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ON SUB-RESOLVENTS AND SUB-MULTIPLICATIVE FUNCTIONALS OF A MARKOV PROCESS

by Emil Popescu

The aim of this paper is to present properties of the sub-resolvents and the sub-multiplicative functionals, starting from known results for resolvents and multiplicative functionals.

In section 1 are given some properties of the sub-resolvents of kernels on a measurable space. In section 2 is presented a result referring to the sub-resolvents exactly subordinated to a resolvent. This result is analogous with that one known for subordinate resolvents. In the last section are transposed a series of results from [1] , referring to the multiplicative functionals of a Markov process, to the sub-multiplicative functionals of the process.

1. Sub-resolvents

Throughout this section $(E, \mathcal{E})$ will be a measurable space.

Definition 1.1.

A family $\mathcal{V} = \{V^{\alpha} : \alpha > 0\}$ of kernels on $(E, \mathcal{E})$ is called sub-resolvent (of kernels) if

i) $V^{\beta} \leqslant V^{\alpha} \leqslant V^{\beta} + (\beta - \alpha) V^{\alpha} V^{\beta}$ for any $\alpha, \beta > 0$ and $\alpha < \beta$.

ii) $V^{\alpha} V^{\beta} = V^{\beta} V^{\alpha}$ for any $\alpha, \beta > 0$.

The sub-resolvent $\{V^{\alpha} : \alpha > 0\}$ is called sub-Markov (resp.Markov) if for any $\alpha > 0$ we have

$$\alpha V^{\alpha} 1 \leqslant 1 \text{ (resp. } \alpha V^{\alpha} 1 = 1)$$

In the sequel we shall consider only sub-Markov sub-resolvents without to specify this thing. Then

$$V^{\alpha}(x, E) \leqslant \frac{1}{\alpha}$$

for any x, hence V^{α} is a bounded kernel on $(E, \mathcal{E})$. From i) it follows that

$$\alpha \longrightarrow V^{\alpha}(x, .)$$

is decreasing and continuous on $(0, \infty)$. Consequently we can define the initial kernel of the sub-resolvent $\mathcal{V}$:

$$V(x, .) = \sup_{\alpha} V^{\alpha}(x, .) = \lim_{\alpha \to 0} V^{\alpha}(x, .)$$

For any $\alpha > 0$ we have

$$VV^{\alpha} = V^{\alpha} V$$

and

$$V \leqslant V^{\alpha} + \alpha V^{\alpha} V$$

If $\beta > 0$ and we define

$$U^{\alpha} = V^{\alpha + \beta}$$

for $\alpha > 0$, then $\{U^{\alpha} : \alpha > 0\}$ is a sub-resolvent with the initial kernel

$$U = \sup_{\alpha} U^{\alpha} = V^{\beta}$$

U is a bounded kernel.

If $\{V^{\alpha} : \alpha > 0\}$ is a sub-resolvent on $(E, \mathcal{E})$, then it is also a sub-resolvent on $(E, \mathcal{E}^{*})$ because

$$x \longrightarrow U^{\alpha}(x, A)$$

is $\mathcal{E}^{*}$ - measurable whenever $A \in \mathcal{E}^{*}$.

From now on, if f is a numerical nonnegative $\mathcal{E}$ - measurable function on E we shall write $f \in \mathcal{E}_{+}$. Moreover, if f is bounded we shall write $f \in b\,\mathcal{E}_{+}$.

Definition 1.2.

Let $f \in \mathcal{E}_{+}$ and $\alpha \geqslant 0$. f is called α -supermedian with respect to the sub-resolvent $\{V^{\alpha} : \alpha > 0\}$ (briefly α - V supermedian) if

$$\beta V^{\alpha + \beta} f \leqslant f$$

for any $\beta > 0$.

f is called α -excessive with respect to the sub-resolvent $\{V^{\alpha} : \alpha > 0\}$ (briefly α - V excessive) if f is α - V supermedian and

$$\lim_{\beta \to \infty} \beta V^{\alpha + \beta} f = f$$

We denote by $\mathcal{S}_{\mathcal{V}}^{\alpha}$ (resp. $\mathcal{E}_{\mathcal{V}}^{\alpha}$) the set of the α -V supermedian (resp. α -V excessive) functions.

The next proposition gives some properties of supermedian and excessive functions with respect to a sub-resolvent.

Proposition 1.1.

Let $\mathcal{V} = \{V^{\alpha} : \alpha > 0\}$ a sub-resolvent of kernels on $(E, \mathcal{E})$. The following assertions hold:

i) $\mathcal{S}_{\mathcal{V}}^{\alpha}$ and $\mathcal{E}_{\mathcal{V}}^{\alpha}$ are convex cones; moreover if f, g $\in \mathcal{S}_{\mathcal{V}}^{\alpha}$ then $f \wedge g = \min(f,g) \in \mathcal{S}_{\mathcal{V}}^{\alpha}$

ii) If (f_n) is an increasing sequence in $\mathcal{S}_{\mathcal{V}}^{\alpha}$ (resp.

