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§o. Introduction

Let X be a complex space and $A\subset X$ a closed subset. A is called complete locally pluripolar [4] if for any $x_0\in A$ there is an open neighbourhood U of x_0 and a plurisubharmonic function $\varphi:U\rightarrow[-\infty,\infty)$ such that $A\cap U=\{\varphi=-\infty\}$. Clearly any closed analytic subset of X is complete locally pluripolar(see [4] for other non-trivial examples).

In this note we consider the problem of the existence of q-complete open neighbourhoods for a complete locally pluripolar set $A\subset X$.

Definition 1. A closed subset A of a complex space X is called q-complete if there exists a smooth strongly q-convex function φ on a neighbourhood of A such that $\varphi|_A$ is an exhaustion function.

For example it is known [8] that a finite dimensional q-complete complex subspace A of X (in the sense of Andreotti and Grauert [1]) is a q-complete set.

We can now state our main result :

Theorem 1. Let X be a complex space and $A\subset X$ a complete locally pluripolar set. If A is q-complete then A has a fundamental system of q-complete open neighbourhoods.

As a direct consequence we obtain :

Corollary 1. If A is a q-complete finite dimensional complex subspace of X then A has a fundamental system of q-complete open neighbourhoods.

When q=1 the above corollary was proved by Siu [1o].

Corollary 2. Let X be a q-complete space and $A\subset X$ a complete locally pluripolar subset. Then A has a fundamental system of q-complete open neighbourhoods.

We now briefly discuss the proof of Theorem 1.

When $A\subset X$ is complete locally pluripolar the complement $Y=X\setminus A$ is "locally hyperconcave" i.e. for any $x_0\in\partial Y=A$ there is a neighbourhood U of x_0 and a plurisubharmonic function $\varphi: U\cap Y\to\mathbb{R}$ such that $\varphi(z)\to-\infty$ if $z\to z'\in\partial Y$. In [2] we have considered "locally hyperconvex" open subsets on complex spaces. The method developed there, especially the technique of patching up plurisubharmonic functions with bounded differences, turns out to be useful also in this "hyperconcave case" to patch up strongly q-convex functions defined locally and get global ones.

§1. Proof of the main result

We shall need the following :

Definition 2. Let X be a complex space, φ a smooth strongly q-convex function on X, $x\in X$ any point and $\{\varphi_i\}_{i\in I}$ smooth functions defined in a neighbourhood U of x. We say that $\{\varphi_i\}_{i\in I}$ have the same positivity directions as φ at x if there exist an open neighbourhood V of x, $V\subset U$, an embedding $V\hookrightarrow\hat{V}\subset\mathbb{C}^N$, smooth extensions $\hat{\varphi}$ of φ, $\hat{\varphi}_i$ of φ_i on $\hat{V}$ and a vector space $E\subset\mathbb{C}^N$, $\dim E=N+q-1$, such that the Levi forms $L(\hat{\varphi};z)\big|_{E(z)}$, $L(\hat{\varphi}_i;z)\big|_{E(z)}$ $z\in\hat{V}$ are positive definite where $E(z)\subset T_z\mathbb{C}^N$ is parallel to E.

Let now $\{U_\lambda\}_{\lambda\in L}$ be open subsets of X such that $\{U_\lambda\}_{\lambda\in L}$ is locally finite on X and let $\varphi_\lambda: U_\lambda\to\mathbb{R}$ be continuous functions such that the following conditions are satisfied :

1) $U_\lambda=\bigcup_{j\in L_\lambda}U_{\lambda,j}$ where $U_{\lambda,j}$ are open subsets of U_λ and $\{U_{\lambda,j}\}_{j\in L_\lambda}$ is locally finite on U_λ

2) For each $U_{\lambda,j}$ there exist finitely many smooth functions $\varphi^k_{\lambda,j}$ $k=1,\dots,k_{\lambda,j}$ such that $\varphi_\lambda\big|U_{\lambda,j}=\max_k\varphi^k_{\lambda,j}$

3) Let $x\in\bigcup_{\lambda\in L}U_\lambda$ be any point. By 1) and 2) there exist only finitely many functions $\varphi^k_{\lambda,j}$ whose domain of definition $U_{\lambda,j}$ contains x. We assume that all these functions have the same positivity directions as φ at x.

