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THE AUTOMORPHISM GROUP OF A FREE PRODUCT OF GROUPS AND SIMPLE C^*-ALGEBRAS

V. Niţică, A. Török

1. Introduction

The aim of this paper is to introduce a new class of discrete groups G for which the reduced C^*-algebra $C^*_r(G)$ is simple with unique trace.

Powers ([P]) showed that the reduced C^*-algebra $C^*_r(F_2)$ of the free nonabelian group on two generators in simple with unique trace. His result was generalized by Choi ([C]), for the free product $Z_2 * Z_3$, and by Paschke and Salinas ([P-S]) for a free product $G_1 * G_2$, where G_1 has at least two elements and G_2 at least three elements. De la Harpe ([H]) introduced the class of Powers groups, which contains all the previous examples. He showed that the Powers groups have the reduced C^*-algebra simple with unique trace. In [B-N], the results of de la Harpe are generalized for a larger class of groups, called weak Powers groups.

Another direction in looking for discrete groups for which the reduced C^*-algebras are simple with unique trace is given by Theorem 3 in [A-L].

Theorem (Akemann and Lee). Let G be a discrete group which contains a normal free nonabelian subgroup with trivial centralizer. Then the reduced C^*-algebra of G is simple with unique trace.

The natural question, asked by de la Harpe ([H]), is the following.

Question. Can one replace the free group in the previous theorem by a (weak) Powers group?

In [N-T], it is proved that the answer is positive for some free products of indecomposable groups. (In this paper, a group is called indecomposable if it is not a free product of nontrivial groups.) Here we prove that one can replace the free group in the Theorem of Akemann and Lee by any proper free product (i.e. of nontrivial groups), which is not the infinite dihedral groups (i.e. $Z_2 * Z_2$). This can also be done in Corollary 5 of [A-L] which deals with a group that has as normal subgroups a family of nonabelian free groups such that the intersection of their centralizers is trivial.

The paper is divided in four sections. In § 2 we introduce some notations, we state the principal theorem and we show how the above described results can be obtained from it. In § 3 we prove some combinatorial results related to free products, and in § 4 there is the proof of the principal theorem.

The whole proof relies on an estimate of the norm of "free" convolutors. This was firstly obtained by Akemann and Ostrand ([A-O]). In order to use this estimate, one needs free families in a group. In [A-L], one encounters this problem in a free nonabelian group, where the Nielsen-Schreier Subgroup Theorem (namely that any subgroup of a free nonabelian group in free) stands in the background. Since in a free product the Kurosh Subgroup Theorem is not useful enough, we are obliged to give long arguments in order to prove that some families are free.

2. Notations and the principal results

We begin with some general notations. All groups are discrete.

In the following, let H be an arbitrary group. Then:

- By e we always denote the identity element of H. $H^* := H \setminus \{e\}$ and ord h stands for the order of the element $h \in H$;

- C[H] stands for the group algebra of H. We identify the elements of H and their image in the group algebra;

- For any $X = \sum_{h \in H} c_h h \in C[H]$ (the sum has finite support), denote

$\operatorname{supp} X := \{ h \in H \mid c_h \neq 0 \}$;

- $C^*_r(H)$ denotes the reduced C^*-algebras of H, that is the closure of C[H] in the norm we obtain regarding its elements as left convolutors on the Hilbert space $l^2(H)$. This norm will be denoted by $\| \cdot \|_H$. (Therefore there is an inclusion $C[H] \hookrightarrow C^*_r(H)$ with dense range). Note that the elements $h \in H$ become unitary elements of $C^*_r(H)$;

- $\tau : C^*_r(H) \to C$ is the canonical trace, i.e. the continuous extension of $\tau_o : C[H] \to C$, $\tau_o(\sum_{h \in H} c_h h) = c_e$;

- If $h, k \in H$, their commutator will be denoted by

$[h,k] = hkh^{-1}k^{-1}$

and $\mathrm{Ad}\, h \in \mathrm{Int}(H) \subset \mathrm{Aut}(H)$ is given by

$\mathrm{Ad}\, h(k) = hkh^{-1}$;

- Given a nonvoid set $M \subset H$, the subgroup it generates is denoted by $\langle M \rangle$ and its centralizer is

$Z_H(M) = \{ h \in H \mid [h,x] = e, \text{ for all } x \in M \}$;

- In order to simplify the notations, we shall write $\overline{h}$ instead of $\mathrm{Ad}\, h$, for all $h \in H$. Therefore there is a surjective morphism

$\overline{} : H \to \mathrm{Int}(H)$.

It is easy to check that its restriction to a subgroup $K \subset H$ is one-to-one if and only if $Z_H(K) = \{ e \}$.

- Given a subgroup $K \subset H$, by an averaging process (of $C^*_r(H)$) with elements of K we shall mean a C-linear map $\theta : C^*_r(H) \to C^*_r(H)$ given by:

(1) $\theta(h) = \frac{1}{n}\sum_{i=1}^{n} k_i h k_i^{-1}$, $h \in H$

where $\{k_i\}_{1\le i\le n} \subset K$ is a fixed set. (As a matter of fact, one defines by (1) θ only on C[H], but since

$$\|\theta(X)\|_H \le \|X\|_H \ , \text{ for all } X \in C[H],$$

θ can be extended by continuity to $C_r^*(H)$.)

It is easy to see that:

- if θ, θ' are averaging processes of $C_r^*(H)$ with elements of the subgroup $K \subset H$, then:

a) $\|\theta(X)\|_H \le \|X\|_H$, for all $X \in C_r^*(H)$;

b) $\theta \circ \theta'$ is also an averaging process with elements of K;

- if H_1, H_2 are groups and $H_1 \subset H_2$, then there is an isometric embedding $C_r^*(H_1) \hookrightarrow C_r^*(H_2)$.

We shall use the function sign : $R \longrightarrow \{-1,0,1\}$, given by

$$\operatorname{sign}(x) = \begin{cases} -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ 1, & \text{if } x > 0 . \end{cases}$$

We can now fix the setting. Let A and B be nontrivial groups, $\Gamma = A * B$ their free product, $G = \mathrm{Aut}(\Gamma)$, the group of automorphisms of Γ , and $\pi : A * B \to A \times B$ the canonical morphisms, with components π_1 and π_2. Denote $\mathrm{Ker}\,\pi = F$. It is known ([S], I, 1.3, Prop. 4) that F is a free group with basis

$$\{ [a,b] \mid a \in A^*, b \in B^* \} .$$

Therefore, if $A * B \ne Z_2 * Z_2$, then F is nonabelian.

For a nonvoid set $M \subset \Gamma$, we shall denote

$$\widetilde{M} := \bigcup_{w \in \Gamma} w M w^{-1}.$$

Since $Z_\Gamma(\Gamma) = \{e\}$, the map:

$$x \in \Gamma \mapsto \bar{x} \in \mathrm{Int}(\Gamma)$$

is an isomorphism.

