n} + \beta^2\xi_{m-1,n} \quad , \quad m \geq 1,\ n \geq 0;$$
$$\xi_{m,n}\chi_1 = \xi_{m,n+1} + (k-2)\xi_{m,n} + \beta^2\xi_{m,n-1} \quad , \quad m \geq 0,\ n \geq 1.$$

ii) For any $\xi \in \mathcal{E}_{1i}^l$, $l \geq 2$, $i=1,2$ or $\xi \in \mathcal{E}_1^1$ and $m,n \geq 0$

$$\chi_m\xi_{o,n} = \xi_{m,n} - \xi_{m-1,n}.$$

iii) For any $\xi \in \mathcal{E}_{i1}^l$, $l \geq 2$, $i=1,2$ or $\xi \in \mathcal{E}_1^1$ and $m,n \geq 0$

$$\xi_{m,o}\chi_n = \xi_{m,n} - \xi_{m,n-1} .$$

iv) For any $\xi\in\mathcal{C}^{l}_{2i}$, $l\geq 2$, $i=1,2$ and $m,n\geq 0$.

$$\chi_m\xi_{o,n}=\xi_{m,n}+(k-2)\xi_{m-1,n}-(k-1)\xi_{m-2,n}.$$

v) For any $\xi\in\mathcal{C}^{l}_{i2}$, $l\geq 2$, $i=1,2$ and $m,n\geq 0$

$$\xi_{m,o}\chi_n=\xi_{m,n}+(k-2)\xi_{m,n-1}-(k-1)\xi_{m,n-2}\,.$$

vi) For any $\xi\in\mathcal{C}^{2}_{2}$ and $n\geq 0$

$$\chi_1\xi_{o,n}=\xi_{1,n}+(k-2)\xi_{o,n}-(k-1)\xi_{o,n-1} \quad ;$$

$$\xi_{n,o}\chi_1=\xi_{n,1}+(k-2)\xi_{n,o}-(k-1)\xi_{n-1,o} \quad ;$$

$$\chi_n\xi=\xi_{n,o}+(k-2)\xi_{n-1,o}-(k-1)\xi_{n-2,o} \quad ;$$

$$\xi\chi_n=\xi_{o,n}+(k-2)\xi_{o,n-1}-(k-1)\xi_{o,n-2}.$$

Proof. The proof of i) is routine, these relations being the analogue of (1.2). By Lemma 1.1, taking into account that $B_1\xi=-\xi$ for $\xi\in\mathcal{C}^{l}_{1i}$, $l\geq 2$, $i=1,2$ and $C_1\xi=(k-2)\xi$ for $\xi\in\mathcal{C}^{l}_{2i}$, $l\geq 2$ we obtain

$$\chi_1\xi_{o,n}=\xi_{1,n}-\xi_{o,n} \qquad \text{for } \xi\in\mathcal{C}^{l}_{1i},\ l\geq 2,\ i=1,2 \tag{1.4}$$

$$\chi_1\xi_{o,n}=\xi_{1,n}+(k-2)\xi_{o,n} \qquad \text{for } \xi\in\mathcal{C}^{l}_{2i},\ l\geq 2,\ i=1,2 \tag{1.5}$$

and the analoguous equality for the action of $J\chi_1J$.

If $\xi\in\mathcal{C}^{1}_{1}$, then $B_1\xi+\xi=0$, hence according to part iii) in Lemma 1.1 we obtain

$$\chi_1\xi_{o,n}=\xi_{1,n}-\xi_{o,n} \qquad \text{for } \xi\in\mathcal{C}^{1}_{1},\ n\geq 0\,. \tag{1.6}$$

Take now $\xi=\sum_{|v|=1}\lambda_v v\in\mathcal{C}_2^1$. Then $B_1\xi+\xi=(k-1)\xi$ and by the definition of B_1 one obtains for each $v\in E_1$

$$(k-1)\lambda_v=\sum_{\substack{|a|=1\\|av|\leq 1}}\lambda_{av}.$$

In particular $\lambda_v=\lambda_{v^{-1}}$ for any $v\in E_1$, hence $J\xi=\xi$ and by Lemma 1.1

$$\chi_1\xi_{o,n}=\xi_{1,n}+(k-2)\xi_{o,n}-(k-1)\xi_{o,n-1}\qquad\text{for } \xi\in\mathcal{C}_2^1, n\geq 0. \quad (1.7)$$

Finally, the statement follows by induction, combining (1.4)-(1.7) with the recurrence relations of χ_n' s (1.1) and (1.2). ▨

COROLLARY 1.3. Let ξ be a vector in $M_o^l\ominus\mathcal{G}_1$, $l\geq 1$. Then $\mathrm{Span}\{\chi_m\xi\chi_n\}_{m,n\geq 0}=\mathrm{Span}\{\xi_{m,n}\}_{m,n\geq 0}$. Moreover, for $\xi\in\mathcal{C}_{1\ 1}^l$, $l\geq 2$ or $\xi\in\mathcal{C}_1^1$ and $m,n\geq 0$ one has

$$\chi_m\xi\chi_n=\xi_{m,n}-(\xi_{m,n-1}+\xi_{m-1,n})+\xi_{m-1,n-1}\qquad;$$

$$\xi_{m,n}=\sum_{r=0}^{m}\sum_{s=0}^{n}\chi_r\xi\chi_s\quad.$$

LEMMA 1.4. Let ξ, ξ' be vectors in M_o^l, $l\geq 1$.

i) For $\xi\in\mathcal{C}_{ij}^l$, $l\geq 2$, $i,j=1,2$ or $\xi\in\mathcal{C}_1^1$ and $m,n,m',n'\geq 0$ one has

$$\langle\xi_{m,n},\xi'_{m',n'}\rangle=\delta_{m,m'}\,\delta_{n,n'}\,\beta^{2(m+n)}\langle\xi,\xi'\rangle.$$

ii) For $\xi\in\mathcal{C}_2^1$, $m,n,m',n'\geq 0$ one has

$$\langle\xi_{m,n},\xi'_{m',n'}\rangle=\delta_{m+n,m'+n'}\,\beta^{2(m+n)}\left(-\frac{1}{N-1}\right)^{|m-m'|}\langle\xi,\xi'\rangle.$$

(δ_{ij} denotes the Kronecker symbol).