If $\{(U_\lambda,\varphi_\lambda)\}_{\lambda\in L}$ satisfy conditions 1), 2), 3) we write

$\{\varphi_\lambda\}_{\lambda\in L}\in P_\varphi(\{U_\lambda\}_{\lambda\in L};X)$.

The following lemma is the analoguous of Lemma 3 in [2] in the "hyperconcave case".Although the ideas are almost the same we shall give the details of proof for the sake of completeness.

Lemma 1. Let X be a complex space, $A\subset X$ a complete locally pluripolar set and assume that there exists a smooth strongly q-convex function φ on X.Let $\{K_\nu\}_{\nu\in\mathbb{N}}$ be compact subsets of $A, K_\nu\subset K_{\nu+1}$ $\nu\in\mathbb{N}$ and $A=\bigcup_{\nu\in\mathbb{N}}K_\nu$.Then there exist :

1) Open subsets U_ν of X with $K_\nu\subset U_\nu$ and $\{U_\nu\setminus A\}_{\nu\in\mathbb{N}}$ is locally finite on $X\setminus A$.
2) functions $\psi_\nu:U_\nu\to[-\infty,\infty)$ with $\exp\psi_\nu$ continuous, $A\cap U_\nu=\{\psi_\nu=-\infty\}$, $\psi_i-\psi_j$ are bounded on $U_i\cap U_j\setminus A$ for any $i,j\in\mathbb{N}$ and $\{\psi_\nu|_{U_\nu\setminus A}\}_{\nu\in\mathbb{N}}\in$ $\in P_\varphi(\{U_\nu\setminus A\}_{\nu\in\mathbb{N}};X\setminus A)$.

Proof

Step 1 We assume first that A can be defined by $\{(D_i,\varphi_i)\}_{i\in\mathbb{N}}$ with $D_i\subset\subset X$, $\{D_i\}_{i\in\mathbb{N}}$ is locally finite, $A\subset\bigcup_{i\in\mathbb{N}}D_i$, $\varphi_i:D_i\to[-\infty,\infty)$ are plurisubharmonic functions, $A\cap D_i=\{\varphi_i=-\infty\}$, $\exp\varphi_i$ is continuous, φ_i is smooth outside A and $\varphi_i-\varphi_j$ are bounded on $D_i\cap D_j\setminus A$ (we shall see at the next step that this is always the case ;when A is analytic this is obvious since one can take $\varphi_i=\log(|f_{i,1}|^2+\ldots+|f_{i,k_i}|^2)$ where $f_{i,1},\ldots,f_{i,k_i}$ generate the ideal $\mathcal{J}$ of A on D_i and D_i are small Stein open sets).

We fix $\nu\in\mathbb{N}$, set $K:=K_\nu$ and we show how we can construct $U:=U_\nu$ and $\psi:=\psi_\nu$ (it will follow from our construction $\psi_i-\psi_j$ are bounded on $U_i\cap U_j\setminus A$ and $\{\psi_\nu|_{U_\nu\setminus A}\}_{\nu\in\mathbb{N}}\in P_\varphi(\{U_\nu\setminus A\}_{\nu\in\mathbb{N}};X\setminus A)$).