$\mathcal{E}_V^\alpha$) then $f = \lim f_n$ is in $\mathcal{S}_V^\alpha$ (resp. $\mathcal{E}_V^\alpha$).

iii) If $f \in \mathcal{S}_V^\alpha$, then the function $\beta \to \beta V^{\alpha+\beta} f$ is increasing.

iv) If $f \in \mathcal{S}_V^\alpha$ then $V^\alpha f \in \mathcal{E}_V^\alpha$

v) Let $f \in \mathcal{S}_V^\alpha$. Then $\hat{f} = \lim_{\beta\to\infty} \beta V^{\alpha+\beta} f$ is the largest α - V excessive function dominated by f and $V^\beta f = V^\beta \hat{f}$ for any $\beta > 0$.

<u>Proof</u>. i) and ii) are immediate from definition.

iii) Let f be α - V supermedian bounded function and $\beta > \eta$. Then

$$\beta V^{\alpha+\beta} - \eta V^{\alpha+\eta} \geqslant \beta V^{\alpha+\eta} + \beta(\eta - \beta) V^{\alpha+\beta} V^{\alpha+\eta} - \eta V^{\alpha+\eta} =$$

$$=(\beta - \eta) V^{\alpha+\eta} (I - \beta V^{\alpha+\beta})$$

From this inequality we observe that $\beta \to \beta V^{\alpha+\beta} f$ is increasing. Let $f \in \mathcal{S}_V^\alpha$ and $f_n = f \wedge n$. Then $f_n \uparrow f$ and f_n is α-V supermedian bounded. Therefore for any n, if $\beta > \eta$ then

$$\beta V^{\beta+\alpha} f_n \geqslant \eta V^{\eta+\alpha} f_n$$

and letting $n \to \infty$ we obtain ii).

iv) It is enough to show for f bounded. We have

$\beta V^{\alpha+\beta} V^\alpha f = V^\alpha (\beta V^{\alpha+\beta} f) \leqslant V^\alpha f$ for any $\beta > 0$. On the other hand

$$V^\alpha f \leqslant V^{\alpha+\beta} f + \beta V^{\alpha+\beta} V^\alpha f$$

It follows that

$$V^\alpha f \leqslant \lim_{\beta\to\infty} \beta V^{\alpha+\beta} V^\alpha f$$

v) From iii) we deduce that if f is α-V supermedian then there exists

$$\hat{f}(x) = \lim_{\beta\to\infty} \beta V^{\alpha+\beta} f(x)$$

Of course, $\hat{f} \in \mathcal{E}_+$, $\hat{f} \leqslant f$ and $\hat{f}$ is α-V supermedian.

Let $g \in \mathcal{E}_V^\alpha$ and $g \leqslant f$. Then $\beta V^{\alpha+\beta} g \leqslant \beta V^{\alpha+\beta} f$ and letting $\beta \to \infty$ we find that $g \leqslant \hat{f}$. Thus, to complete

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the proof, we must show that $\hat{f}$ is α-V excessive. Suppose first that f is bounded. Then we have:

$$V^\beta \hat{f} = V^\beta(\lim_{\eta\to\infty} \eta V^{\alpha+\eta} f) = \lim_{\eta\to\infty} \eta V^\beta V^{\alpha+\eta} f \geqslant \lim_{\eta\to\infty} \frac{\eta}{\alpha+\eta-\beta}(V^\beta f - V^{\alpha+\eta} f) = V^\beta f$$

Since $\hat{f} \leqslant f$ it follows $V^\beta \hat{f} = V^\beta f$ for any $\beta > 0$. Consequently $\beta V^{\alpha+\beta}\hat{f} = \beta V^{\alpha+\beta} f \uparrow \hat{f}$ as $\beta \to \infty$ and so $\hat{f}$ is α-V excessive. If f is unbounded, let $f_n = f \wedge n$. Then $\beta V^{\alpha+\beta} f_n$ is increasing in both β and n. Therefore $\hat{f} = \lim_{\beta\to\infty} \beta V^{\alpha+\beta} f = \lim_{n\to\infty} \lim_{\beta\to\infty} \beta V^{\alpha+\beta} f_n = \lim_{n\to\infty} \hat{f}_n$. But the limit of an increasing sequence of α-V excessive functions is α-V excessive.

<u>Definition 1.3.</u> A family $\mathcal{W} = \{W^\alpha : \alpha > 0\}$ of kernels on $(E, \mathcal{E})$ is called super-resolvent if

i) $W^\alpha \geqslant W^\beta + (\beta - \alpha) W^\alpha W^\beta$ for all $\alpha, \beta > 0$ and

ii) $W^\alpha W^\beta = W^\beta W^\alpha$ for all $\alpha, \beta > 0$.

The super-resolvent $\{W^\alpha\}$ is called sub-Markov if

$$\alpha W^\alpha 1 \leqslant 1$$

for any $\alpha > 0$.

From definition it follows that for any $f \in \mathcal{E}_+$ we have

$$\alpha < \beta \implies W^\alpha f \geqslant W^\beta f,$$

and therefore the map W from $\mathcal{E}_+$ into $\mathcal{E}_+$ defined by

$$Wf := \sup_\alpha W^\alpha f$$

is a kernel on $(E, \mathcal{E})$ which is called the initial kernel of the super-resolvent $\{W^\alpha\}$. For any $\alpha > 0$ we have

$$WW^\alpha = W^\alpha W \text{ and } W \geqslant W^\alpha + \alpha WW^\alpha.$$

We can define the α-W supermedian and α-W excessive functions of a super-resolvent similar to definition 1.2.