Proof. Since $\langle \xi_{m,n}, \xi_{m',n'}\rangle = \langle q_{m+n+1}(\chi_m \xi \chi_n), q_{m'+n'+1}(\chi_{m'}\xi\chi_{n'})\rangle$ it is enough to check the statement when $m+n=m'+n'$. Assume that $m'-m=n-n'=r\geq 0$. Then

$$\langle \xi_{m,n}, \xi'_{m',n'}\rangle = \langle \xi_{m,n}, \chi_{m'}\xi'\chi_{n'}\rangle = \langle \chi_{m'}\xi_{m,n}\chi_{n'}, \xi'\rangle =$$

$$= \langle q_1(\chi_{m'}\xi_{m,n}\chi_{n'}), \xi'\rangle . \qquad (1.8)$$

Note that $\chi_{m'}\xi_{m,n}\chi_{n'-1}, \chi_{m'-1}\xi_{m,n}\chi_{n'-2} \in \bigcup_{j\geq 1+1} M_o^j$, in particular $q_1(\chi_{m'}\xi_{m,n}\chi_{n'-1}) = q_1(\chi_{m'-1}\xi_{m,n}\chi_{n'-2}) = 0$. Thus, (1.1) and (1.2) yield

$$q_1(\chi_{m'}\xi_{m,n}\chi_{n'}) = q_1(\chi_{m'}\xi_{m,n}\chi_1\chi_{n'-1}), \text{ for } n'\geq 1 \qquad (1.9)$$

and

$$q_1(\chi_{m'}\xi_{m,n}\chi_{n'}) = q_1(\chi_{m'-1}\chi_1\xi_{m,n}\chi_{n'}), \text{ for } m'\geq 1.$$

Combining Lemma 1.2 and (1.9) and taking into account that $q_1(\chi_{m'}\xi_{m,n+1}\chi_{n'-1}) = q_1(\chi_{m'}\xi_{m,n}\chi_{n'-1}) = 0$ we obtain

$$q_1(\chi_{m'}\xi_{m,n}\chi_{n'}) = \beta^2 q_1(\chi_{m'}\xi_{m,n-1}\chi_{n'-1}), \ n,n'\geq 1.$$

A similar formula is still true on the left side. Iterating the previous formulas we find

$$q_1(\chi_{m'}\xi_{m,n}\chi_{n'}) = \beta^{2(m+n')} q_1(\chi_r\xi_{o,r}) = \beta^{2(m+n')} q_1(\chi_{r-1}\chi_1\xi_{o,r}), r\geq 1,$$

where we have used the fact that $q_1(\chi_{r-1}\xi_{o,r})=q_1(\chi_{r-2}\xi_{o,r})=0$.

When $r\geq 1$ and $\xi\in\mathcal{C}^1_{ij}$, $l\geq 2$, $i,j=1,2$ or $\xi\in\mathcal{C}^1_1$, Lemma 1.2 yields

$$q_1(\chi_{m'}\xi_{m,n}\chi_{n'})=\beta^{2(m+n')}(q_1(\chi_{r-1}\xi_{l,r})+\varepsilon q_1(\chi_{r-1}\xi_{o,r}))=0$$

($\varepsilon=-1$ for $\xi\in\mathcal{C}^1_{1i}$, $l\geq 1$, $i=1,2$ and $\varepsilon=k-2$ for $\xi\in\mathcal{C}^1_{2i}$, $l\geq 2$, $i=1,2$).

When $r\geq 1$ and $\xi\in\mathcal{C}^1_2$, we get by Lemma 1.2

$$q_1(\chi_r\xi_{o,r})=q_1(\chi_{r-1}\chi_1\xi_{o,r})=q_1(\chi_{r-1}\xi_{1,r})+(k-2)q_1(\chi_{r-1}\xi_{o,r})-$$
$$-(k-1)q_1(\chi_{r-1}\xi_{o,r-1})=-(k-1)q_1(\chi_{r-1}\xi_{o,r-1}).$$

Consequently

$$q_1(\chi_{m'}\xi_{m,n}\chi_{n'})=\beta^{2(m+n')}(-(k-1))^r q_1(\xi)=\beta^{2(m+n)}\cdot(-\frac{1}{N-1})^r\cdot\xi$$

and the statement ii) follows by (1.8). ▨

The following elementary lemma is quite probably folklore but for the sake of completeness we sketch the proof.

LEMMA 1.5. Let a be a real number with $|a|<1$. Then, for any integer $l\geq 0$ and any complex numbers $\lambda_0,\ldots,\lambda_l$

$$\frac{1-a^2}{1+2|a|+2a^2}\left(\sum_{i=0}^{l}|\lambda_i|^2\right)\leq\sum_{i,j=0}^{l}\lambda_i\bar{\lambda}_j a^{|i-j|}\leq\frac{1+|a|}{1-|a|}\left(\sum_{i=0}^{l}|\lambda_i|^2\right).$$

In particular, the vectors $\{\xi_{m,n}\}_{m,n\geq 0}$ are linearly independent when $\xi\in\mathcal{C}^2_1$, $\xi\neq 0$ and $N\geq 3$.

Proof. A direct calculation shows that the operator B_l given by the matrix with entries $b_{ij}=a^{|i-j|}$, $i,j=0,\ldots,l$ is invertible and

$$B_l^{-1}=(1-a^2)^{-1}((1+a^2).I-a(N_l+N_l^*)-a^2D_l), \tag{1.10}$$

where N_l is the nilpotent operator with $n_{ij}=\delta_{i,j+1}$ and D_l is the diagonal operator with $d_{ij}\neq 0$ only when $i=j=0$, $i=j=l$ and $d_{oo}=d_{ll}=1$. The first inequality follows readily by (1.10). The second inequality is obtained directly by

$$B_l=I+\sum_{i=1}^{l}a^i(N_l^i+N_l^{*i}) \quad . \qquad ∎$$

LEMMA 1.6. i) For any $l\geq 1$, the projections p_{l-1} and q_l commute and $\mathcal{S}_l$ is the range of $p_{l-1}q_l$.

ii) For any orthogonal vectors $\xi_1,\xi_2\in\mathcal{C}_1^1\cup\mathcal{C}_2^1\cup\bigcup_{\substack{l\geq 2\\ i,j=1,2}}\mathcal{C}_{ij}^l$, the projections p_{ξ_1} and p_{ξ_2} are orthogonal.

iii) When $N=k=2$, $\alpha_l=\dim(M_o^l\ominus\mathcal{S}_l)$ is zero for any $l\geq 2$ and one for $l=1$. In the other cases $\alpha_l\geq 1$ for any $l\geq 1$ and

$$\alpha_l=N(N-1)^{l-1}(k-1)^l-(1+l\alpha_1+(l-1)\alpha_2+\ldots+2\alpha_{l-1}).$$

Proof. i) Since $\mathcal{S}_l$ is included into the range of $p_{l-1}\wedge q_l$, it is enough to check that for any $\xi\in M_o^\alpha$, $\alpha\leq l-1, m,n\geq 0$

$$q_l(\chi_m\xi\chi_n)\in\mathcal{S}_l \tag{1.11}$$

When $\alpha=0$, this follows by $q_l(\chi_l)=q_l(\chi_1\chi_{l-1})$. Arguing by induction, let $1\leq\alpha\leq l-1$ and assume that (1.11) is true for $\alpha-1$. Let ξ be a vector in M_o^α. If $\xi\in\mathcal{S}_\alpha$, then $\chi_m\xi\chi_n$ is in the range of $p_{\alpha-1}$, hence $q_l(\chi_m\xi\chi_n)\in\mathcal{S}_l$ by the previous assumption. If $\xi\in\mathcal{C}_{ij}^\alpha$, $i,j=1,2$, then $\chi_m\xi\chi_n\in\mathrm{Span}\{\xi_{r,s}\}_{r+s\leq m+n}$ by Lemma 1.2, hence it is enough to check that $q_l(\xi_{r,s})=q_lq_{r+s+\alpha}(\xi_{r,s})\in\mathcal{S}_l$. Thus $r+s+\alpha=l$ and according to

Lemma 1.2, $q_l(\xi_{r,s})=q_l(\xi_{r,s-1}\chi_1)$ with $\xi_{r,s-1}\in M_o^{l-1}$ for $s\geq 1$ and $q_l(\xi_{r,s})=q_l(\chi_1\xi_{r-1,s})$ with $\xi_{r-1,s}\in M_o^{l-1}$ for $r\geq 1$.