Let $D_1,\ldots,D_m$ be open subsets, $D_i\subset\subset X$, $K\subset\bigcup_{i=1}^m D_i$ and $\varphi_i:D_i\to[-\infty,\infty)$ plurisubharmonic functions with $\exp\varphi_i$ continuous, φ_i smooth on $D_i\setminus A$, $A\cap D_i=\{\varphi_i=-\infty\}$ and $\varphi_i-\varphi_j$ are bounded on $D_i\cap D_j\setminus A$ $i,j\in\{1,\ldots,m\}$ (they are chosen from the given $\{(D_i,\varphi_i)\}_{i\in\mathbb{N}}$).Choose open subsets $D_i''\subset\subset D_i'\subset\subset D_i$ with $K\subset\bigcup_{i=1}^m D_i''$ and $p_i'\in C_o^\infty(X)$, $p_i'\geq o$, supp $p_i'\subset D_i'$, $p_i'|_{D_i''}=1$. Let V_i' be an open neighbourhood of $\partial D_i'$ such that $V_i'\subset D_i$ and $p_i'=o$

on V'_i. Since $\varphi_j-\varphi_i$ are bounded on $V'_j\cap D''_i\setminus A$ there is a large constant $\lambda_i>0$ with $\lambda_i p'_i>\varphi_j-\varphi_i$ on $V'_j\cap D''_i\setminus A$. For any $i=1,\dots,m$ we set $p_i=\lambda_i p'_i$. Because $p_j=0$ on V'_j we get (*) $p_i+\varphi_i\geq p_j+\varphi_j$ on $V'_j\cap D''_i$. Let $C:=C_\nu>0$ be a constant (to be chosen later large enough). We set $I=\{1,\dots,m\}$ and if $x\in U=\bigcup_{i=1}^{m}D''_i$ we put $I(x)=\{i\in I \mid x\in D'_i\}$. For any $x\in U$ we set $u(x)=\max_{i\in I(x)}\{p_i(x)+\varphi_i(x)\}$ and define $\psi:=C\varphi+u$. Let $x_0\in U$ be any point. If D_{x_0} is a sufficiently small neighbourhood of x_0 it follows from (*) that $u|_{D_{x_0}}=\max_{i\in I(x_0)}\{p_i+\varphi_i\}$ hence we get (**) $\psi|_{D_{x_0}}=\max_{i\in I(x_0)}\{\varphi_i+C\varphi+p_i\}$. We set $\psi_\nu:=\psi$. Clearly $\exp\psi_\nu$ is continuous and $A\cap U_\nu=\{\psi_\nu=-\infty\}$. On the other hand $\psi_i-\psi_j$ are bounded on $U_i\cap U_j\setminus A$. Indeed the defining formula (**) for ψ_i involves the plurisubharmonic functions $\varphi_{i,1},\dots,\varphi_{i,k_i}$ (and other bounded functions on U_i) and similarly the defining formula for ψ_j involves the plurisubharmonic functions $\varphi_{j,1},\dots,\varphi_{j,m_j}$ (and other bounded functions on U_j). Since the differences $\varphi_{i,t_i}-\varphi_{j,s_j}$ are chosen from the beginning bounded outside A (the functions φ_{i,t_i}, φ_{j,s_j} are chosen from the initial given $\{(D_i,\varphi_i)\}_{i\in\mathbb{N}}$) it follows that $\psi_i-\psi_j$ are bounded on $U_i\cap U_j\setminus A$. By shrinking U_ν we may assume that $\{U_\nu\setminus A\}_{\nu\in\mathbb{N}}$ is locally finite on $X\setminus A$ so condition 1) is satisfied. Also shrinking U_ν if necessary we may assume that U_ν can be covered by finitely many domains such that on these domains ψ_ν is given by (**). Now we make the following remark : Let $(p_\lambda)_{\lambda\in\mathbb{N}}$ be functions in $C^\infty_0(X)$. Then there exist sufficiently large constants $a_\lambda>0$ such that $(p_\lambda+a_\lambda\varphi)_{\lambda\in\mathbb{N}}$ have the same positivity directions as φ at any point $x\in X$. Hence if the $C_\nu>0$ defining ψ_ν are sufficiently large $\{\psi_\nu|_{U_\nu\setminus A}\}_{\nu\in\mathbb{N}}\in P_\varphi(\{U_\nu\setminus A\}_{\nu\in\mathbb{N}};X\setminus A)$ because all functions φ_i are smooth and plurisubharmonic outside A.