<u>Proposition 1.2</u>. Let $\{U^\alpha : \alpha > 0\}$ be a resolvent and $\{V^\alpha : \alpha > 0\}$ a sub-resolvent on $b\mathcal{E}_+$ such that $V^\alpha f \leqslant U^\alpha f$ for all $\alpha > 0$, $f \in b\mathcal{E}_+$ and $U^\alpha V^\beta = V^\beta U^\alpha$ for all $\alpha, \beta > 0$. Then

there exists $\{W^\alpha : \alpha > 0\}$ super-resolvent on $b\mathcal{E}+$ such that

$$U^\alpha f = V^\alpha f + W^\alpha f$$

for all $\alpha > 0$ and $f \in b\mathcal{E}+$.

Proof. Let $W^\alpha f := U^\alpha f - V^\alpha f$. Then for any $\alpha, \beta > 0$, $\alpha < \beta$ and $f \in b\mathcal{E}+$ we have

$$W^\alpha f \geqslant W^\beta f + (\beta - \alpha) W^\alpha W^\beta f.$$

Indeed,

$$U^\alpha f - V^\alpha f \geqslant U^\beta f - V^\beta f + (\beta - \alpha)(U^\alpha - V^\alpha)(U^\beta f - V^\beta f)$$

since

$$V^\beta f - V^\alpha f - (\beta - \alpha) V^\alpha V^\beta f + (\beta - \alpha)(V^\alpha U^\beta f + U^\alpha V^\beta f) \geqslant 0$$

$$V^\beta f - V^\alpha f + (\beta - \alpha) V^\alpha U^\beta f \geqslant V^\beta f - V^\alpha f + (\beta - \alpha) V^\alpha V^\beta f \geqslant 0$$

and $U^\alpha V^\beta f \geqslant V^\alpha V^\beta f$. Of course, the relation $U^\alpha V^\beta = V^\beta U^\alpha$ implies $W^\alpha W^\beta = W^\beta W^\alpha$.

Definition 1.4. A sub-semigroup (super-semigroup) on $(E, \mathcal{E})$ is a family $P = \{P_t : t \geqslant 0\}$ of kernels on $(E, \mathcal{E})$ such that

i) $P_{t+s} \leqslant (\geqslant)\ P_t P_s$, for any $t, s \geqslant 0$

ii) $P_t P_s = P_s P_t$

iii) For any function $f \in \mathcal{E}_+$, the map

$$(t, x) \longrightarrow P_t f(x)$$

is measurable on the measurable product space $\mathbb{R}_+ \times E$.

The sub-semigroup (super-semigroup) P is called Markov (resp. sub-Markov) if $P_t 1 = 1$ (resp. $P_t 1 \leqslant 1$) for any $t \geqslant 0$. In the sequel, we shall consider only sub-Markov sub-semigroups (super-semigroups) without to specify this thing. It is not assumed in definition that $P_0 = I$; of course, if P is sub-semigroup, then $P_0^2 = P_0$.

Proposition 1.3. Let $P = \{P_t : t \geqslant 0\}$ be a sub-semigroup (super-semigroup) on $(E, \mathcal{E})$ and for any $\alpha > 0$ let V^α be the kernel on $(E, \mathcal{E})$ defined by

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$$V^{\alpha} f(x) = \int_0^{\infty} e^{-\alpha t} P_t f(x) dt, \ f \in \mathcal{E}_+ .$$

Then the family $\mathcal{V} = \{ V^{\alpha} : \alpha > 0 \}$ is a sub-resolvent (resp. super-resolvent) on $(E, \mathcal{E})$ which will be called the sub-resolvent (super-resolvent) associated with the sub-semigroup (super-semigroup) P.

The proof is immediate.

Remark. We can define, similar to the case of semi-groups, the α - supermedian and α - excessive functions with respect to the sub-semigroup P. We denote by $\mathcal{E}_P^{\alpha}$ the set of α - excessive functions with respect to the sub-semigroup P. Then $\mathcal{E}_P^{\alpha} \subset \mathcal{E}_{\mathcal{V}}^{\alpha}$. Indeed, let $f \in \mathcal{E}_P^{\alpha}$. Since $e^{-\alpha t} P_t f \leq f$ for any $t \geq 0$ we get

$$\beta V^{\alpha+\beta} f = \int_0^{\infty} \beta e^{-(\alpha+\beta)t} P_t f \, dt \leq f.$$

On the other hand,

$$\beta V^{\alpha+\beta} f = \int_0^{\infty} \beta e^{-(\alpha+\beta)t} P_t f \, dt = \int_0^{\infty} e^{-u} e^{-\alpha \frac{u}{\beta}} P_{\frac{u}{\beta}} f \, du$$

increases to f as $\beta \to \infty$.

2. Sub-resolvents Exactly Subordinate to a Resolvent.

Let $(E, \mathcal{E})$ be a measurable space. We denote by $B = b\mathcal{E}$ the Banach space of all bounded $\mathcal{E}$-measurable functions under the supremum norm. Let $\mathcal{U} = \{U^\alpha : \alpha > 0\}$ be a sub-Markov resolvent on B. Of course, a sub-resolvent $\mathcal{V}$ on B is a family of positive linear operators on B which verifies the relations i) and ii) from the definition 1.1. for any $f \in B_+$.