Part ii) is an immediate consequence of i) and of Lemma 1.4.

In order to prove iii) remark first that for N=k=2, $M_o^l=\mathcal{S}_l$ when $l\geq 2$ (since both have dimension two and since $\dim \mathcal{S}_1=1$, $\alpha_1= =2-1=1$). In this case $M_o^1\ominus\mathcal{S}_1=\mathcal{E}_2^1$.

In the other cases, since $q_l(\chi_1\chi_{l-1})=q_l(\chi_{l-1}\chi_1)$ we get $\dim\mathcal{S}_l\leq 2\ \mathrm{Card}\ E_{l-1}-1$ and

$$\alpha_l=\mathrm{Card}\ E_l-\dim\mathcal{S}_l\geq N(N-1)^{l-2}(k-1)^{l-1}\left((N-1)(k-1)-2\right)+1\geq 1.$$

Moreover, in this case it is possible to compute precisely α_l. By Lemma 1.2 the projections p_ξ and q_l commute whenever $\xi\in\mathcal{E}_{ij}^r$, $i,j=1,2$, $r\leq l-1$ and $\dim(p_\xi q_l)=l-r+1$ since the vectors $\{\xi_{m,n}\}_{m+n=l}$ are linearly independent for $\xi\in\mathcal{E}_{ij}^r$, $i,j=1,2,r\geq 1$ by Lemmas 1.4 and 1.5. Thus we obtain

$$\alpha_l=\mathrm{Card}\ E_l-\dim(p_{l-1}q_l)=N(N-1)^{l-1}(k-1)^l-\Big(1+\sum_{r=1}^{l-1}(l-r+1)\alpha_r\Big).\quad ▨$$

Remark. If $k\geq 3$, then for any $l\geq 2$, $i,j=1,2$, $\mathcal{E}_{ij}^l\neq\{0\}$. Indeed, assume that $x_1,y_1\in G_1^*$, $x_1\neq y_1$, $x_2,y_2\in G_2^*$, $x_2\neq y_2$ and $x\in G_3^*$. Then

$$(x_1-y_1)x^{l-2}(x_2-y_2)\in\mathcal{E}_{11}^l;$$
$$\sum_{a\in G_1^*}a\Big(x_3^{l-2}\sum_{b\in G_2^*}b-x_2^{l-2}\sum_{c\in G_3^*}c\Big)\in\mathcal{E}_{22}^l\ ;$$
$$\sum_{a\in G_1^*}ax_3^{l-2}(x_2-y_2)\in\mathcal{E}_{12}^l.$$

Let us remark also that in this case $\mathcal{E}_1^1$ is spanned by the vectors x_r-x_s, where x_r and x_s are distinct elements of the same set G_n^*, $n=1,\ldots,N$, $\mathcal{E}_2^1$ is spanned by the vectors $\sum_{a\in G_m^*}a-\sum_{b\in G_n^*}b$, where $m\neq n$ and $\dim\mathcal{E}_1^1=N(k-2)$, $\dim\mathcal{E}_2^1=N-1$.

If k=2, then for any $l\geq 1$, $B_l=C_l=0$. Consequently $M_o^l\ominus\mathcal{G}_l=$ $=\mathcal{E}_{22}^l$ for $l\geq 2$ and $M_o^1\ominus\mathcal{G}_1=\mathcal{E}_2^1$.

LEMMA 1.7. If $N>k\geq 3$, then for any vectors $\xi\in\mathcal{E}_{11}^l$, $l\geq 1$ and $\xi'\in\mathcal{E}_{22}^{l'}$, $l'\geq 2$, with $\|\xi\|_2=\|\xi'\|_2=1$, the operator

$$T_o:\mathrm{Span}\{\chi_m\xi\chi_n\}_{m,n\geq 0}\longrightarrow \mathrm{Span}\{\chi_m\xi'\chi_n\}_{m,n\geq 0}$$

defined by

$$T_o(\xi_{m,n})=\xi'_{m,n}+(k-1)(\xi'_{m,n-1}+\xi'_{m-1,n})+(k-1)^2\xi'_{m-1,n-1},\ m,n\geq 0,$$

extends to a bounded invertible operator $T:\overline{\mathcal{A}\xi}^{\|\ \|_2}\longrightarrow\overline{\mathcal{A}\xi'}^{\|\ \|_2}$ such that for any $m,n\geq 0$, $T(\chi_m\xi\chi_n)=\chi_m\xi'\chi_n$. In particular the cyclic projections p_ξ and $p_{\xi'}$ have the same central support in $\mathcal{A}'$.

Proof. According to Corollary 1.3 and to Lemma 1.4, $\{\xi_{m,n}\}_{m,n\geq 0}$ is an orthonormal baiss in $\overline{\mathcal{A}\xi}^{\|\ \|_2}$. Thus T_o is a well-defined operator and by Corollary 1.3 we get

$$T_o(\chi_m\xi\chi_n)=T_o(\xi_{m,n}-\xi_{m-1,n})-T_o(\xi_{m,n-1}-\xi_{m-1,n-1}).$$

Let us remark that by Lemma 1.2 and by the definition of T_o

$$\begin{aligned}T_o(\xi_{m,n}-\xi_{m-1,n})&=\xi'_{m,n}+(k-2)\xi'_{m-1,n}-(k-1)\xi'_{m-2,n}+\\&+(k-1)(\xi'_{m,n-1}+(k-2)\xi'_{m-1,n-1}-(k-1)\xi'_{m-2,n-1})=\\&=\chi_m(\xi'_{o,n}+(k-1)\xi'_{o,n-1})\end{aligned}$$

and

$$T_o(\xi_{m,n-1}-\xi_{m-1,n-1})=\chi_m(\xi'_{o,n-1}+(k-1)\xi'_{o,n-2}) \text{ for } m,n\geq 0.$$

Hence, for any $m,n\geq 0$ we get

$$T_o(\chi_m\xi\chi_n)=\chi_m(\xi'_{o,n}+(k-2)\xi'_{o,n-1}-(k-1)\xi'_{o,n-2})=\chi_m\xi'\chi_n.$$

Let us define the operators

$$S_L,S_R:\overline{\mathcal{A}\xi''}^{\|\,\|_2}\longrightarrow\overline{\mathcal{A}\xi''}^{\|\,\|_2};\ S_L(\xi_{m,n})=\xi_{m-1,n},S_R(\xi_{m,n})=\xi_{m,n-1}$$
$$U:\overline{\mathcal{A}\xi''}^{\|\,\|_2}\longrightarrow\overline{\mathcal{A}\xi'''}^{\|\,\|_2},\ U(\xi_{m,n})=\xi'_{m,n},\ \text{for } m,n\geq 0.$$

By Lemma 1.4 U is a unitary and $\|S_L\|=\|S_R\|=\beta^{-1}$. Since $T=U(I+(k-1)S_L)(I+(k-1)S_R)$ and $\|(k-1)S_L\|=\|(k-1)S_R\|=\frac{k-1}{\beta}=\sqrt{\frac{k-1}{N-1}}<1$, it follows that T is bounded and invertible. Clearly $T\in\mathcal{A}'$, hence the polar decomposition of T yields a partial isometry $v\in\mathcal{A}'$ with $v^*v=p_\xi$, $vv^*=p_{\xi'}$. In particular $z(p_\xi)=z(p_{\xi'})$. ▨

Remark. If S is the unilateral shift on $l^2(\mathbb{N})$, then $I+S^*$ is one-to-one. When $N=k\geq 3$, the operators $I+(k-1)S_L$ and $I+(k-1)S_R$ are unitary equivalent with $(I+S^*)\otimes I$, hence $T\in\mathcal{A}'$ is this time one-to-one and with dense range; therefore $z(p_\xi)=z(p_{\xi'})$ in this case also.