Step 2 We prove now the existence of $\{(D_i,\varphi_i)\}_{i\in\mathbb{N}}$ with the properties stated at step 1. We first remark that by a result in ([4],p.19) φ_i may be assumed smooth outside A, $\exp\varphi_i$ continuous and $A\cap D_i=\{\varphi_i=-\infty\}$. To get the condition of boundness we shall modify the functions φ_i by composing them with a suitable smooth increasing convex function τ. To show the existence of τ we follow the method in [2].

Statement A (analoguous to lemma 1 in [2]). Let $\{a_i\}_{i\in\mathbb{N}}$ be a sequence of negative real numbers with $a_i\downarrow-\infty$. Then there exists a smooth increasing (convex) function $\tau:(-\infty,0)\to(-\infty,0)$ such that $\lim_{i\to\infty}\tau(a_i)=-\infty$ and $\tau(a_i)-\tau(a_{i+1})<1$ for any $i\in\mathbb{N}$. To see this we define

$$\tau(t)=\begin{cases}\left(\frac{a_1}{a_2}+\dots\frac{a_i}{a_{i+1}}\right)-i-\frac{t}{a_{i+1}} & a_{i+1}\le t\le a_i \quad i\ge 1\\ \frac{a_1}{a_2}-\frac{t}{a_2}-1 & a_1\le t<0\end{cases}$$

Clearly τ satisfies all the required conditions except the smoothness. However τ may easily be smoothed.

Statement B Let $f_n:(-\infty,0)\to(-\infty,0)$ $n\in\mathbb{N}$ be increasing functions such that $\lim_{t\to-\infty}f_n(t)=-\infty$ for any $n\in\mathbb{N}$. Then there exists a smooth increasing convex function $\tau:(-\infty,0)\to(-\infty,0)$ with $\lim_{t\to-\infty}\tau(t)=-\infty$ and $\tau\circ f_n-\tau\circ f_m$ is bounded for any $n,m\in\mathbb{N}$. This may be proved as follows (see [7]): Choose a sequence $\{\lambda_i\}_{i\in\mathbb{N}}$ of negative numbers with $\lambda_i\downarrow-\infty$ and $\min\{f_n(\lambda_i)\mid n\le i\}>\max\{f_n(\lambda_{i+1})\mid n\le i+1\}$ for any $i\in\mathbb{N}$. If we define $a_i=\min\{f_n(\lambda_i)\mid n\le i\}$ for odd i and $a_i=\max\{f_n(\lambda_i)\mid n\le i\}$ for even i then $a_i\downarrow-\infty$. If τ is the function constructed at statement A corresponding to $\{a_i\}_{i\in\mathbb{N}}$ then $\tau\circ f_n-\tau\circ f_m$ is bounded for any $m,n\in\mathbb{N}$.

We can now prove the existence of τ such that $\tau\circ\varphi_i-\tau\circ\varphi_j$ are bounded for any $i,j\in\mathbb{N}$. Let $\{U_i\}_{i\in\mathbb{N}}$, $\{V_i\}_{i\in\mathbb{N}}$ be locally finite open coverings of A with $V_i\subset\subset U_i\subset\subset X$, $\varphi_i:U_i\to[-\infty,0)$ plurisubharmonic functions, $A\cap U_i=\{\varphi_i=-\infty\}$, $\exp\varphi_i$ continuous and φ_i smooth outside A. For any i,j such that $V_i\cap V_j\ne\emptyset$ we set $B_{ij}(t)=\sup\{\varphi_i(x)\mid x\in V_i\cap V_j\ \varphi_j(x)\le t\}$. B_{ij} is increasing and $\lim_{t\to-\infty}B_{ij}(t)=-\infty$. By statement B there is a smooth increasing convex function τ with $\lim_{t\to-\infty}\tau(t)=-\infty$ and $\tau\circ B_{ij}-\tau$ is bounded for any $i,j\in\mathbb{N}$. If $x\in V_i\cap V_j\setminus A$ $B_{ij}(\varphi_j(x))\ge\varphi_i(x)$ hence $\tau(\varphi_i(x))-\tau(\varphi_j(x))\le\tau(B_{ij}(\varphi_j(x)))-\tau(\varphi_j(x))=$bounded. The proof of Lemma 1 is complete.