Definition 2.1. A sub-resolvent $\mathcal{V} = \{V^\alpha : \alpha > 0\}$ on B is called sub-resolvent subordinate to $\{U^\alpha\}$ if

$$V^\alpha f \leqslant U^\alpha f$$

for any $\alpha > 0$ and $f \in B_+$.

Proposition 2.1. Let $\mathcal{V} = \{V^\alpha : \alpha > 0\}$ a sub-resolvent on B subordinate to $\{U^\alpha\}$. If $f \in B_+$ and $\alpha > 0$, then $U^\alpha f - V^\alpha f$ is α-U supermedian.

Proof. For $f \in B_+$, $(U^\alpha f - V^\alpha f) \in B_+$ for any $\alpha > 0$.

$$U^\alpha f - V^\alpha f - \beta U^{\beta+\alpha}(U^\alpha f - V^\alpha f) = U^\alpha f - V^\alpha f - \beta U^{\beta+\alpha} U^\alpha f + \beta U^{\beta+\alpha} V^\alpha f \geqslant U^{\alpha+\beta} f - V^{\alpha+\beta} f \geqslant 0$$

for any $\alpha, \beta > 0$.

Definition 2.2. A sub-resolvent $\mathcal{V} = \{V^\alpha : \alpha > 0\}$ is called exactly subordinate to the resolvent $\{U^\alpha\}$ if it is subordinate to $\{U^\alpha\}$ and, in addition, $U^\alpha f - V^\alpha f$ is α-U excessive for all $f \in B_+$ and $\alpha > 0$.

Theorem 2.1. Let $\{V^\alpha\}$ be a sub-resolvent on B subordinate to $\{U^\alpha\}$. Then

$$W^\alpha f(x) = \lim_{\beta \to \infty} \beta U^\beta V^\alpha f(x)$$

exists for all $f \in B_+$, $\alpha > 0$, $x \in E$ and the family $\{W^\alpha : \alpha > 0\}$ is a sub-resolvent exactly subordinate to $\{U^\alpha\}$. Moreover, for each $\alpha > 0$ and $x \in E$ we have on B_+:

$$V^\alpha(x,.) \leqslant W^\alpha(x,.).$$

We denote by E_V the set of the points $y \in E$ for which exists $\gamma > 0$ such that

$$V^{\gamma}1(y) + (\gamma - \alpha)\, V^{\gamma}V^{\alpha}1(y) = V^{\alpha}1(y)$$

for any $\alpha > 0$.

Let $x \in E_V$ be such that exists $\gamma > 0$ with the property

$$\beta U^{\beta} V^{\gamma}1(x) \longrightarrow V^{\gamma}1(x)$$

as $\beta \to \infty$. Then

$$V^{\alpha}(x,.) = W^{\alpha}(x,.)$$

for any $\alpha > 0$. In particular equality holds at any $x \in E_V$ for which

$$\beta V^{\beta}1(x) \longrightarrow 1$$

as $\beta \to \infty$.

Proof. We shall adapt the proof of the theorem (4.9) from [1], Chap.III, to the case of sub-resolvents. If $f \in B_+$, from the resolvent equation of $\{U^{\alpha}\}$ we have

$$U^{\alpha}f - \beta U^{\beta}V^{\alpha}f = U^{\beta+\alpha}f + \beta U^{\beta+\alpha}U^{\alpha}f - \beta U^{\beta+\alpha}V^{\alpha}f -$$

$$- \beta\alpha U^{\beta+\alpha}U^{\beta}V^{\alpha}f = \beta U^{\beta+\alpha}(U^{\alpha}f - V^{\alpha}f) + O(\tfrac{1}{\beta})$$

since $\|U^{\beta}\| \leqslant \frac{1}{\beta}$. Since $U^{\alpha}f - V^{\alpha}f$ is α-Usupermedian (prop. 2.1.), it follows that exists $\lim_{\beta\to\infty} \beta U^{\beta}V^{\alpha}f$ and $U^{\alpha}f - W^{\alpha}f$ is the α-U excessive regularization of $U^{\alpha}f - V^{\alpha}f$. Therefore

$$W^{\alpha}f = \lim_{\beta\to\infty} \beta U^{\beta}V^{\alpha}f$$

exists for all $f \in B_+$. We observe that $W^{\alpha}f \leqslant U^{\alpha}f$ for any $f \in B_+$, since

$$W^{\alpha}f \leqslant \lim_{\beta\to\infty} \beta U^{\beta}U^{\alpha}f \leqslant \lim_{\beta\to\infty}(\beta+\alpha)U^{\beta+\alpha}U^{\alpha}f = U^{\alpha}f$$

On the other hand, for any $f \in B_+$,

$$W^{\alpha}f = \lim_{\eta\to\infty} \eta\, U^{\eta}V^{\alpha}f \geqslant \lim_{\eta\to\infty} \eta U^{\alpha+\eta}V^{\alpha}f \geqslant \lim_{\eta\to\infty} \eta\, V^{\alpha+\eta}V^{\alpha} \geqslant$$

$$\geqslant \lim_{\eta\to\infty}(V^{\alpha}f - V^{\alpha+\eta}f) = V^{\alpha}f$$

Now, we have:

$$V^{\alpha}f - V^{\beta}f \leqslant (\beta-\alpha)V^{\beta}V^{\alpha}f \leqslant (\beta-\alpha)\, V^{\beta}W^{\alpha}f$$

for any $f \in B_+$.