LEMMA 1.8. If $N\geq k\geq 3$, then for any vectors $\xi\in\mathcal{C}^{l}_{11}$, $l\geq 1$ and $\xi'\in\mathcal{C}^{l'}_{12}$, $l'\geq 2$ with $\|\xi\|_2=\|\xi'\|_2=1$, the operator

$$T_o:\mathrm{Span}\{\chi_m\xi\chi_n\}_{m,n\geq 0}\longrightarrow\mathrm{Span}\{\chi_m\xi'\chi_n\}_{m,n\geq 0}$$

defined by $T_o(\xi_{m,n})=\xi'_{m,n}+(k-1)\xi'_{m,n-1}$, $m,n\geq 0$,

extends to a bounded operator $T:\overline{\mathcal{A}\xi''}^{\|\,\|_2}\longrightarrow\overline{\mathcal{A}\xi'''}^{\|\,\|_2}$ such that for any

$m,n\geq 0$, $T(\chi_m\xi\chi_n)=\chi_m\xi'\chi_n$. Moreover, T is invertible for $N>k$ and is one-to-one with dense range for $N=k$. In particular $z(p_\xi)=z(p_{\xi'})$.

Proof. The proof is analogue to that of Lemma 1.7. In this case $T=U(I+(k-1)S_L)$. ▨

Clearly, by the some way we get that $z(p_\xi)=z(p_{\xi'})$ for any $\xi\in\mathcal{C}^l_{11}$, $l\geq 1$ and $\xi'\in\mathcal{C}^l_{21}$, $l'\geq 2$, with $\xi,\xi'\neq 0$.

A similar argument shows that for any $\xi\in\mathcal{C}^l_{22}$, $l\geq 2$ and $\xi'\in\mathcal{C}^{l'}_2$, $\xi,\xi'\neq 0$, the projections p_ξ and $p_{\xi'}$ are Murray-von Neumann equivalent in $\mathcal{A}'$. However, in this case the computation is a bit more complicated.

LEMMA 1.9. If $N\geq k\geq 2$ and $N\geq 3$, then for any vectors $\xi\in\mathcal{C}^l_2$ and $\xi'\in\mathcal{C}^l_{22}$, $l\geq 2$, $\|\xi\|_2=\|\xi'\|_2=1$, the operator

$$T_o:\mathrm{Span}\{\chi_m\xi\chi_n\}_{m,n\geq 0}\longrightarrow \mathrm{Span}\{\chi_m\xi'\chi_n\}_{m,n\geq 0}$$

defined by

$$T_o(\xi_{m,n})=\xi'_{m,n}+(k-1)\xi'_{m-1,n-1},\quad m,n\geq 0$$

extends to a bounded invertible operator $T:\overline{\mathcal{A}\xi}^{\|\ \|_2}\longrightarrow\overline{\mathcal{A}\xi'}^{\|\ \|_2}$ such that for any $m,n\geq 0$, $T(\chi_m\xi\chi_n)=\chi_m\xi'\chi_n$. In particular $z(p_\xi)=z(p_{\xi'})$.

Proof. By Lemma 1.5 the vectors $\{\xi_{m,n}\}_{m,n\geq 0}$ are linearly independent in $\mathcal{A}\xi$ and the operator $R:\overline{\mathcal{A}\xi}^{\|\ \|_2}\longrightarrow\overline{\mathcal{A}\xi'}^{\|\ \|_2}$, $R(\xi_{m,n})=\xi'_{m,n}$, $m,n\geq 0$ is bounded and invertible. Denote $\eta_{m,n}=\sum_{j\geq 0}(-1)^j(k-1)^j\,\xi_{m-j,n-j}$ (the sum is finite since $\xi_{r,s}=0$ for $r<0$ or $s<0$) for $m,n\geq 0$ and $\eta_{m,n}=0$ for $m<0$ or $n<0$. By the very definition of T_o it is clear that

$$T_o(\eta_{m,n})=\xi'_{m,n} \quad \text{for any } m,n\geqslant 0 \ . \tag{1.12}$$

Moreover, for any $m,n\geqslant 0$ we get

$$\xi_{m,o}\chi_n=\eta_{m,n}+(k-2)\eta_{m,n-1}-(k-1)\eta_{m,n-2} \ . \tag{1.13}$$

Indeed, $\xi_{m,0}=\eta_{m,0}$ and for $n=1$ we obtain

$$\xi_{m,0}\chi_1=\xi_{m,1}-(k-1)\xi_{m-1,0}+(k-2)\xi_{m,0}=\eta_{m,1}+(k-2)\eta_{m,0}. \tag{1.14}$$

Note also that for any $m,n\geqslant 0$

$$\eta_{m,n}\chi_1=\Big(\sum_{j=0}^{n-1}(-1)^j(k-1)^j\xi_{m-j,n-j}+(-1)^n(k-1)^n\xi_{m-n,o}\Big)\chi_1=$$

$$=\sum_{j=0}^{n-1}(-1)^j(k-1)^j(\xi_{m-j,n-j+1}+(k-2)\xi_{m-j,n-j}+\beta^2\xi_{m-j,n-j-1})+$$

$$+(-1)^n(k-1)^n(\xi_{m-n,1}-(k-1)\xi_{m-n-1,o}+(k-2)\xi_{m-n,o})=$$

$$=\eta_{m,n+1}+(k-2)\eta_{m,n}+\beta^2\eta_{m,n-1} \ . \tag{1.15}$$

Thus (1.1), (1.14) and (1.15) yield

$$\xi_{m,o}\chi_2=(\eta_{m,1}+(k-2)\eta_{m,o})\chi_1-(k-2)(\eta_{m,1}+(k-2)\eta_{m,o})-$$

$$-N(k-1)\xi_{m,o}=\eta_{m,2}+(k-2)\eta_{m,1}+\beta^2\eta_{m,o}+(k-2)\eta_{m,1}+$$

$$+(k-2)^2\eta_{m,o}-(k-2)\eta_{m,1}-(k-2)^2\eta_{m,o}-N(k-1)\eta_{m,o}=$$

$$=\eta_{m,2}+(k-2)\eta_{m,1}-(k-1)\eta_{m,o}.$$

Now (1.13) follows by induction on n as before combining (1.2), (1.15) and part i) of Lemma 1.2.