Remark 1. Let X be a complex space, $\varphi: X\to\mathbb{R}$ a smooth strongly q-convex function and $\psi: X\to\mathbb{R}$ a function in $P_\varphi(\{X\};X)$. We want to approximate ψ by a smooth strongly q-convex function ψ_1 such that ψ_1 has the same positivity directions as φ.

We choose locally finite open coverings of X $\{V_i\}_{i\in I}$, $\{U_i\}_{i\in I}$, $V_i\subset\subset U_i\subset\subset X$ and smooth functions $\{\rho_i\}_{i\in I}$ defined near $\overline{U}_i$ such that the following conditions are satisfied :

1) $|\psi-\rho_i|<\varepsilon_i$ on U_i where $\varepsilon_i>o$ is any given sequence of positive numbers.

2) $\rho_i>\psi$ on V_i and $\rho_i<\psi$ on ∂U_i

3) let $x\in X$ be any point and $U_{i_1},\dots,U_{i_k}$ the open sets of the covering $\{U_i\}_{i\in I}$ which contain x. Then there exist a neighbourhood V of x, $V\subset U_{i_1}\cap\dots\cap U_{i_k}$, an embedding $V\hookrightarrow\hat{V}\subset\mathbb{C}^N$, a vector space $E\subset\mathbb{C}^N$ with dim $E=N+q-1$ and on $\hat{V}$ smooth extensions $\hat{\varphi}$ of φ, $\hat{\rho}_{i_1},\dots,\hat{\rho}_{i_k}$ of $\rho_{i_1},\dots,\rho_{i_k}$ such that the Levi forms $L(\hat{\varphi};z)|_{E(z)}, L(\hat{\rho}_{i_1};z)|_{E(z)}, \dots, L(\hat{\rho}_{i_k};z)|_{E(z)}$, $z\in\hat{V}$ are positive definite, where $E(z)\subset T_z\mathbb{C}^N$ is parallel to E.

The existence of $\{\rho_i\}_{i\in I}$ follows easily by a perturbation argument with smooth functions which are positive on V_i and negative on ∂U_i and using locally max-regularization.

If we set $\rho(x)=\sup\limits_{x\in U_i}\rho_i(x)$ then ρ is a continuous function (near a given point x_o it follows from 2) that $\rho(x)=\sup\limits_{i\in I(x_o)}\rho_i(x)$ where $I(x_o)=\{i\in I|\,x_o\in U_i\}$), $|\rho-\psi|<\eta$ where η is any given continuous positive function on X, (*) $\rho|_{\partial U_i}>\rho_i|_{\partial U_i}$ and ρ has property 3).

Under these hypothesis the approximation of ρ by ψ_1 follows from the patching technique developed by Richberg in [9] (as remarqued by Diederich and Fornaess in [3])

However in our case we can give a direct argument as follows :

Let $\lambda:\mathbb{R}\to\mathbb{R}_+$ be a smooth function, supp$\lambda\subset[-1,1]$, $\int_{\mathbb{R}}\lambda(t)dt=1$ and let $\varepsilon: X\to(o,\infty)$ be a smooth function.