Operating on this relation by ηU^{η} and letting $n \to \infty$ we obtain

$$W^{\alpha}f - W^{\beta}f \leqslant (\beta-\alpha)W^{\beta}W^{\alpha}f \ , \ f \in B_+.$$

Since $\{U^{\alpha}\}$ is resolvent and $U^{\alpha}f - W^{\alpha}f$ is the α-U excessive regularization of $U^{\alpha}f - V^{\alpha}f$, $f \in B_+$ it follows that

$$U^{\beta}(U^{\alpha}f - V^{\alpha}f) = U^{\beta}(U^{\alpha}f - W^{\alpha}f)$$

for any $\beta > 0$ and hence that

$$U^{\beta}V^{\alpha}f = U^{\beta}W^{\alpha}f$$

Furthermore for $f \in B_+$ we have

$$0 = U^{\beta}(W^{\alpha} - V^{\alpha})f \geqslant V^{\beta}(W^{\alpha} - V^{\alpha})f \geqslant 0.$$

It follows that

$$V^{\beta}W^{\alpha}f = V^{\beta}V^{\alpha}f.$$

Let us show that

$$W^{\alpha}W^{\beta}f = W^{\beta}W^{\alpha}f,$$

for all $f \in B_+$ and $\alpha, \beta > 0$. Indeed,

$$W^{\alpha}W^{\beta}f = \lim_{\eta\to\infty} \eta U^{\eta}V^{\alpha}W^{\beta}f = \lim_{\eta\to\infty} \eta U^{\eta}V^{\alpha}V^{\beta}f = \lim_{\eta\to\infty} \eta U^{\eta}V^{\beta}V^{\alpha}f =$$
$$= \lim_{\eta\to\infty} \eta U^{\eta}V^{\beta}W^{\alpha}f = W^{\beta}W^{\alpha}f.$$

Therefore $\{W^{\alpha}\}$ is a sub-resolvent. Since $U^{\alpha}f - V^{\alpha}f$ is α-U excessive for any $f \in B_+$, it follows that $\{W^{\alpha}\}$ is exactly subordinate to $\{U^{\alpha}\}$.

Let us show the second part of the theorem. We show already that $V^{\alpha}(x,.) \leqslant W^{\alpha}(x,.)$ for any x and $\alpha > 0$. Let $x \in E_V$ be such that

$$\beta U^{\beta}V^{\gamma}1(x) \longrightarrow V^{\gamma}1(x)$$

as $\beta \to \infty$, for some fixed $\gamma > 0$. From definition $\beta U^{\beta}V^{\gamma}1 \longrightarrow W^{\gamma}1$ as $\beta \to \infty$. It follows that $V^{\gamma}(x,.) = W^{\gamma}(x,.)$. Therefore,

$$\beta U^{\beta}V^{\gamma}f(x) \longrightarrow W^{\gamma}f(x) = V^{\gamma}f(x),$$

for any $f \in B_+$. Let $\alpha > 0$ be and

$$V^{\alpha}1 - V^{\gamma}1 \leqslant (\gamma-\alpha)\, V^{\gamma}V^{\alpha}1.$$

Operating on this by βU^{β} and letting $\beta \to \infty$ we obtain:

$$W^{\alpha}1 - W^{\gamma}1 \leqslant (\gamma-\alpha)\, W^{\gamma}V^{\alpha}1$$

In the point x we have

$$W^{\gamma}1(x) = V^{\gamma}1(x),$$

$$W^{\alpha}1(x) \leq W^{\gamma}1(x) + (\gamma-\alpha)W^{\gamma}V^{\alpha}1(x) = V^{\gamma}1(x) +$$
$$+ (\gamma-\alpha)\, V^{\gamma}V^{\alpha}1(x) = V^{\alpha}1(x)$$

since $x \in E_V$. Therefore,

$$V^{\alpha}(x,.) = W^{\alpha}(x,.)$$

on B_+, for any $\alpha > 0$.

In particular, if $\beta V^{\beta}1(x) \longrightarrow 1$ as $\beta \to \infty$, then

$$\beta U^{\beta}1(x) - \beta V^{\beta}1(x) \longrightarrow 0.$$

Let $\alpha > 0$ be. Of course $\alpha V^{\alpha}1 \leq 1$. We have:

$$(\beta U^{\beta} - \beta V^{\beta})V^{\alpha}1(x) \leq \frac{1}{\alpha}(\beta U^{\beta} - \beta V^{\beta})\,1(x) \longrightarrow 0$$

as $\beta \to \infty$. But, from Prop. 1.1., iv) it follows that $V^{\alpha}1 \in \mathcal{C}^{\alpha}_{V}$.

Hence

$$\beta U^{\beta}V^{\alpha}1(x) \longrightarrow V^{\alpha}1(x)$$

as $\beta \to \infty$. Therefore, it follows as above

$$W^{\alpha}(x,.) = V^{\alpha}(x,.)$$

and the proof of theorem is complete.