In virtue of (1.12), (1.13) and Lemma 1.2 one has

$$T_o(\xi_{m,o}\chi_n)=\xi'_{m,n}+(k-2)\xi'_{m,n-1}-(k-1)\xi'_{m,n-2}=\xi'_{m,o}\chi_n,\ m,n\geq 0,\ \text{hence}$$

for any $m,n\geq 0$ we obtain

$$T_o(\chi_m\xi\chi_n)=T_o(\xi_{m,o}\chi_n)+(k-2)T_o(\xi_{m-1,o}\chi_n)-(k-1)T_o(\xi_{m-2,o}\chi_n)=$$
$$=(\xi'_{m,o}+(k-2)\xi'_{m-1,o}-(k-1)\xi'_{m-2,o})=\chi_m\xi'\chi_n.$$

Finally, we remark that denoting $D=S_LS_R:\ \overline{\mathcal{A}\xi''}^{\|\ \|_2}\longrightarrow\overline{\mathcal{A}\xi'}^{\|\ \|_2}$ one obtains $D(\xi'_{m,n})=\xi'_{m-1,n-1}$, $m,n\geq 0$ and $T=(I_{\overline{\mathcal{A}\xi'}^{\|\ \|_2}}+(k-1)D)R$. Since R is invertible and by Lemma 1.4 $\|(k-1)D\|=\dfrac{k-1}{\beta^2}=\dfrac{1}{N-1}<1$, it follows that T is bounded and invertible. ▨

Remark. For any $l,l'\geq 1$, $i,j=1,2$ and $\xi\in\mathcal{C}^l_{ij}$, $\xi'\in\mathcal{C}^{l'}_{ij}$ or $\xi\in\mathcal{C}^l_i$, $\xi'\in\mathcal{C}^{l'}_i$, $\xi,\xi'\neq 0$ it is clear that $z(p_\xi)=z(p_{\xi'})$ since $\chi_m\xi\chi_n$ and $\chi_m\xi'\chi_n$ satisfy the same recurrence relations.

PROPOSITION 1.10. The radial algebra A is a Cartan M.A.S.A. in $M=\mathcal{L}(\mathbb{Z}_2*\mathbb{Z}_2)$.

Proof. Denote by x_1 and x_2 the generators of $\mathbb{Z}_2*\mathbb{Z}_2$ and set $x=x_1-x_2\in\mathcal{L}(\mathbb{Z}_2*\mathbb{Z}_2)$. Then $\chi_1x=-x\chi_1$. In this case the spectrum of χ_1 is $[-2,2]$ and the Plancherel measure is $d\mu(t)=\dfrac{dt}{\pi\sqrt{4-t^2}}$. Thus $\alpha(\chi_1)=-\chi_1$ extends to an automorphism α of A and for any $a\in A$, $ax=x\alpha(a)$. Note also that $x^*x=x^2=2-\chi_2$ and it is easily see that $2-\chi_2$ is one-to-one, hence $s(2-\chi_2)=1$. Now, the polar decomposition $x=v|x|$ yields a unitary $v\in\mathcal{L}(\mathbb{Z}_2*\mathbb{Z}_2)$. Since $s(|x|)=1$, it follows that $av=v\alpha(a)$ for all $a\in A$, hence v normalise A and $M=\{x,\chi_1\}''=N_M(A)''$. ▨

THEOREM 1.11. Suppose that $G=G_1*\ldots*G_N$ is the free product of N groups each having order $k\leq N$. Then:

i) For $N=k=2$ the radial von Neumann subalgebra A is a Cartan M.A.S.A. in $\mathcal{L}(\mathbb{Z}_2*\mathbb{Z}_2)$. In particular $\mathcal{A}=(AVJAJ)''$ is a M.A.S.A.

ii) In all the other cases $\mathcal{A}'$ is of the homogeneous type I_∞ on $(I-p_1)l^2(G)$. In particular A is a singular M.A.S.A.

Proof. It has been proved in [17] that A is a M.A.S.A. in $\mathcal{L}(G)$ iff $N\geq k$.

When $N=k=2$, A is a Cartan subalgebra in $\mathcal{L}(\mathbb{Z}_2*\mathbb{Z}_2)$ by Proposition 1.10. In this case $\mathbb{Z}_2*\mathbb{Z}_2$ is amenable, hence $\mathcal{L}(\mathbb{Z}_2*\mathbb{Z}_2)$ is isomorphic to the II_1 hyperfinite factor $\mathcal{R}$ by Connes theorem or writting the dihedral group as a semi-direct product $\mathbb{Z}_2*\mathbb{Z}_2=\mathbb{Z}\rtimes\mathbb{Z}_2$ and constructing directly matrix units in $\mathcal{L}(\mathbb{Z}_2*\mathbb{Z}_2)$. By [12] it follows that $\mathcal{A}=(AVJAJ)''$ is maximal abelian. The last assertion follows also by Lemma 1.6 since in this case $\mathcal{E}^l_{ij}=\mathcal{E}^1_1=\{0\}$, $l\geq 2$, $i,j=1,2$ and $\mathcal{E}^1_2$ is spanned by the vector $\xi=x_1-x_2$ (x_1 and x_2 are the generators of $\mathbb{Z}_2*\mathbb{Z}_2$).

In the other cases (G nonamenable), take for any $l\geq 1$ an orthonormal basis $\{\xi^l_r\}_{1\leq r\leq\alpha_l}$ in $M^l_0\ominus\mathcal{Y}_l$ such that $\xi^1_r\in\mathcal{E}^1_1\cup\mathcal{E}^1_2$ for $1\leq r\leq\alpha_1$ and $\xi^l_r\in\bigcup_{i,j=1,2}\mathcal{E}^l_{ij}$ for $l\geq 2$, $1\leq r\leq\alpha_l$. One can actually deduce by Lemmas 1.4 and 1.6 that the projections $\{p_{\xi^l_r}\}_{\substack{1\leq i\leq\alpha_l\\ 1\leq l}}$ are mutual orthogonal and $\sum_{l\geq 1}\sum_{i=1}^{\alpha_l}p_{\xi^l_i}=I-p_1$. Since $p_1\in\mathcal{A}$ (see e.g. [12]), Lemmas 1.7, 1.8 and 1.9 yield for any $l\geq 1$, $i=1,\ldots,\alpha_l$ that $z(p_{\xi^l_i})=I-p_1$. ▨

COROLLARY 1.12. Each vector $\xi\in\mathcal{E}^1_1\cup\mathcal{E}^1_2\cup\bigcup_{\substack{i,j=1,2\\ l\geq 2}}\mathcal{E}^l_{ij}$, $\xi\neq 0$ has the central support $z(p_\xi)=I-p_1$ in $\mathcal{A}'$.