For any $x_o \in X$ we set $I(x_o)=\{i_1,\dots,i_k\}$, $k=k(x_o)$ and define

$$\psi_1(x_o)=\int_{\mathbb{R}^k}\max(\rho_{i_1}(x_o)-\varepsilon(x_o)t_1,\dots,\rho_{i_k}(x_o)-\varepsilon(x_o)t_k)\lambda(t_1)\dots\lambda(t_k)dt_1\dots dt_k .$$

From (*) and the properties of λ it follows that in a neighbourhood of x_o we have

$$\psi_1(x)=\int_{\mathbb{R}^k}\max(\rho_{i_1}(x)-\varepsilon(x)t_1,\dots,\rho_{i_k}(x)-\varepsilon(x)t_k)\lambda(t_1)\dots\lambda(t_k)dt_1\dots dt_k$$

if $\varepsilon(x)$ is small enough, hence for small ε ψ_1 is smooth.
Locally ψ_1 can be extended to $\hat{\psi}_1$ given by :

$$\hat{\psi}_1(z)=\int_{\mathbb{R}^k}\max(\hat{\rho}_{i_1}(z)-\hat{\varepsilon}(z)t_1,\dots,\hat{\rho}_{i_k}(z)-\hat{\varepsilon}(z)t_k)\lambda(t_1)\dots\lambda(t_k)dt_1\dots dt_k$$

where "$\wedge$" denotes local extension. Consequently, if the second derivatives of ε are sufficiently small (in some fixed local embeddings which satisfy 3)) ψ_1 is a smooth strongly q-convex function with the same positivity directions as φ , which is the desired approximation of ρ .

We are now in a position to prove :

Theorem 1. Let X be a complex space and $A\subset X$ a complete locally pluripolar set. If A is q-complete then A has a fundamental system of q-complete open neighbourhoods.

Proof

We may assume that there exists a smooth strongly q-convex function $\varphi > o$ on X such that $\varphi|_A$ is an exhaustion function.
Step 1 We show that there exist a neighbourhood U of A and a smooth strongly q-convex function $\tilde{\psi}$ on $U\setminus A$, with the same positivity directions as φ , and such that $\tilde{\psi}(x)\to -\infty$ if $x\to x_o\in A$.
To obtain $\tilde{\psi}$ we use Lemma 1 and a patching technique essentially due to Stehlé [11].

Let V be an open neighbourhood of A such that $\{x\in V \mid \varphi(x)<\nu\}\subset\subset X$ for any $\nu\in\mathbb{N}$ and let $\lambda : X\to\mathbb{R}_+$ be a continuous function with $A=\{\lambda=o\}$. Choose a sequence $\varepsilon_\nu\downarrow o$ such that $P_\nu=\{x\in V \mid \varphi(x)<\nu \;\; \lambda(x)<\varepsilon_\nu\}\subset\subset X$.

If ε_ν decreases rapidily to zero form Lemma 1 we may assume that there exist functions $\psi_\nu : P_\nu \to [-\infty,\infty)$ with $\exp \psi_\nu$ continuous, $A\cap P_\nu = \{\psi_\nu = -\infty\}$, $\psi_i - \psi_j$ is bounded on $P_i \cap P_j \setminus A$ for any $i,j \in \mathbb{N}$ and $\{\psi_\nu|_{P_\nu \setminus A}\}_{\nu\in\mathbb{N}} \in P_\varphi(\{P_\nu \setminus A\}_{\nu\in\mathbb{N}}; V\setminus A)$. Let $\alpha_\nu > 0$ be numbers such that $-\alpha_\nu < \psi_\nu - \psi_{\nu+1} < \alpha_\nu$ on $P_\nu \cap P_{\nu+1} \setminus A$ and choose constants $c_\nu > 0$ such that $c_\nu > 12\alpha_\nu + 13c_{\nu-1}$ for any $\nu \in \mathbb{N}$.