3. Sub-multiplicative Functionals of a Markov Process

Let $X = (\Omega, \mathcal{M}, \mathcal{M}_t, X_t, \theta_t, P^x)$ be a Markov process with state space $(E, \mathcal{E})$ in the meaning of [1].

Definition 3.1. A family $M = \{M_t : 0 \leq t < \infty\}$ of real-valued random variables on $(\Omega, \mathcal{F})$ is called a sub-multiplicative functional of X provided:

i) $M_t \in \mathcal{F}_t$ for any $t \geq 0$.

ii) $M_{t+s} \leq M_t \cdot (M_s \circ \theta_t) = M_s \cdot (M_t \circ \theta_s)$ a.s. for any $t, s \geq 0$

iii) $0 \leq M_t(\omega) \leq 1$ for all t and ω.

M is called right continuous (or continuous) if $t \longrightarrow M_t(\omega)$ is right continuous (or continuous) almost surely. We observe that for all t and s, $M_{t+s} \leq M_s$ a.s. The relationship $M_o \leq M_o \cdot (M_o \circ \theta_o) = M_o^2 \leq M_o$ a.s., hence $M_o = M_o^2$ a.s. implies that almost surely M_o is either zero or one. Like at multiplicative functionals, a point $x \in E$ is called permanent for M if $P^x(M_o = 1) = 1$. We denote by E_M the set of permanent points which is universally measurable. If X is normal, then $x \in E \setminus E_M$ if and only if $P^x(M_o = 0) = 1$.

In the sequel, we give a few examples of sub-multiplicative functionals:

1. $M_t = \exp(-t^2)$ and $E_M = E$

2. Let T be a terminal time. We define

$$M_t(\omega) = \begin{cases} \exp(-t^2) & \text{if } t < T(\omega) \\ 0 & \text{if } t \geq T(\omega) \end{cases}$$

$\{M_t\}$ is a right continuous sub-multiplicative functional and $E_M = \{x \in E \mid P^x(T > 0) = 1\}$

3. Let X progressively measurable with respect to $\{\mathcal{F}_t\}$ and let $f \in b\,\mathcal{E}_+$. Then

$$M_t = \exp(-t \int_0^t f(X_s)\, ds)$$

is a continuous sub-multiplicative functional with $E_M = E$.

Definition 3.2. A family $S = \{S_t : 0 \leq t < \infty\}$ of real-valued random variables on $(\Omega, \mathcal{F})$ is called a super-multiplicative functional of X if

i) $S_t \in \mathcal{F}_t$ for any $t \geqslant 0$.

ii) $S_{s+t} \geqslant S_t \cdot (S_s \circ \theta_t) = S_s \cdot (S_t \circ \theta_s)$ a.s. for any t, $s \geqslant 0$.

iii) $0 \leqslant S_t(\omega) \leqslant 1$ for all t and ω.

Proposition 3.1. Let N_t be a multiplicative functional of X and $\{M_t\}$ a sub-multiplicative functional of X such that $M_t \leqslant N_t$ for any $t \geqslant 0$ and

$$M_t \cdot N_s \circ \theta_t + N_t \cdot M_s \circ \theta_t = M_s \cdot N_t \circ \theta_s + N_s \cdot M_t \circ \theta_s$$

for all t, $s \geqslant 0$.

Then there exists a super-multiplicative functional of X, $\{S_t\}$, such that

$$N_t = M_t + S_t,$$

for any $t \geqslant 0$.

Proof. Let $S_t := N_t - M_t \geqslant 0$. $\{S_t\}$ verifies the conditions of def. 3.2. Indeed, i) and iii) are evident while ii) one checks through computation from the relationships of hypothesis.

As example, let the multiplicative functional $N_t = 1$ and let the sub-multiplicative functional $M_t = \exp(-t^2)$. Then $S_t = 1 - \exp(-t^2)$ is a super-multiplicative functional of X.

If M is a sub-multiplicative functional of X, we define for any $t \geqslant 0$ an operator Q_t on $b\mathcal{E}^*_+$ by

$$Q_t f(x) = E^x\{f(X_t) M_t\} .$$

Like at multiplicative functionals, Q_t is a positive linear operator from $b\mathcal{E}^*_+$ to $b\mathcal{E}^*_+$ such that $Q_t \leqslant P_t$, where P_t is the transition operator of X. We have:

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$$Q_{t+s} f(x) \leqslant Q_s Q_t f(x)$$
$$Q_s Q_t f(x) = Q_t Q_s f(x)$$
$$Q_t 1 \leqslant 1 .$$

It follows that $\{Q_t : t \geqslant 0\}$ is a sub-semigroup on $b\mathcal{E}^*_+$, called the sub-semigroup generated by M.

One observe that $Q_o f(x) = E^x\{f(X_o)M_o\}$. If X is normal then $Q_o f(x) = I_{E_M}(x) f(x)$. If E_Δ is a metric space, X is right continuous and M is right continuous, then $Q_t 1(x) = E^x\{M_t ; X_t \in E\}$ tends to $Q_o 1(x) = E^x\{M_o ; X_o \in E\}$ as $t \to 0$.