2. MORE ABOUT THE STRUCTURE OF THE ALGEBRA $\mathcal{A}$

In this section we use the recurrence formulas in order to give an explicit description for the spatial structure of the algebra $\mathcal{A}=(AVJAJ)''$ on each cyclic projection p_ξ. For $\xi\in\mathcal{E}_1^1\cup\mathcal{E}_2^1\cup\bigcup_{\substack{i,j=1,2\\ l\geq 2}}\mathcal{E}_{ij}^l$, $\xi\neq 0, m,n\geq 0$ denote $\xi^o_{m,n}=\|\xi_{m,n}\|^{-1}\cdot\xi_{m,n}=\beta^{-(m+n)}\xi_{m,n}$. Denote also by $\{e_n\}_{n\geq 0}$ the canonical orthonormal basis in $l^2(\mathbb{N})$ and by S the unilateral shift on $l^2(\mathbb{N})$, $Se_n=e_{n+1}$, $n\geq 0$. The projection onto $\mathbb{C}e_o$ is $P_o=[S^*,S]$. By Lemma 1.4 $\{\xi_{m,n}\}_{m,n\geq 0}$ is an orthonormal basis in $\overline{\mathcal{A}\xi}^{\|\ \|_2}=\overline{\mathrm{Span}\{\chi_m\xi\chi_n\}_{m,n\geq 0}}^{\|\ \|_2}$ whenever $\xi\in\mathcal{E}_1^1\cup\bigcup_{\substack{i,j=1,2\\ l\geq 2}}\mathcal{E}_{ij}^l$, $\xi\neq 0$ and $U:$

$:\overline{\mathcal{A}\xi}^{\|\ \|_2}\rightarrow l^2(\mathbb{N})\times l^2(\mathbb{N})$, $U_\xi(\xi^o_{m,n})=e_m\otimes e_n$ is an unitary operator. According to Lemma 1.2, we obtain for each $l\geq 2$, $i=1,2$

$$U_\xi\chi_1 p_\xi U_\xi^*=((k-2)I+\beta(S+S^*))\otimes I \qquad \text{for } \xi\in\mathcal{E}^l_{2i} \qquad (2.1)$$

$$U_\xi J\chi_1 J\ p_\xi U_\xi^*=I\otimes((k-2))I+\beta(S+S^*)) \qquad \text{for } \xi\in\mathcal{E}^l_{i2} \qquad (2.2)$$

$$U_\xi\chi_1 p_\xi U_\xi^*=((k-2)I-(k-1)P_o+\beta(S+S^*))\otimes I \qquad \text{for } \xi\in\mathcal{E}^l_{1i} \qquad (2.3)$$

$$U_\xi J\chi_1 J p_\xi U_\xi^*=I\otimes((k-2)I-(k-1)P_o+\beta(S+S^*)) \qquad \text{for } \xi\in\mathcal{E}^l_{i1} \qquad (2.4)$$

The formulas (2.3) and (2.4) are still valid also for $\xi\in\mathcal{E}_1^1$. It remainds to find explicit formulas for $\chi_1 p_\xi$ and $J\chi_1 J p_\xi$ in the case $\xi\in\mathcal{E}_2^1$. Let us recall that according to Lemma 1.4 one has

$$\langle\xi^o_{m,n},\xi^o_{m',n'}\rangle=\delta_{m+n,m'+n'}\cdot a^{|m-m'|}\ , \text{ where } a=-(N-1)^{-1}.$$

In order to orthonormalise the vectors $\{\xi^o_{m,n}\}_{m,n\geq 0}$, denote as in Lemma 1.5 $B_l=[a^{|i-j|}]_{0\leq i,j\leq l}$. It is well-known that the positive definite matrix B_l has the Choletsky factorization $B_l=C_l^*C_l$, where C_l is the triangular matrix with entries $c_{oj}=a^j$, $c_{ij}=\sqrt{1-a^2}\cdot a^{j-i}$, $1\leq i\leq j\leq l$. Moreover, C_l is invertible and $C_l^{-1}=D_l$, with the only non-zero entries $d_{oo}=1, d_{jj}=(1-a^2)^{-1/2}$, $d_{j-1,j}=-a(1-a^2)^{-1/2}$, $1\leq j\leq l$.

It follows that the vectors $\{\eta_{m,n}\}_{m,n\geqslant 0}$ given by

$$\eta_{o,n}=\xi^{o}_{o,n} \tag{2.5}$$

$$\eta_{m,n}=(1-a^2)^{-1/2}(\xi^{o}_{m,n}-a\xi^{o}_{m-1,n+1}),\ m\geqslant 1,\ n\geqslant 0 \tag{2.6}$$

define an orthonormal basis in $\overline{\mathcal{A}\xi}^{\|\ \|_2}$. Moreover

$$\xi^{o}_{m,n}=a^m\eta_{o,m+n}+(1-a^2)^{1/2}\cdot\sum_{j=0}^{m-1}a^j\eta_{m-j,n+j}. \tag{2.7}$$

Rewritting the first two relations in Lemma 1.2 vi) as

$$\chi_1\xi^{o}_{o,n}=\beta\xi^{o}_{1,n}+(k-2)\xi^{o}_{o,n}-\sqrt{\frac{k-1}{N-1}}\,\xi^{o}_{o,n-1} \tag{2.8}$$

$$\xi^{o}_{m,o}\chi_1=\beta\xi^{o}_{m,1}+(k-2)\xi^{o}_{m,o}-\sqrt{\frac{k-1}{N-1}}\,\xi^{o}_{m-1,o} \tag{2.9}$$

and taking into account (2.5)-(2.7), one may readily verify the following four relations

$$\eta_{m,n}\chi_1=\beta\eta_{m,n+1}+(k-2)\eta_{m,n}+\beta\eta_{m,n-1}\quad \text{for } m,n\geqslant 0 \tag{2.10}$$

and

$$\chi_1\eta_{o,n}=\beta\eta_{1,n}+(k-2)\eta_{o,n}-\sqrt{\frac{k-1}{N-1}}(N-1-\sqrt{N(N-2)})\eta_{1,n}-$$
$$-\sqrt{\frac{k-1}{N-1}}(\eta_{o,n+1}+\eta_{o,n-1}) \tag{2.11}$$

$$\chi_1\eta_{1,n}=\beta\eta_{2,n}+(k-2)\eta_{1,n}+\beta\eta_{o,n}-\sqrt{\frac{k-1}{N-1}}(N-1-\sqrt{N(N-2)})\eta_{o,n} \tag{2.12}$$

$$\chi_1\eta_{m,n}=\beta\eta_{m+1,n}+(k-2)\eta_{m,n}+\beta\eta_{m-1,n}\quad\text{for } m\geqslant 2,\ n\geqslant 0. \tag{2.13}$$

Thus, denoting $U_\xi:\overline{\mathcal{A}\xi}^{\|\ \|_2}\rightarrow l^2(\mathbb{N})\times l^2(\mathbb{N})$, $U_\xi(\eta_{m,n})=e_m\otimes e_n$, we obtain for any $\xi\in\mathcal{C}^1_2$, $\xi\neq 0$

$$U_\xi J\chi_1 Jp_\xi U_\xi^{*}=I\otimes((k-2)I+\beta(S+S^{*})) \tag{2.14}$$

$$U_\xi \chi_1 p_\xi U_\xi^* = ((k-2)I+\beta(S+S^*)-\sqrt{\frac{k-1}{N-1}}(N-1-\sqrt{N(N-2)})(P_o S\otimes S^* P_o))\otimes I$$
$$-\sqrt{\frac{k-1}{N-1}}\, P_o\otimes(S+S^*).$$