We set $U := \bigcup_{\nu\in\mathbb{N}} \{x \in V \mid \varphi(x) < \nu - \frac{1}{2} \quad \lambda(x) < \varepsilon_\nu\}$ and $h_\nu = c_\nu\left[\varphi - (\nu - \frac{1}{3})\right]$.

We define the function ψ on U by :

$$\psi(x) = \begin{cases} \psi_1(x) & \text{if } \varphi(x) \le \frac{1}{2} \\ \max(\psi_\nu(x) + h_{\nu-1}(x), \psi_{\nu+1}(x) + h_\nu(x)) & \text{if } \nu - \frac{1}{2} \le \varphi(x) \le \nu - \frac{1}{4} \quad \nu \ge 1 \\ \psi_{\nu+1}(x) + h_\nu(x) & \text{if } \nu - \frac{1}{4} \le \varphi(x) \le \nu + \frac{1}{2} \quad \nu \ge 1 \end{cases}$$

By the inequalities $c_\nu > 12\alpha_\nu + 13c_{\nu-1}$ ψ is well defined. Since $\{\psi_\nu|_{P_\nu\setminus A}\}_{\nu\in\mathbb{N}} \in P_\varphi(\{P_\nu \setminus A\}_{\nu\in\mathbb{N}}; V\setminus A)$ it follows from the definition of ψ that $\psi|_{U\setminus A} \in P_\varphi(\{U\setminus A\}; U\setminus A)$. By remark 1 there exists a smooth strongly q-convex function $\tilde\psi$ on $U\setminus A$ such that $|\tilde\psi - \psi| < 1$ and $\tilde\psi$ has the same positivity directions as φ. Clearly $\tilde\psi(x) \to -\infty$ when $x \to x_0 \in A$ (because ψ has this property) so $\tilde\psi$ has the required properties.

Step 2 We prove now Theorem 1. By step 1 we may assume that there exists a function $\tilde\psi : X \to [-\infty,\infty)$, $A = \{\tilde\psi = -\infty\}$, $\exp\tilde\psi$ is continuous and $\tilde\psi|_{X\setminus A}$ is smooth strongly q-pseudoconvex with the same positivity directions as φ. Let V be an open neighbourhood of A such that $\{x\in V \mid \varphi(x) < n\} \subset\subset X$ for any $n \in \mathbb{N}$. Choose real numbers $\lambda_n \downarrow -\infty$ with $\{x \in V \mid \varphi(x) < n \quad \tilde\psi(x) < \lambda_n\} \subset\subset V$ for any $n\in\mathbb{N}$ and let $\lambda : [0,\infty) \to \mathbb{R}$ be a smooth increasing convex function such that $\lambda|_{[n-1,n]} \ge -\lambda_n$ $n \in \mathbb{N}$. We define $D := \{x \in V \mid \tilde\psi(x) + \lambda(\varphi(x)) < 0\}$ and we show that D is q-complete. Let $\tau : (-\infty, 0) \to \mathbb{R}_+$ be a increasing smooth convex function such that τ vanishes near $-\infty$ and $\lim_{t\to 0} \tau(t) = +\infty$. Then $\varphi + \tau\circ(\tilde\psi + \lambda\circ\varphi)$ is a smooth strongly q-convex exhaustion function on D (because $\tilde\psi$ has the same positivity directions as φ) hence D is q-complete.

Remark 2. It follows from the proof of Theorem 1 and Lemma 1 that for a given q-complete space X and a complete locally pluripolar set $A \subset X$ there exists a smooth strongly q-convex function ψ on $X \setminus A$ such that $\psi(x) \to -\infty$ when $x \to x_0 \in A$. When X is Stein this means that a complete locally pluripolar set A is complete globally pluripolar (when X is an arbitrary domain of $\mathbb{C}^n$ such a result does not hold [4]). A result of this type for pluripolar sets (i.e. locally contained in $\{\psi_i = -\infty\}$) is proved in [6] (in this case $A \subset \{\psi = -\infty\}$).

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