The principal result, the proof of which will be given in § 4, is the following:

Theorem 1. Let A and B be nontrivial groups such that $A * B \neq Z_2 * Z_2$. Then for any finite nonvoid set $M \subset G^* = \mathrm{Aut}(A * B) \setminus \{ \mathrm{id}_{A*B} \}$ and any value $\varepsilon > 0$, there is an averaging process θ of $C_r^*(G)$ with elements of $\mathrm{Int}(A * B)$ such that

$$\| \theta(g) \|_G < \varepsilon \quad \text{for all } g \in M.$$

In the sequel, we shall prove the results announced in § 1.

We need the following lemma, the proof of which is standard ([H], Proposition 3):

Lemma 2. Let H be a group with the property that for any finite nonvoid set $M \subset H^*$ and any value $\varepsilon > 0$, there is an averaging process θ with elements of H such that:

$$\| \theta(h) \|_H < \varepsilon, \quad \text{for all } h \in M.$$

Then $C_r^*(H)$ is simple with unique trace. ∎

Corollary 3. Let A and B be nontrivial groups such that $A * B \neq Z_2 * Z_2$. If a (discret) group H contains $A * B$ as a normal subgroup will trivial centralizer, then $C_r^*(H)$ is simple with unique trace.

Proof. Since $\Gamma = A * B$ is normal in H, there is an inclusion:

(2) $\qquad \mathrm{Int}(\Gamma) \subset H' \subset \mathrm{Aut}(\Gamma)$

where $H' = \{ \mathrm{Ad}\, h_{|\Gamma} \mid h \in H \}$. Moreover H and H' are isomorphic because $Z_H(\Gamma) = \{ e \}$. Therefore, the problem reduces to prove that $C_r^*(H')$ is simple with unique trace for a group H' satisfying (2). But this follows easily by Theorem 1, due to Lemma 2. ∎

Remark. It is not hard to see that the groups H that satisfy the hypothesis of Corollary 3 are in one-to-one correspondence with the subgroups of $Out(A * B) := Aut(A * B)/Int(A * B)$.

Corollary 4. Denote by $\mathcal{F}$ the family of groups that are proper free products and are not equal to $Z_2 * Z_2$. Let H be a group having as normal subgroups a family $\{\Gamma_i\}_{i\in I}$ of groups (I any nonvoid set), such that $\Gamma_i \in \mathcal{F}$ for all $i \in I$ and

$$\bigcap_{i\in I} Z_H(\Gamma_i) = \{e\}.$$

Then $C_r^*(H)$ is simple with unique trace.

Proof. We shall prove that H satisfies the hypothesis of Lemma 2.

Note first the following

$$(3)\quad \left\{ \begin{array}{l} \text{for any } i \in I\text{, given a nonvoid finite set } M \subset H \setminus Z_H(\Gamma_i) \\ \text{and a value } \varepsilon > 0\text{, there is an averaging process } \Theta \text{ with elements} \\ \text{of } \Gamma_i \text{ such that} \\ \qquad \|\Theta(h)\|_H < \varepsilon \quad \text{for all } h \in M. \end{array} \right.$$

To see this, denote $M' = \{ Ad\, h_{|\Gamma_i} \mid h \in M\}$. Then $M' \subset Aut(\Gamma_i)$ because Γ_i is normal in H, and $id_{\Gamma_i} \notin M'$ because $M \cap Z_H(\Gamma_i) = \emptyset$. By Theorem 1 we obtain an averaging process Θ'

$$\Theta'(\alpha) = \frac{1}{n}\sum_{k=1}^{n} \overline{g}_k\, \alpha\, \overline{g}_k^{-1}, \quad \alpha \in Aut(\Gamma_i),$$

with $\{g_k\}_{k=1,2,\dots,n} \subset \Gamma_i$ such that

$$\|\Theta'(h')\|_{G_i} < \varepsilon, \text{ for all } h' \in M',$$

where $G_i = Aut(\Gamma_i)$.

Define the averaging process Θ of $C_r^*(H)$ with elements of Γ_i by

$$\Theta(h) = \frac{1}{n}\sum_{k=1}^{n} g_k\, h\, g_k^{-1}, \quad h \in H.$$

Since, for all $h \in H$,

$$\|\theta'(\mathrm{Ad}\,h|_{\Gamma_i})\|_{G_i} = \|\frac{1}{n}\sum_{k=1}^{n} \bar{g}_k (\mathrm{Ad}\,h|_{\Gamma_i}) \bar{g}_k^{-1}\|_{G_i} =$$

$$= \|[\frac{1}{n}\sum_{k=1}^{n} \overline{g_k\,\mathrm{Ad}\,h(g_k^{-1})}]\,(\mathrm{Ad}\,h|_{\Gamma_i})\|_{G_i} = \|\frac{1}{n}\sum_{k=1}^{n} \overline{g_k\,\mathrm{Ad}\,h(\bar{g}_k^{-1})}\|_{G_i} =$$

$$= \|\frac{1}{n}\sum_{k=1}^{n} g_k h g_k^{-1} h^{-1}\|_{\Gamma_i} = \|(\frac{1}{n}\sum_{k=1}^{n} g_k h g_k^{-1})h^{-1}\|_H = \|\frac{1}{n}\sum_{k=1}^{n} g_k h g_k^{-1}\|_H =$$

$$= \|\theta(h)\|_H,$$

we see that (3) holds.

Consider now a finite nonvoid set $M \subset H^*$ and a value $\varepsilon > 0$. There is a finite family $\{i_1,\dots,i_N\} \subset I$ such that

$$M \cap (\bigcap_{k=1}^{N} Z_H(\Gamma_{i_k})) = \emptyset.$$

Denote $M_n = M \setminus \bigcap_{k=1}^{n} Z_H(\Gamma_{i_k})$ for $n = 1,2,\dots,N$ and $M_0 = \emptyset$ (hence $M_N = M$). Deleting some indices i_k, we may assume that

$$M_{n+1} \setminus M_n \neq \emptyset, \quad \text{for } 0 \leq n < N.$$

We shall prove by induction on n, for $n = 1,2,\dots,N$ that:

$$(4)_n \qquad \left\{ \begin{array}{l} \text{there is an averaging process } \theta_n \text{ with elements of } \langle \bigcup_{k=1}^{n} \Gamma_{i_k} \rangle \\ \text{such that} \\ \qquad \|\theta_n(h)\|_H < \varepsilon, \quad \text{for all } h \in M_n. \end{array} \right.$$

This is true for $n=1$ due to (3). Assume $(4)_n$ is true for some n, $1 \leq n < N$. Since

$$(5) \qquad M_{n+1} \setminus M_n \subset \bigcap_{k=1}^{n} Z_H(\Gamma_{i_k}) \setminus Z_H(\Gamma_{i_{n+1}})$$

we obtain from (3) an averaging process $\theta_{(n+1)}$ with elements of $\Gamma_{i_{n+1}}$ such that

$$\|\theta_{(n+1)}(h)\|_H < \varepsilon, \quad \text{for all } h \in M_{n+1} \setminus M_n.$$

Define $\theta_{n+1} = \theta_{(n+1)} \circ \theta_n$, an averaging process with elements of $\langle \bigcup_{k=1}^{n+1} \Gamma_{i_k} \rangle$. Since

$$\|\theta_{n+1}(h)\|_H = \|\theta_{(n+1)}(\theta_n(h))\|_H \leq \|\theta_n(h)\|_H < \varepsilon, \quad \text{for all } h \in M_n$$

and

$$\theta_{n+1}(h) = \theta_{(n+1)}(h), \quad \text{for all } h \in M_{n+1} \setminus M_n,$$

(by (5), Θ_n is an averaging process with elements that commute with $M_{n+1} \setminus M_n$, hence $\Theta_{n|M_{n+1}\setminus M_n} = id_{M_{n+1}\setminus M_n}$), we see that $(4)_{n+1}$ holds.