We make the notation $B = b\mathcal{E}^*$. B is a Banach space under the supremum norm. We have $P_t B \subset B$ for any $t \geqslant 0$ ([1]). The sub-semigroup $\{Q_t : t \geqslant 0\}$ of nonnegative linear operators on B_+ is called subordinate to $\{P_t\}$ if $Q_t f \leqslant P_t f$ for any $t \geqslant 0$ and $f \in B_+$. In the sequel, if $M = \{M_t\}$ is a right continuous sub-multiplicative functional of X, then

$$Q_t f(x) = E^x\{f(X_t)M_t\}$$

and

$$V^\alpha f(x) = E^x \int_0^\infty e^{-\alpha t} f(X_t) M_t dt$$

will be the sub-semigroup, resp. the sub-resolvent corresponding to M on B_+. If we denote by $\{P_t\}$ and $\{U^\alpha\}$ the semigroup and respective the resolvent of process X then $\{Q_t\}$ is subordinate to $\{P_t\}$ while $\{V^\alpha\}$ is subordinate to $\{U^\alpha\}$ on B_+.

Definition 3.3. Let $(E, \mathcal{E})$ be a measurable space. The function $P_t(x, A)$ defined for $t \geqslant 0$, $x \in E$, $A \in \mathcal{E}$ is called a semi-transition function on $(E, \mathcal{E})$ provided

i) $A \to P_t(x, A)$ is a probability measure on $\mathcal{E}$ for any t and x.

ii) $x \to P_t(x, A)$ is in $\mathcal{E}$ for each t and A

iii) $P_{t+s}(x,A) \leqslant \int P_t(x, dy) P_s(y, A) = \int P_s(x,dy) P_t(y, A)$

for all t, x and A.

We shall call semi-Markov process a process which satisfies the conditions from the definition of a Markov process in the meaning of [1] , excepting the Axiom M (Markov property) which one replace by the following condition:

$$E^x\{f \circ X_{t+s};\Lambda\} \leqslant E^x\{E^{X_t}(f\circ X_s);\Lambda\} = E^x\{E^{X_s}(f\circ X_t);\Lambda\}$$

for all x, t, s, $f \in b\mathcal{E}_+$ and $\Lambda \in \mathcal{M}_t$.

Proposition 3.2. For the semi-Markov process X, define $N_t(x, A) = P^x(X_t \in A)$, $x \in E_\Delta$, $A \in \mathcal{E}_\Delta$. Then $N_t(x, A)$, $0 < t < \infty$ is a semi-transition function for the process $\{X_t\}$ over $(\Omega, \mathcal{M}, P^x)$ with values in $(E_\Delta, \mathcal{E}_\Delta)$.

The proof is immediate from the conditions of def. 3.3.

Definition 3.4. The semi-Markov process X is called strong semi-Markov provided that for each stopping time T with respect to $\{\mathcal{M}_t\}$ and $f \in b\mathcal{E}_+$ one has:

i) $X_T \in \mathcal{M}_T/\mathcal{E}^*_\Delta$

ii) $E^x f(X_{t+T}) \leqslant E^x\{E^{X(T)}[f(X_t)]\}$.

Let $X = (\Omega, \mathcal{M}, \mathcal{M}_t, X_t, \theta_t, P^x)$ be. A semi-Markov process Y with state space $(E_0, \mathcal{E}^*_0)$ where $E_0 \in \mathcal{E}^*$, $\mathcal{E}^*_0 = \mathcal{E}^*|_{E_0}$ is called a semi-subprocess of X if its transition sub-semi-group $\{Q_t\}$ is dominated by $\{P_t\}$: $Q_t f(x) \leqslant P_t f(x)$ for all $x \in E_0$, $f \geqslant 0$, $f \in b\mathcal{E}^*_0$ and f is taken to vanish on $E \setminus E_0$. We define $\overline{Q}_t$ on $B_+ = b\mathcal{E}^*_+$ by setting

$$\overline{Q}_t f(x) = \begin{cases} Q_t f\big|_{E_0}(x) & \text{if } x \in E_0 \\ 0 & \text{if } x \in E \setminus E_0 \end{cases}$$

where $f \in B_+$. Then $\{\overline{Q}_t\}$ is a sub-semi-group of nonnegative linear operators on B_+ and $\overline{Q}_t = Q_t$ on $b\mathcal{E}^*_0$. Y is semi-subprocess of X if and only if $\{\overline{Q}_t\}$ is subordinate to $\{P_t\}$.

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Let X be normal and let M be a right continuous sub-multiplicative functional of X. If X satisfies this conditions, one can construct a semi-subprocess of X whose transition sub-semigroup is generated of M. Indeed, we observe that the construction for subprocesses from [1], III.3. one can make as well in our case. We denote with $\hat{X} = (\hat{\Omega}, \hat{\mathcal{M}}, \hat{\mathcal{M}}_t, \hat{X}_t, \hat{\theta}_t, \hat{P}^x)$ the process constructed. We have the following result:

Theorem 3.1. $\hat{X}$ is a semi-Markov process with state space $(E, \mathcal{E}^*)$ such that

$$\hat{E}^x\{f(\hat{X}_t)\} = E^x\{f(X_t)M_t\}$$

for any $f \in B$, that is $\hat{X}$ is a semi-subprocess of X whose transition sub-semigroup is generated by M.