The recurrence relation of χ_n's describe the action of the operators χ_1 and $J\chi_1 J$ on $\overline{\mathcal{A}.1}^{\|\ \|_2} = \overline{\mathrm{Span}\{\chi_n\}_{n\geq 0}}^{\|\ \|_2}$. More precisely, denoting by U_1 the unitary $U_1 : \overline{\mathcal{A}.1}^{\|\ \|_2} \to l^2(\mathbb{N})$, $U_1(\chi_n)=\|\chi_n\|_2 . e_n$, $n\geq 0$, where $\|\chi_n\|_2^2 = N(k-1)\beta^{2(n-1)}$, $n\geq 1$ one obtains

$$U_1 J\chi_1 J p_1 U_1^* = U_1 \chi_1 p_1 U_1^* = (k-2)(I-P_o)+\beta(S+S^*)+$$
$$+\sqrt{k-1}\,(\sqrt{N}-\sqrt{N-1})(P_o S^* + S P_o) \qquad (2.15)$$

Putting all these facts together and according to Lemma 1.7-1.9 one obtains

THEOREM 2.1. Let $G=G_1 * \ldots * G_N$, each G_i having order $k\leq N$, $k\geq 2$, $N\geq 3$. Then

i) There exists an unitary

$$U : l^2(G) \to l^2(\mathbb{N}) \oplus ((l^2(\mathbb{N})\oplus l^2(\mathbb{N}))\otimes l^2(\mathbb{N})) \text{ such that}$$

$$U\chi_1 U^* = S_1 \oplus ((S_2+S_3)\otimes I) \quad , \text{ with}$$
$$S_1 = (k-2)(I-P_o)+\beta(S+S^*)+\sqrt{k-1}\,(\sqrt{N}-\sqrt{N-1})(P_o S^* + S P_o) \quad ;$$
$$S_2 = (k-2)I+\beta(S+S^*) \quad ;$$
$$S_3 = (k-2)I-(k-1)P_o+\beta(S+S^*) \ .$$

ii) For each $\xi \in \mathcal{C}_1^1 \cup \mathcal{C}_2^1 \cup \bigcup_{\substack{i,j=1,2 \\ 1\geq 2}} \mathcal{C}_{ij}^1$, $\xi \neq 0$, the von Neumann algebras $\mathcal{A}p_\xi$ and $A\otimes A$ are isomorphic. In particular $\mathcal{A}$ is isomorphic to $A \oplus (A\otimes A)$.

iii) For each $\xi \in \mathcal{C}_1^1 \cup \bigcup_{\substack{i,j=1,2 \\ 1\geq 2}} \mathcal{C}_{ij}^1$, $\xi\neq 0$, $\mathcal{A}p_\xi$ and $A\otimes A$ are spatial isomorphic (A acts on $L^2(A,\tau)$, ξ identified with $1\otimes 1$, $1\in L^2(A,\tau)$).

Let us consider now the case $G=\mathbb{F}_N$ and let us recall some facts from [14], reformulated in the following lemmas (in this case $\xi^o_{m,n}=(2N-1)^{-(m+n)/2}\xi_{m,n}$).

LEMMA I. i) For $\xi\in M^l_o$, $l\geq 2$, one has

$$\chi_1\xi^o_{m,n}=\sqrt{2N-1}\,(\xi^o_{m+1,n}+\xi^o_{m-1,n}) \qquad \text{for } m\geq 1,\ n\geq 0 \tag{2.16}$$

$$\xi^o_{m,n}\chi_1=\sqrt{2N-1}\,(\xi^o_{m,n+1}+\xi^o_{m,n-1}) \qquad \text{for } m\geq 0,\ n\geq 1. \tag{2.17}$$

ii) For $M^l_o\ominus\mathcal{S}_l$, $l\geq 2$, the previous two relations are still true for all $m,n\geq 0$.

iii) For $\xi\in M^1_o\ominus\mathcal{S}_1$ such that $J\xi=\varepsilon\xi$, $\varepsilon\in\{-1,1\}$ (since $M^1_o\ominus\mathcal{S}_1$ is J-invariant and $J^2=1$, $M^1_o\ominus\mathcal{S}_1=\mathcal{C}_1\oplus\mathcal{C}_{-1}$, where $\mathcal{C}_\varepsilon=\{\xi\in M^1_o\ominus\mathcal{S}_1 : J\xi=\varepsilon\xi\}$ one has

$$\chi_1\xi^o_{o,n}=\sqrt{2N-1}\,\xi^o_{o,n+1}-\frac{\varepsilon}{\sqrt{2N-1}}\,\xi^o_{o,n-1} \tag{2.18}$$

$$\xi^o_{n,o}\chi_1=\sqrt{2N-1}\,\xi^o_{n+1,o}-\frac{\varepsilon}{2N-1}\,\xi^o_{n-1,o} \qquad \text{for } m,n\geq 0. \tag{2.19}$$

LEMMA II i) For $\xi\in M^l_o\ominus\mathcal{S}_l$, $l\geq 2$, $m,m',n,n'\geq 0$ one has

$$\langle\xi^o_{m,n},\xi^o_{m',n'}\rangle=\delta_{m,m'}\cdot\delta_{m,n'}\ .$$

ii) For $\xi\in\mathcal{C}_\varepsilon$, $\varepsilon\in\{-1,1\}$, $m,m',n,n'\geq 0$ one has

$$\langle\xi^o_{m,n},\xi^o_{m',n'}\rangle=\delta_{m+n,m'+n'}(-\varepsilon(2N-1))^{-|m-m'|}$$

Therefore, for each $\xi\in M^l_o\ominus\mathcal{S}_l$, $l\geq 2$ we get

$$U_\xi\,\chi_1\,p_\xi\,U^*_\xi=\sqrt{2N-1}\,(S+S^*)\otimes I$$

$$U_\xi\,J\chi_1 J p_\xi\,U^*_\xi=\sqrt{2N-1}\,.\,I\otimes(S+S^*),$$

where U_ξ is defined as in the case $G=G_1*\ldots*G_N$.

For $\xi\in\mathscr{C}_\varepsilon$ we repeate the calculation of (2.10)-(2.13) but with (2.18) and (2.19) in place of (2.8) and (2.9). Noticing that (2.5)-(2.7) are still true with $a=-(2N-1)^{-1/2}$ we find in this

$$U_\xi J\chi_1 Jp_\xi U_\xi^*=\sqrt{2N-1}\,.\,I\otimes(S+S^*)$$
$$U_\xi\chi_1 p_\xi U_\xi^*=\left(\sqrt{2N-1}\,(S+S^*)-\frac{2N-1-2\sqrt{N(N-1)}}{\sqrt{2N-1}}(P_oS^*+SP_o)\right)\otimes I-$$
$$-\frac{\varepsilon}{\sqrt{2N-1}}P_o\otimes(S+S^*)$$

The recurrence relations of χ_n's [2]

$$\chi_1^2=\chi_2+2N\chi_o$$
$$\chi_1\chi_n=\chi_n\chi_1=\chi_{n+1}+(2N-1)\chi_{n-1},\ n\geqslant 2$$

yield in this case

$$U_1\chi_1p_1U_1^*=U_1J\chi_1p_1U_1^*=\sqrt{2N-1}\,(S+S^*)+(\sqrt{2N}-\sqrt{2N-1})(P_oS^*+SP_o)\qquad(2.20)$$

where $U_1:\overline{\mathcal{A}_1}^{\|\;\|_2}\longrightarrow l^2(\mathbb{N})$, $U(\chi_n)=\|\chi_n\|_2\cdot e_n$, $n\geqslant 0$ and $\|\chi_n\|_2^2=2N(2N-1)^{n-1}$, $n\geqslant 1$. Thus we obtain the following statement