Therefore $(4)_N$ is true and the corollary follows. ∎

3. Combinatorial lemmas

Let $\Gamma = A * B$ be as above a free product of nontrivial groups. Each $g \in \Gamma^*$ can be uniquely written in the reduced form as $g = g_1 \dots g_m$, where for $1 \leq j \leq m$, g_j is an element of A^* or B^* and two adjacent g_j's are not both in A^* or both in B^*. In this case, define the length of g to be $|g| = m$. (We set $|e| = 0$). We also define the beginnings of g to be

$$L(g) := \{ e, g_1, g_1 g_2, \dots, g_1 g_2 \dots g_m \}$$

and the ends of g to be

$$R(g) := \{ e, g_m, g_{m-1} g_m, \dots, g_1 g_2 \dots g_m \} .$$

For $1 \leq k \leq m$, we consider the k-beginning of g to be $l_k(g) := g_1 \dots g_k$, and the k-end of g to be $r_k(g) := g_{m-k+1} \dots g_m$. The 1-beginning and the 1-end will be also denoted by $l(g)$, respectively $r(g)$.

For $v_1, v_2, \dots, v_k \in \Gamma$, $k \geq 2$, we say that the product $v_1 v_2 \dots v_k$ is reduced if none of the factors is e and, for all $i = 1, \dots, k-1$, the 1-end of v_i and the 1-beginning of v_{i+1} are in different groups. We say that the product $v_1 v_2 \dots v_k$ is reduced mod $\{ v_j \mid j \in J \}$, where $J \subset \{1, \dots, n\}$, it some v_j's, $j \in J$, may equal e and after deleting those equal to e, we obtain a reduced product. By a statement like "the product $(v_1 v_2) v_3$ is reduced" we mean "the product $w v_3$ is reduced, where $w = v_1 v_2$".

Let w_1, w_2 be two words in Γ^*. We say that in the product $w_1 w_2$ there is a consolidation if $r(w_1)$ and $l(w_2)$ are both in the same group (A or B) and $r(w_1) l(w_2) \neq e$. We say that there is a cancellation it $r(w_1) l(w_2) = e$.

Let X be a subset of a group H. Then X is a free family if and only if $X \cap X^{-1} = \phi$ and no product $w = x_1 x_2 \ldots x_n$ is equal to e, where $n \geq 1$, $\{x_1, \ldots, x_n\} \subset X \cup X^{-1}$ and $x_i x_{i+1} \neq e$, for all $1 \leq i \leq n - 1$. ($X^{-1} := \{x^{-1} \mid x \in X\}$).

The following remark will be useful in § 4. It is exercise 12 in [M-K-S], Section 1.4.

Remark. If $\{a,b\}$ is a free subset in a group H, then $\{a^n b^{-n}\}_{n \geq 1}$ is also free.

Let $H \subset \Gamma$ be a nontrivial subgroup. We say that $g \in \Gamma^*$ begins (ends) with an element of H if $H^* \cap L(g) \neq \phi$ (respectively $H^* \cap R(g) \neq \phi$). Denote

$\Gamma(H) = \{g \in \Gamma^* \mid$ g neither begins nor ends with an element of $H\}$.

We define a function $q_H : \Gamma \setminus H \to H$ in the following may: for any $g \in \Gamma \setminus H$, there are unique elements $w \in \Gamma(H)$ and $h_1, h_2 \in H$ such that h_1 is of maximal length and $g = h_1 w h_2$ is reduced mod$\{h_1, h_2\}$; then $q_H(g) := h_1 h_2$.

Lemma 5. Let A and B be nontrivial groups such that $A * B \neq Z_2 * Z_2$, and let $\alpha \in \mathrm{Aut}(A * B)$. Then $\alpha_{|F} = id_F$ implies $\alpha = id_{A*B}$.

Proof. By Proposition 3 (the case n = 2) of [N-T], the conclusion holds if one of the groups A or B has an element of infinite order.

The only case that cannot be dealt with the above result is when A and B are indecomposable and both A and B differ from Z. (If $A = A_1 * A_2$ is a proper decomposition, then ord$(a_1 a_2)$ is infinite for any $a_1 \in A_1^*$, $a_2 \in A_2^*$.) But in this case the conclusion can be obtained by a straightforward checking using the set of generators of Aut(A * B) given by Fuchs-Rabinowich ([F-R]). A cleaner proof of this last case can be found in ([Co], Proposition 1.4). ∎

Lemma 6. Let $a \in A^*$ and $w \in \Gamma \setminus A$ be elements of infinite order. Assume

$s \in \mathbf{N}^*$ is such that

$$q_A(w) \notin \{a^{ns} \mid n \in Z\}.$$

Then $\{a^s, w^s\}$ is a free family in Γ.

Proof. If $w \in B^*$, the conclusion is clear. Therefore we assume from now on that $|w| \geq 2$. Then:

(6) $\qquad l(w^n) = l(w),\ r(w^n) = r(w)$, for all $n \in N^*$,

as will be checked at the end of the proof.

Write $w = a_1 w_1 a_2$ reduced mod $\{a_1, a_2\}$ with $a_1, a_2 \in A$, $w_1 \in \Gamma(A)$. By (6), one obtains that

$$(7) \qquad w^{ns} = \begin{cases} a_1 w_n a_2,\ w_n \in \Gamma(A) \text{ if } n \geq 1 \\ \\ a_2^{-1} w_n a_1^{-1},\ w_n \in \Gamma(A) \text{ if } n \leq -1. \end{cases}$$

Let z be any word in a^s and w^s. Conjugating it by a high power of w^s, we can put it in the form:

$$z = w^{sn_1} a^{sm_2} w^{sn_2} a^{sm_2} \dots a^{sm_{p-1}} w^{sn_p}$$

where $p \geq 2$, $n_i, m_j \in Z^*$ for $1 \leq i \leq p$, $i \leq j \leq p-1$. Using (7), we obtain that

$$(8) \qquad z = c_1 w_{n_1} c_2 w_{n_2} \dots c_{p-1} w_{n_p} c_p$$

where $c_1 \in \{a_1, a_2^{-1}\}$, $c_p \in \{a_2, a_1^{-1}\}$ and c_i, $i = 2,\dots,p-1$ are elements of A of the form $a_2 a^{sm} a_1$, $a_2 a^{sm} a_2^{-1}, a_1^{-1} a^{sm} a_1$ or $a_1^{-1} a^{sm} a_2^{-1}$. By the hypothesis, $c_i \neq e$ for $i = 2,\dots,p-1$, therefore in (8) z is reduced mod$\{c_1, c_p\}$, hence $z \neq e$.