The proof is that from [1] for subprocesses, with the mention that in the place of Markov property for $\hat{X}$ one verifies the property of $\hat{X}$ concerning to be semi-Markov. This can be shown using the fact that M is a sub-multiplicative functional of X.

We now suppose that X is strong Markov, M is a right continuous sub-multiplicative functional of X and let $\hat{X}$ be the semi-subprocess of X constructed above, corresponding to M.

Definition 3.5. Let M be a sub-multiplicative functional of X. M is called strong sub-multiplicative if M is right continuous and

$$E^x\{f(X_{t+T})M_{t+T}\} \leq E^x\{E^{X(T)}[f(X_t)M_t]\, M_T\}$$

for all x, t, $f \in b\mathcal{E}_+$, $\{\mathcal{M}_t\}$ stopping times T.

The following proposition is analogous of the prop. (3.12) from [1] .

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Proposition 3.3. Let X be a strong Markov process and let M be a strong sub-multiplicative functional of X. Then the semi-subprocess $\hat{X}$ corresponding to M is strong semi-Markov.

From now on, X will be a standard process with state space $(E, \mathcal{E})$.

Proposition 3.4. Let M be a right continuous sub-multiplicative functional of X and let $\{Q_t\}$, $\{V^\alpha\}$ be the sub-semigroup and the sub-resolvent corresponding to M. Let $\{W^\alpha\}$ be the exactly subordinate resolvent corresponding to $\{V^\alpha\}$ in theorem 2.1. Then:

$$V^\alpha(x,.) = W^\alpha(x,.)$$

for any $x \in E_M \cap E_V$.

Proof. Since $x \in E_M$ it follows that $Q_t 1(x) \longrightarrow 1$ as $t \longrightarrow 0$. Therefore $\beta V^\beta 1(x) \longrightarrow 1$ as $\beta \longrightarrow \infty$. The conclusion result from the theorem 2.1.

Proposition 3.5. Let $M = \{M_t\}$ be a right continuous sub-multiplicative functional of X such that

$$P^x\left[X_T \in E \setminus (E_M \cap E_V);\ M_T > 0\right] = 0$$

for all x and $\{\mathcal{M}_t\}$ stopping times T. Then

$$E^x \int_0^\infty e^{-\alpha t} f(X_{t+T}) M_{t+T}\, dt \leqslant E^x\left\{E^{X(T)}\left[\int_0^\infty e^{-\alpha t} f(X_t) M_t\, dt\right] M_T\right\}$$

for all x, t, T and $f \in b\mathcal{E}_+$.

Proof. Let $\{V^\alpha\}$ denote the corresponding sub-resolvent to M. Let $\{W^\alpha\}$ be the exactly subordinate sub-resolvent associated with $\{V^\alpha\}$ in theorem 2.1. For each continuous and positive f and $\alpha > 0$, the map $W^\alpha f = U^\alpha f - (U^\alpha f - W^\alpha f)$ is bounded, nearly Borel measurable, finely continuous and equal with $V^\alpha f$ on $E_M \cap E_V$ (according to prop. 3.4.). The proof follows as the proof of the theorem (4.12) from [1], III, using the fact that M is sub-multiplicative.

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For M strong sub-multiplicative functional the result (4.16) of [1] , Chap. III becomes:

Let T an $\{\mathcal{M}_t\}$ stopping time, $Y \in b\mathcal{F}$ and $R \in \mathcal{F}$, $R \geqslant 0$.

Then

$$E^x \{ (Y \circ \theta_T) M(T + R \circ \theta_T);\ A \} \leqslant E^x \{ E^{X(T)} (Y\ M_R) M_T;\ A \}$$

for all $A \in \mathcal{M}_T$ and x.

Using this result we obtain the following proposition, similar to prop. (4.21).

Proposition 3.6.

Let M be a strong sub-multiplicative functional of X.Then

$$P^x\ (X_T \in E \setminus E_M;\ M_T > 0\ ) = 0$$

for any x and for any $\{\mathcal{M}_t\}$ stopping time T.

The proof is that from [1] observing that is sufficient to use only the fact that M is strong sub-multiplicative. If $R = \inf \{ t : M_T = 0 \}$ then $M_R = 0$ almost surely from the right continuity of M. Let T be any $\{\mathcal{M}_t\}$ stopping time. Then, using the above result, we have:

$$E^x \{ M(T + R \circ \theta_T);\ T < \zeta \} \leqslant$$

$$\leqslant E^x \{ E^{X(T)} [M_R]\ M_T;\ T < \zeta \}$$

Since the right side of the inequality is low, it follows that

$$E^x \{ M(T + R \circ \theta_T);\ T < \zeta \} = 0.$$

Consequently $T + R \circ \theta_T \geqslant R$ almost surely on $\{ T < \zeta \}$ and so

$$P^x (X_T \in E \setminus E_M;\ M_T > 0\ ) = 0$$

Remark.

The propositions 3.5 and 3.6 give for the sub-multiplicative functionals, the analogous of the following result: any multiplicative functional is regular if and only if it is strong multiplicative.

On the other hand a right continuous sub-multiplicative functional M of which the corresponding sub-resolvent $\{V^{\alpha}\}$ is exactly subordinate to the resolvent $\{U^{\alpha}\}$ satisfies the relationship from the proposition 3.5.

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