THEOREM 2.2. Let $G=\mathbb{F}_N$ be the free group on $N\geqslant 2$ generators. Then

i) There exists an unitary

$U:l^2(G)\longrightarrow l^2(\mathbb{N})\oplus(l^2(\mathbb{N})\otimes l^2(\mathbb{N}))$ such that

$U\chi_1U^*=S_o\oplus(\sqrt{2N-1}\,(S+S^*)\otimes I)$, where

$S_o=\sqrt{2N-1}\,(S+S^*)+(\sqrt{2N}-\sqrt{2N-1})(P_oS^*+SP_o)$.

ii) For each $\xi\in\mathcal{E}_1\cup\mathcal{E}_{-1}\cup\bigcup_{l\geq 2}(M_0^l\ominus\mathcal{S}_1)$, $\xi\neq 0$, the von Neumann algebras $\mathcal{A}p_\xi$ and $A\otimes A$ are isomorphic. In particular $\mathcal{A}$ is isomorphic to $A \oplus (A\otimes A)$.

iii) For each $\xi\in\bigcup_{l\geq 2}(M_0^l\ominus\mathcal{S}_1)$, $\xi\neq 0$, $\mathcal{A}p_\xi$ and $A\otimes A$ are spatial isomorphic (A acts on $L^2(A,\tau)$, ξ identified with $1\otimes 1$, $1\in L^2(A,\tau)$).

Denote the vector state associate to each $\xi\in l^2(G)$, $\|\xi\|=1$ by ω_ξ. Then the functional $\mu_\xi:\mathbb{C}[X]\to\mathbb{C}$, $\mu_\xi(X^n)=\omega_\xi(\chi_1^n)=\langle\chi_1^n\xi,\xi\rangle$ is a probability measure on $\mathbb{R}$ with compact support. Finally, we shall compute $d\mu_\xi$ for $\mathcal{E}_1^1\cup\mathcal{E}_2^1\cup\bigcup_{\substack{i,j=1,2\\ l\geq 2}}\mathcal{E}_{ij}^l$ when $G=G_1*\dots*G_N$ and for $\xi\in\mathcal{E}_1\cup\mathcal{E}_{-1}\cup\bigcup_{l\geq 2}(M_0^l\ominus\mathcal{S}_1)$ when $G=\mathbb{F}_N$. For $\xi=1$ these are the Plancherel measures computed in [4] for the first case and in [1], [3], [11] for the second:

$$d\mu_1(t)=\frac{N\sqrt{4(N-1)(k-1)-(t+2-k)^2}}{2\pi(t+N)(N(k-1)-t)}\,dt \quad \text{for } \chi_1\in\mathcal{L}(G_1*\dots*G_N),\ N\geq k$$

$$d\mu_1(t)=\frac{N\sqrt{4(2N-1)-t^2}}{\pi((2N)^2-t^2)}\,dt \qquad \text{for } \chi_1\in\mathcal{L}(\mathbb{F}_N).$$

In fact, according to (2.15) and (2.20), the Plancherel measure is the measure $\mu_{\tilde S}:\mathbb{C}[X]\to\mathbb{C}$, $\mu_{\tilde S}(X^n)=\mathrm{Tr}(\tilde S^n P_0)$ associated to some finite-dimensional perturbation $\tilde S$ of $S+S^*$. Our approach here in the case $G=G_1*\dots*G_N$ is to apply Theorem 3 in [4] in order to find explicitely the probability measure $\mu_{\tilde S}$ associated to the one-dimensional self-adjoint perturbation $\tilde S=a(I-P_0)+\alpha P_0+\beta(S+S^*)$ of $aI+\beta(S+S^*)$, $a,\alpha,\beta\in\mathbb{R}$.

LEMMA 2.3. The continuous part of $d\mu_{\tilde S}$ is

$$d\mu^c_{\tilde S}(t)=\frac{\sqrt{4\beta^2-(t-a)^2}}{\pi((a-\alpha)(t-a)+(a-\alpha)^2+\beta^2)}\,dt$$

The discrete part of $d\mu_{\tilde{S}}$ is

$$d\mu^d_{\tilde{S}}(t)=2\left(1-\frac{b^2}{(a-\alpha)^2}\right)_+\delta\left(\frac{\alpha+b^2}{\alpha-a}\right)\quad \text{for } \alpha\neq a$$

(Here $a_+=(a+|a|)/2$ and $\delta(x)$ denotes the Dirac measure of mass one concentrated in x).

Proof. Consider the sequence of polynomials defined by

$$p_o(t)=1,\ p_1(t)=t-\alpha,\ p_{n+1}(t)=(t-a)p_n(t)-\beta^2p_{n-1}(t)\ \text{ for } n\geq 1.$$

Clearly $p_o(\tilde{S})e_o=e_o$, $p_1(\tilde{S})e_o=be_1$, $p_2(\tilde{S})e_o=b(\tilde{S}-a)e_1-b^2e_o=b^2e_2$. By induction we find $p_n(\tilde{S})e_o=b^ne_n$ for $n\geq 1$, hence

$$\mu_{\tilde{S}}(p_np_m)=\beta^{m+n}\delta_{m,n}\quad \text{for all } m,n\geq 0\ . \tag{2.21}$$

Thus Theorem 3 in [4] applies and the statement follows.

According to Theorems 2.1, 2.2 and to Lemma 2.3 we obtain

COROLLARY 2.4. Let $G=\mathbb{F}_N$. Then for each $\zeta\in\mathcal{C}_1\cup\mathcal{C}_{-1}\cup\bigcup_{l\geq 2}(M_o^l\ominus\mathcal{S}_l)$, $\zeta\neq 0$ one has

$$d\mu_\zeta(t)=\frac{\sqrt{4(2N-1)-t^2}}{\pi(2N-1)}$$

COROLLARY 2.5. Let $G=G_1*\dots*G_N$, each G_i having order $k\leq N$. Then

i) $d\mu_\zeta(t)=\dfrac{\sqrt{4(N-1)(k-1)-(t+2-k)^2}}{\pi(N-1)(k-1)}\,dt$ for $\zeta\in\mathcal{C}^1_2\cup\bigcup_{\substack{i=1,2\\ l\geq 2}}\mathcal{C}^l_{2i}$

ii) $d\mu_\zeta(t)=\dfrac{\sqrt{4(N-1)(k-1)-(t+2-k)^2}}{\pi(k-1)(t+N)}\,dt$ for $\zeta\in\mathcal{C}^1_1\cup\bigcup_{\substack{i=1,2\\ l\geq 2}}\mathcal{C}^l_{1i}$

and the similar results for $J\chi_1 J$.

Note that in order to compute $d\mu_\xi$ in Corollary 2.4 and in part i), Corollary 2.5, we need only $d\mu_{S+S^*}$ and one can readily obtain $d\mu_{S+S^*}(t) = \frac{1}{\pi}\sqrt{4-t^2}$ by a direct calculation of the moments, by Lemma 2.3 or using the principal function of S, as in [18].

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