It remains to prove (6).

If $|w|$ is even, then w begins and ends with letters from different groups, hence in the product $w\,w \dots w = w^n$ there appears neither consolidation nor cancellation. Therefore (6) holds.

If $|w|$ is odd, then w begins and ends with letters from the same group. If in $w\,w$ there is only a consolidation, since $|w| \geq 2$, one sees also easily that (6) holds.

Otherwise, one can show by induction that w has a reduced form $w = v\tilde{w}v^{-1}$ with $\tilde{w}$, $v \in \Gamma^*$, such that ord$\tilde{w}$ is infinite and in $\tilde{w}\tilde{w}$ there is only a consolidation. If $|\tilde{w}| \geq 2$, we have seen above that $l(\tilde{w}^n) = l(\tilde{w})$, $r(\tilde{w}^n) = r(\tilde{w})$. Therefore $w^n = v(\tilde{w}^n)v^{-1}$ is reduced for $n \in N^*$, and this holds even if $|\tilde{w}| = 1$. Hence $l(w^n) = l(v)$, $r(w^n) = r(v^{-1})$ and then (6) holds too. ∎

Lemma 7. Let $w, v \in \Gamma^*$. Assume that vw^{-1} is reduced, and that vw^{-1} begins with w^{-1} (i.e. $w^{-1} \in L(vw^{-1})$). Then in the reduced form of wvw^{-1} there appear all the letters (possibly their inverses) which appear in w and in v.

Proof. Since $w^{-1} \in L(vw^{-1})$, there is an element $h \in \Gamma^*$ such that $vw^{-1} = w^{-1}h$, both products being reduced. There are two cases:

Case I. $v \geq w$; then $w^{-1} \in L(v)$ and $w^{-1} \in R(h)$, therefore

$$v = w^{-1}u,\ h = uw^{-1}$$

for some $u \in \Gamma$, where both $w^{-1}u$ and uw^{-1} are reduced mod$\{u\}$. Since $wvw^{-1} = h$, the conclusion is fulfiled.

Case II. $v < w$; then $v \in L(w^{-1})$ and $h \in R(w^{-1})$, therefore

$$w^{-1} = vu,\ w^{-1} = uh$$

for some $u \in \Gamma^*$, where both vu and uh are reduced. Hence $v^{-1} \in R(w)$. Note that the product vv is reduced ($l(u) = l(v)$ due to the above relations, and we known that the product vu is reduced). Let $n \geq 1$ be maximal such that $v^{-n} \in R(w)$. Then $w = xv^{-n}$ reduced mod$\{x\}$, hence $vw^{-1} = v^{n+1}x^{-1}$. Since w is completely cancelled in the product $w(vw^{-1}) = (xv^{-n})(v^{n+1}x^{-1})$ we see that x must be cancelled in the product $x(vx^{-1})$ (i.e. $x^{-1} \in L(vx^{-1})$). Note that vx^{-1} is reduced mod$\{x\}$. If $|x| \geq |v|$, then $v^{-1} \in R(x)$, i.e. $x = yv^{-1}$ reduced mod$\{y\}$, contradicting thus the choice of n. Therefore $|x| < |v|$, hence $x^{-1} \in L(v)$, i.e. $v = x^{-1}z$ reduced mod$\{x,z\}$. From all these, we obtain that

$$v = x^{-1}z,\ w = xv^{-n} = x(z^{-1}x)^n,\ wvw^{-1} = zx^{-1},$$

all products being reduced mod$\{x,z\}$, hence the conclusion holds in this case too.■

Lemma 8. Let $a \in A^*$, $b \in B^*$ and $\alpha \in \mathrm{Aut}(A * B)$ be such that $\alpha(a)$, $\alpha(b) \in \widetilde{A}$ and $\alpha([a,b]) \notin F$. Then $\alpha(ab) \notin \langle ab\rangle$.

Proof. Let $\alpha(a) = wa_1w^{-1}$, $\alpha(b) = va_2v^{-1}$, where $a_1, a_2 \in A^*$. Replacing, possibly, a_1 and a_2 by their conjugate with elements of A, we may assume that wa_1w^{-1} and va_2v^{-1} are in reduced form mod$\{w,v\}$.

One has six cases:

Case I. w = e or (and) v = e. There are the following subcases :

a) $\alpha(a) = a_1$, $\alpha(b) = a_2$, $\alpha(ab) = a_1a_2$;

b) $\alpha(a) = wa_1w^{-1}$, $w \in \Gamma^*$, $\alpha(b) = a_2$, $\alpha(ab) = wa_1w^{-1}a_2$;

c) $\alpha(a) = a_1$, $\alpha(b) = va_2v^{-1}$, $v \in \Gamma^*$, $\alpha(ab) = a_1va_2v^{-1}$.

Subcase a) is trivial.

For subcase b), there are three situations, depending on the fact that the product $w^{-1}a_2$ has a consolidation, a cancellation or it is reduced. If there appears a consolidation, than $l(\alpha(ab))$, $r(\alpha(ab)) \in A$, hence $\alpha(ab) \notin \langle ab\rangle$. If there appears a cancellation, then $w = a_2\widetilde{w}$ is in reduced form for some $\widetilde{w} \in \Gamma(A)$, so that $\alpha(ab) = a_2\widetilde{w}a_1\widetilde{w}^{-1}$ is reduced. The assumption $\alpha(ab) \in \langle ab\rangle$ will then imply that the only letters appearing in $\alpha(a)$ and $\alpha(b)$ are $a^{\pm 1}$, $b^{\pm 1}$, hence $\alpha([a,b]) \in F$, contradiction. Finally, it $w^{-1}a_2$ is reduced, then $\alpha(ab) = wa_1w^{-1}a_2$ is reduced, hence the same argument as above shows that $\alpha(ab) \notin \langle ab\rangle$.

Subcase c) is similar to subcase b).

Case II. The product $w^{-1}v$ is in reduced form. Since $\alpha(ab) = wa_1w^{-1}va_2v^{-1}$ is in reduced form, the assumption $\alpha(ab) \in \langle ab\rangle$ will again imply that the only letters appearing in $\alpha(a)$ and $\alpha(b)$ are $a^{\pm 1}$, $b^{\pm 1}$, hence

$\alpha([a,b]) \in F$, contradiction.

Case III. $v = w \neq e$. Then $\alpha(ab) = wa_1a_2w^{-1}$. Since $a_1a_2 \neq e$ (otherwise $\alpha(ab) = e$), the above expression of $\alpha(ab)$ is reduced, therefore $\alpha(ab) \notin \langle ab \rangle$ because it beings and ends with letters from the same group.

Case IV. In the product $w^{-1}v$ there are cancellations and consolidations, but neither w^{-1} nor v are completely cancelled (i.e. $w \notin L(v)$ and $v^{-1} \notin R(w^{-1})$). Then $\alpha(ab)$ beings and ends with letters from the same group (because $\alpha(ab) = wa_1(w^{-1}v)a_2v^{-1}$ is reduced), hence $\alpha(ab) \notin \langle ab \rangle$.

Case V. In the product $w^{-1}v$, w^{-1} is completely cancelled (i.e. $w \in L(v)$), but $v \neq w \neq e$. Then $v = wv_1$ is reduced for some $v_1 \in \Gamma^*$, and $\alpha(ab) = wa_1v_1a_2v_1^{-1}w^{-1}$, where the product $v_1a_2v_1^{-1}w^{-1}$ is reduced. If $l(v_1) \neq a_1^{-1}$, then $\alpha(ab)$ begins and ends with letters from the same group, hence $\alpha(ab) \notin \langle ab \rangle$. Otherwise, v_1 can be written in the reduced form $v_1 = a_1^{-1}v_2$, $v_2 \in \Gamma^*$. (If $v_2 = e$, then $v_1 \in A^*$. But this is impossible because $r(v_1) = r(v) \in B$.) Then $\alpha(ab) = wv_2a_2v_2^{-1}a_1w^{-1}$, with the product $v_2a_2v_2^{-1}a_1w^{-1}$ reduced. It w is not completely cancelled (i.e. $w^{-1} \notin L(v_2a_2v_2^{-1}a_1w^{-1})$), then $\alpha(ab)$ begins and ends with letters from the same group, hence $\alpha(ab) \notin \langle ab \rangle$. Otherwise we can use Lemma 7. Hence, if $\alpha(ab) \in \langle ab \rangle$, then the letters which appear in $v_2a_2v_2^{-1}a_1$ and in w can only be $a^{\pm 1}$, $b^{\pm 1}$. Therefore $\alpha([a,b]) \in F$, contradiction.

Case VI. In the product $w^{-1}v$, v is completely cancelled (i.e. $v^{-1} \in R(w^{-1})$), but $e \neq v \neq w$. This case is similar with Case V. ∎

Lemma 9. Let $a \in A^*$, $b \in B^*$ and $w_0 \in \Gamma(\langle ab \rangle)$ be such that ord $(w_0(ab)^k)$ is infinite for some $k \in Z$. Then for any ε, $\delta \in \{-1,1\}$:

$$l_2((ab)^{3\varepsilon}w_0(ab)^{3\delta}) = (ab)^{\varepsilon}$$

$$r_2((ab)^{3\varepsilon}w_0(ab)^{3\delta}) = (ab)^{\delta}$$

Particularly, if $r(w_o) \in B$, and either one of the conditions below are satisfied:

(i) $a^2 \neq e$;

(ii) $a^2 = e$, and $w_o \neq b^{-1}$,

then $l((ab)w_o(ab)) \in A$ and $r((ab)w_o(ab)) \in B$.

Proof. One can assume that $r(w_o) \in B$. Indeed, if $r(w_o) \in A$, let's denote $A' = B$, $B' = A$, $a' = b^{-1}$, $b' = a^{-1}$. Then $\Gamma' = A' * B' = \Gamma$, $\langle a'b' \rangle = \langle ab \rangle$, $w_o \in \Gamma(\langle a'b' \rangle)$, $ord(w_o(a'b')^{-k}) = ord(w_o(ab)^k)$ and $r(w_o) \in B'$.

Therefore from now on we assume that $r(w_o) \in B$. There are four possibilities to choose ε and δ.

Case I. $\varepsilon = 1$, $\delta = 1$.

If one of the conditions (i) or (ii) holds, then:

$$l((ab)w_o(ab)) \in A$$

$$r((ab)w_o(ab)) \in B.$$

We list the subcases and give some hints:

1) $|w_o| \geq 2$. Since $r(w_o) \in B$, $w_o ab$ is reduced, and $(ab)^{-1} \notin L(w_o ab)$ because $w_o \in \Gamma(\langle ab \rangle)$.

2) $|w_o| = 1$. Then $w_o = b_1$, where $b_1 \in B^*$, and all follows from conditions (i) and (ii).

If $a^2 = e$ and $w_o = b^{-1}$, then $(ab)w_o(ab) = b$. As $ord(w_o(ab)^k) \neq 2$, one obtains that $b^2 \neq e$. Then:

$$l((ab)^2 w_o(ab)^2) \in A$$

$$r((ab)^2 w_o(ab)^2) \in B.$$

Case II. $\varepsilon = 1$, $\delta = -1$.

In this case one has that

$$l((ab)w_o(ab)^{-1}) \in A$$

$$r((ab)w_o(ab)^{-1}) \in A.$$

To see this, we make an analysis based on $|w_o|$. The appearing subcases can be dealt in a straightforward manner.

If $|w_o| \geq 4$, all is clear because $w_o \in \Gamma(\langle ab\rangle)$ (hence $(ab)^{-1} \notin L(w_o)$ and $ab \notin R(w_o)$).

If $|w_o| = 3$, then $w_o = b_1 a_1 b_2$, where $a_1 \in A^*$, $b_1, b_2 \in B^*$. One has the following subcases:

1) $b_1 \neq b^{-1}$ and $b_2 \neq b$;

2) $b_1 = b^{-1}$ (hence $a_1 \neq a^{-1}$ because $w_o \in \Gamma(\langle ab\rangle)$) and $b_2 \neq b$;

3) $b_1 \neq b^{-1}$ and $b_2 = b$ (hence $a_1 \neq a$ because $w_o \in \Gamma(\langle ab\rangle)$);

4) $b_1 = b^{-1}$ and $b_2 = b$.

If $|w_o| = 2$, then $w_o = a_1 b_1$, where $a_1 \in A^*$, $b_1 \in B^*$. Since $w_o \in \Gamma(\langle ab\rangle)$ the only subcases are:

1) $b_1 = b$;

2) $b_1 = b$, hence $a_1 \neq a$.

If $|w_o| = 1$, then $w_o = b_1$, $b_1 \in B^*$ and the assertion holds.

Case III. $\varepsilon = -1$, $\delta = 1$

Since $r(w_o) \in B$, one has that

$$l((ab)^{-1} w_o (ab)) \in B$$

$$r((ab)^{-1} w_o (ab)) \in B.$$

as can be easily seen both if $|w_o| \geq 2$ and if $|w_o| = 1$.

Case IV. $\varepsilon = -1$, $\delta = -1$

If none of the conditions (iii) - (v) below is satisfied

(iii) $w_o = b$ and $a^2 = e$;

(iv) $w_o = a^2 b$ and $a^2 \neq e$;

(v) $w_o = ab^2$ and $b^2 \neq e$,

then we shall prove that

$$(9)\qquad \begin{cases} l((ab)^{-1}w_o(ab)^{-1}) \in B. \\ r((ab)^{-1}w_o(ab)^{-1}) \in A. \end{cases}$$

If $|w_o| \geq 4$, the assertion holds because $w_o \in \Gamma(\langle ab\rangle)$ (hence $ab \notin L(w_o)$ and $ab \notin R(w_o)$).

If $|w_o| = 3$, then $w_o = b_1 a_1 b_2$, where $a_1 \in A^*$, $b_1, b_2 \in B^*$. Since $w_o \in \Gamma(\langle ab\rangle)$, the only subcases are:

1) $b_2 \neq b$;

2) $b_2 = b$, hence $a_1 \neq a$,

and both give (9).

If $|w_o| = 2$, then $w_o = a_1 b_1$, where $a_1 \in A^*$, $b_1 \in B^*$. The following (not disjoint) possibilities can appear:

1) $a_1 \neq a$ and $b_1 \neq b$;

2) $b_1 = b$ hence $a_1 \neq a^2$ (because (iv) doesn't hold)

3) $a_1 = a$, hence $b_1 \neq b^2$ (becasue (v) doesn't hold),

and all give (9) .

If $|w_o| = 1$, then $w_o = b_1$, $b_1 \in B^*$, and we use the fact that we are not in situation (iii).

If we are in situations (iii), then $(ab)^{-1}w_o(ab)^{-1} = b^{-1}$. As $\mathrm{ord}(w_o(ab)^k) \neq 2$, one obtains that $b^2 \neq e$, hence

$$l((ab)^{-2}w_o(anb)^{-2}) \in B$$
$$r((ab)^{-2}w_o(ab)^{-2}) \in A.$$

The same is true if we are in situation (iv) or (v). For example, if (iv) holds, then

$$(ab)^{-2}w_o(ab)^{-2} = \begin{cases} b^{-1}a^{-1}b^{-2}a^{-1}, \text{ if } b^2 \neq e \\ b^{-1}a^{-2}, \text{ if } b^2 = e. \end{cases} \qquad \blacksquare$$

For the next lemma, we need the following definitions:

Definitions. Let $a \in A^*$, $b \in B^*$, $w \in \Gamma$ and $k \in Z$. We say that w begins (ends) with $(ab)^k$ if $(ab)^k \in L(w)$ and $(ab)^{k+\varepsilon} \notin L(w)$ (respectively $(ab)^k \in R(w)$ and $(ab)^{k+\varepsilon} \notin R(w)$) where $\varepsilon = \mathrm{sign}(k)$ if $k \neq 0$, and $\varepsilon \in \{-1;1\}$ if $k = 0$.

For an arbitrary word $v \in \Gamma^*$, we say that w begins (ends) with v if $v \in L(w)$ (respectively $v \in R(w)$).

(The ambiguity of the above definitions will not be misleading).

Lemma 10. Let $a \in A^*$, $b \in B^*$, $w_0 \in \Gamma(\langle ab\rangle)$ and $k \in N$ be such that $\mathrm{ord}(w_0(ab)^k)$ is infinite and $r(w_0) \in B$ for $k > 0$. Then, for any $n \in N^*$:

1) if $k > 0$, then $[w_0(ab)^k]^n$ begins with w_0 and ends with $(ab)^t$, where $t \in \{k, k-1\}$;

2) if $k = 0$, then w_0^n begins with $(ab)^\delta$ and ends with $(ab)^\varepsilon$; where $\delta, \varepsilon \in \{-1, 0, 1\}$.

Proof. Denote $w = w_0(ab)^k$.

Case I. $\underline{k \geq 2}$. If $a^2 \neq e$ or $a^2 = e$ and $w_0 \neq b^{-1}$ the second part of Lemma 9 shows that:

$$l((ab)w_0(ab)) \in A$$

$$r((ab)w_0(ab)) \in B$$

Then

$$w^n = w_0(ab)^{k-1}(abw_0ab)(ab)^{k-2} \ldots (abw_0ab)(ab)^{k-1}, \quad n \in N^*,$$

is reduced mod$\{(ab)^{k-2}\}$, hence w^n begins with w_0. Since $r_4(abw_0ab) \neq (ab)^2$ for any $w_0 \in \Gamma(\langle ab\rangle)$, w^n ends with $(ab)^{k-1}$ or $(ab)^k$.

If $a^2 = e$ and $w_0 = b^{-1}$, then $b^2 \neq e$, because $\mathrm{ord}(w_0(ab)^k) \neq 2$, and it is easy to see that, for $n \geq 2$, w^n begins with w_0 and ends with $(ab)^{k-1}$. (One has $w = (ab)^{-1} b(ab)^{k-1}$. The only difficulty appears when $k = 2$, but then $\mathrm{ord}\, b$ is infinite and the statement is also true.)

Case II. $\underline{k = 1}$. Since $w_o \in \Gamma(\langle ab \rangle)$, one has that $l(abw_o) \in A$.

If $|w_o| \geq 2$, then $r(abw_o) \in B$, because $r(w_o) \in B$ and $w_o \in \Gamma(\langle ab \rangle)$. Hence $w^n = w_o(abw_o)...(abw_o)ab$ is reduced for $n \geq 1$. So that w^n begins with w_o. Now again $r_4(abw_o ab) \neq (ab)^2$, hence w^n ends with ab.

If $|w_o| = 1$, then either $r(abw_o) \in B$, and we use the above argument, or $w_o = b^{-1}$, $w = b^{-1}ab$ and ord a is infinite. Hence w^n ends with (ab) for $n = 1$ and with $(ab)^o$ for $n \geq 2$, and it always begins with w_o.

Case III. $\underline{k = 0}$. If $|w_o|$ is even, since the product $w^n = w_o w_o ... w_o$ is reduced, the conclusion holds with $\delta = \varepsilon = 0$. The same happens if $|w_o| = 1$. It remains to study the case $|w_o| \geq 3$, $|w_o|$ odd. Then w_o begins and ends with letters from the same group. If in the product $w_o w_o$ there is only a consolidation, we obtain again the conclusion with $\delta = \varepsilon = 0$. If in $w_o w_o$ there appear cancellations, then, like in Lemma 6, w_o has a reduced form $w_o = v\tilde{w}_o v^{-1}$, where $\tilde{w}_o, v \in \Gamma^*$, ord $\tilde{w}_o$ is infinite and in $\tilde{w}_o \tilde{w}_o$ appears only a consolidation.

If $|\tilde{w}_o| \geq 2$, then $l(\tilde{w}_o^n) = l(\tilde{w}_o)$, $r(\tilde{w}_o^n) = r(\tilde{w}_o)$ as in Lemma 6. Hence $w_o^n = v(\tilde{w}_o^n)v^{-1}$ is reduced and $l_u(w_o^n) = l_u(w_o)$; $r_u(w_o^n) = r_u(w_o)$, where $u = |v| + 1 \geq 2$, therefore $w_o^n \in \Gamma(\langle ab \rangle)$.

If $|\tilde{w}_o| = 1$, as ord $\tilde{w}_o$ is infinite, $w_o^n = v(\tilde{w}_o^n)v^{-1}$, $n \in N^*$, is reduced and $l_{|v|}(w_o^n) = l_{|v|}(w_o)$, $r_{|v|}(w_o^n) = r_{|v|}(w_o)$. Hence $w_o^n \in \Gamma(\langle ab \rangle)$ for $|v| \geq 2$. While for $|v| = 1$, $|w_o^n| = 3$, for any $n \geq 1$, hence w_o^n cannot begin or end with $(ab)^s$, $|s| \geq 2$. ∎

Lemma 11. Let $a \in A^*$, $b \in B^*$, $w \in \Gamma^*_{\langle ab \rangle}$ and choose $k_1, k_2 \in Z$ such that $w = (ab)^{k_1} w_o (ab)^{k_2}$, the product is reduced mod$\{(ab)^{k_1}, (ab)^{k_2}\}$, $|k_1|$ is maximal and $w_o \in \Gamma(\langle ab \rangle)$. Then there exist $w'_o \in \Gamma(\langle ab \rangle)$ and $k' \in \{k_1 + k_2, k_1 + k_2 - \text{sign}(k_1 + k_2)\}$ such that $\text{Ad}((ab)^{-k_1})(w) = w'_o (ab)^{k'}$ and the last product is reduced mod$\{(ab)^{k'}\}$.

Proof. If $\mathrm{sign}(k_1+k_2) = \mathrm{sign}\, k_2$ or $k_1+k_2=0$, then the assertion is trivial with $w_o' = w_o$ and $k' = k_1+k_2$.

If $\mathrm{sign}(k_1+k_2) = -\mathrm{sign}\, k_2$, there are two cases:

(i) $k_2>0$, $k_1+k_2<0$;

(ii) $k_2<0$, $k_1+k_2>0$.

Their proofs are similar, so we only prove case (i). Since $k_2>0$, $r(w_o)\in B$. Then $r(w_o b^{-1}a^{-1})\in A$ because $w_o\in \Gamma(\langle ab\rangle)$. Therefore the product

$$(w_o b^{-1}a^{-1})(ab)^{k_1+k_2+1}$$

is reduced mod$\{(ab)^{k_1+k_2+1}\}$. Now the conclusion holds with $w_o' = w_o b^{-1}a^{-1}$ and $k' = k_1+k_2+1$ (since $k_1<0$, $l(w_o)\in B$, hence $w_o'\in\Gamma(\langle ab\rangle)$). ∎

Lemma 12. Let $a\in A^*$, $b\in B^*$ and $w\in\Gamma\setminus\langle ab\rangle$ be such that ord w is infinite and $q_{\langle ab\rangle}(w) = (ab)^s$, $s\in Z$. Then $\{w^N, (ab)^N\}$ is a free family in Γ for any $N\geq 2|s|+8$.

Proof. Let $k_1,k_2\in Z$ be such that $w=(ab)^{k_1}w_o(ab)^{k_2}$, where the product is reduced mod$\{(ab)^{k_1}, (ab)^{k_2}\}$, $|k_1|$ is maximal and $w_o\in\Gamma(\langle ab\rangle)$. Then $s=k_1+k_2$. To prove that $\{w^N, (ab)^N\}$ is a free family, it is enough to prove that $\{[\mathrm{Ad}((ab)^{-k_1})(w)]^N, (ab)^N\}$ is a free family.

By Lemma 11, one has $w' = [\mathrm{Ad}(ab)^{-k_1}](w) = w_o'(ab)^k$, where $w_o'\in\Gamma(\langle ab\rangle)$, $k\in\{s, s-\mathrm{sign}(s)\}$. Moreover, we can assume that $k\geq 0$. Indeed, if $k<0$, then denoting $B'=A$, $A'=b$, $a'=b^{-1}$ and $b'=a^{-1}$, one has $a'b'=(ab)^{-1}$, $w_o'\in\Gamma(\langle a'b'\rangle)$, $w'\in\Gamma\setminus\langle a'b'\rangle$, ord w' is infinite and $w' = w_o'(a'b')^{-k}$.

Therefore we can assume that $w=w_o(ab)^k$ is reduced with $w_o\in\Gamma(\langle ab\rangle)$ and $k\in\{|s|, |s|-1\}\cap N$.

Now, Lemma 10 implies that for all $n\geq 1$

$$(10)\quad \begin{cases} w^{N\,m} = (ab)^{c_n}\, w_n\, (ab)^{d_n} \text{ reduced mod}\{(ab)^{c_n}, (ab)^{d_n}\}, \text{ where } w_n\in\Gamma(\langle ab\rangle) \\ d_n\in\{k, k+1, k-1\},\ c_n\in\{-1,0,1\}. \end{cases}$$

Take a word x in w^N and $(ab)^N$. Conjugating it by a high power of $(ab)^N$, it can be put in the form:

$$x = (ab)^{Ns_1} w^{Nt_1\varepsilon_1} (ab)^{Ns_2} w^{Nt_2\varepsilon_2} \dots w^{Nt_p\varepsilon_p} (ab)^{Ns_{p+1}}$$

where $t_i \in \mathbb{N}^*$, $\varepsilon_i \in \{-1, 1\}$, $1 \le i \le p$, $s_j \in \mathbb{Z}^*$, $1 \le j \le p+1$.

Using (10) we can write

$$x = (ab)^{Ns_1+f_1+g_0} w_{t_1}^{\varepsilon_1} (ab)^{Ns_2+f_2+g_1} w_{t_2}^{\varepsilon_2} \dots w_{t_p}^{\varepsilon_p} (ab)^{Ns_{p+1}+f_{p+1}+g_p}$$

where $g_o = 0$, $f_{p+1} = 0$,

$$f_j = \begin{cases} c_j, & \text{for } \varepsilon_j = 1 \\ -d_j, & \text{for } \varepsilon_j = -1 \end{cases}$$

$$g_j = \begin{cases} d_j, & \text{for } \varepsilon_j = 1 \\ -c_j, & \text{for } \varepsilon_j = -1 \end{cases}, \quad 1 \le j \le p.$$

Denote $Ns_j + f_j + g_{j-1} = n_j$, $\delta_j = \text{sign}(n_j)$, for $1 \le j \le p+1$. If $|n_j| \ge 6$ for all $1 \le j \le p+1$, then x can be written as

$$x = (ab)^{n_1-3\delta_1}[(ab)^{3\delta_1} w_{t_1}^{\varepsilon_1} (ab)^{3\delta_2}](ab)^{n_2-6\delta_2}[(ab)^{3\delta_2} w_{t_2}^{\varepsilon_2} (ab)^{3\delta_3}] (ab)^{n_3-6\delta_3} \dots$$
$$[(ab)^{3\delta_p} w_{t_p}^{\varepsilon_p} (ab)^{3\delta_{p+1}}] (ab)^{n_{p+1}-3\delta_{p+1}}.$$

Due to Lemma 6, this product is reduced mod $\{(ab)^{n_j-6\delta_j} | 2 \le j \le p\}$, hence it is distinct from e.

By (10), $\max_{1\le j\le p+1} |f_j + g_{j-1}| \le 2k + 2 \le 2|s| + 2$. Therefore, if $N \ge 2|s| + 8 \ge$ $\ge \max_{1\le j\le p+1} |f_j + g_{j-1}| + 6$, then $x \ne e$. This shows that $\{w^N, (ab)^N\}$ is a free family in Γ for $N \ge 2|s| + 8$. ∎

4. Proof of the main result

We define, for $n \in \mathbb{N}^*$ and $v \in \Gamma$ the following averaging process:

$$\theta_{n,v}(\alpha) = \frac{1}{n}\sum_{k=1}^{n} \bar{v}^{nk} \alpha\, \bar{v}^{-nk}, \quad \alpha \in \text{Aut}(\Gamma).$$

Theorem 1 will be proved using Lemmas 13 and 14 and the Theorem of Akemann and Lee (which can be deduced too from these two lemmas). All these results rely on the following consequence of a theorem of Akemann and Ostrand ([A-O]): if $\{h_i\}_{1\le i\le n}$ is a free family in a (discrete) group H, then

$$\left\| \sum_{i=1}^{n} h_i \right\|_H = 2\sqrt{n-1}.$$

Lemma 13. Assume $a\in A^*$ and $\alpha \in \mathrm{Aut}(A * B)$ are such that a is an element of infinite order and $\alpha(a)\notin A$ or $\alpha^{-1}(a)\notin A$. If $n\in N^*$ is such that

$$q_A(\{\alpha(a), \alpha^{-1}(a)\}\setminus A) \cap \{a^{nk} | k \in Z^*\} = \phi$$

then $\|\theta_{n,a}(\alpha)\|_G = 2\sqrt{n-1}/n$.

Proof. Denote $\alpha(a) = w_1$, $\alpha^{-1}(a) = w_2$. Since

$$\beta\, \overline{v} = \overline{\beta(v)}\beta, \text{ for } \beta\in \mathrm{Aut}(\Gamma),\ v\in\Gamma,$$

we obtain:

$$\|\theta_{n,a}(\alpha)\|_G = \left\| \left[\overline{\frac{1}{n}\sum_{k=1}^{n} a^{nk}\alpha(a^{nk})^{-1}}\right]\alpha \right\|_G =$$

$$= \left\| \frac{1}{n}\sum_{k=1}^{n} \overline{a^{nk}w_1^{-nk}} \right\|_G = \left\| \frac{1}{n}\sum_{k=1}^{n} a^{nk}w_1^{-nk} \right\|_\Gamma$$

respectively:

$$\|\theta_{n,a}(\alpha)\|_G = \left\| \alpha \left[\frac{1}{n}\sum_{k=1}^{n} \overline{\alpha^{-1}(a)^{nk}a^{-nk}}\right] \right\|_G$$

$$= \left\| \frac{1}{n}\sum_{k=1}^{n} w_2^{nk}a^{-nk} \right\|_\Gamma .$$

By Lemma 6 applied to w_1 or w_2 and by the Remark from § 3, $\{a^{nk}w_1^{-nk}\}_{k\ge 1}$ or $\{w_2^{nk}a^{-nk}\}_{k\ge 1}$ is a free family in Γ (note that $\mathrm{ord}\, w_1 = \mathrm{ord}\, w_2 = \mathrm{ord}\, a$), therefore the conclusion follows from the result of Akemann and Ostrand. ∎

Lemma 14. Assume $a\in A^*$, $b\in B^*$ and $\alpha\in \mathrm{Aut}(A * B)$ are such that one of the following situations holds:

(i) $\alpha(a), \alpha(b)\in\tilde{A}$ and $\alpha([a,b])\notin F$;

(ii) $\alpha^{-1}(a)$, $\alpha^{-1}(b)\in\tilde{A}$ and $\alpha^{-1}([a,b])\notin F$.

If $n\in N^*, n \ge 8$ is such that

$$q_{\langle ab\rangle}(\{\alpha(ab), \alpha^{-1}(ab)\}\setminus\langle ab\rangle) \subset \{(ab)^m \,|\, |m| \le (n-8)/2\} ,$$

then $\|\theta_{n,ab}(\alpha)\|_G = 2\sqrt{n-1}/n$.

Proof. Denote $\alpha(ab) = w_1$, $\alpha^{-1}(ab) = w_2$. As above, we obtain

$$\|\theta_{n,ab}(\alpha)\|_G = \|\frac{1}{n}\sum_{k=1}^{n}(ab)^{nk}w_1^{-nk}\|_\Gamma = \|\frac{1}{n}\sum_{k=1}^{n}w_2^{nk}(ab)^{-nk}\|_\Gamma .$$

Also, $\operatorname{ord} w_1 = \operatorname{ord} w_2 = \operatorname{ord}(ab)$ is infinite. If (i) (respectively (ii)) holds, then, by Lemma 8, $\alpha(ab) \notin \langle ab\rangle$ (respectively $\alpha^{-1}(ab) \notin \langle ab\rangle$) therefore the restriction on n shows that we fit the situation of Lemma 12, hence the family $\{(ab)^{nk}w_1^{-nk}\}_{k\geq 1}$ (respectively $\{w_2^{nk}(ab)^{-nk}\}_{k\geq 1}$) is free in Γ due to the Remark from § 3. The conclusion follows again applying the result of Akemann and Ostrand. ∎

Lemma 15. Assume $\alpha \in \operatorname{Aut}(A * B)$ is such that one of the situations below holds:

(i) $\alpha(A) \subset \tilde{A}$ and $\alpha(B) \subset \tilde{B}$;

(ii) $\alpha(A) \subset \tilde{B}$ and $\alpha(B) \subset \tilde{A}$.

Then $\alpha(F) \subset F$.

Proof. We shall verify only the case (i), the other being similar.

Let $a \in A^*$, $b \in B^*$. According to (i), there are $a' \in A^*$, $b' \in B^*$ and $v, w \in \Gamma$ such that

$$\alpha(a) = va'v^{-1}, \quad \alpha(b) = wb'w^{-1}.$$

Then:

$$\pi(\alpha(a)) = \pi(v)(a',e)\pi(v)^{-1} = (\pi_1(v)a'\pi_1(v)^{-1}, e)$$

$$\pi(\alpha(b)) = (e, \pi_2(w)b'\pi_2(w)^{-1})$$

hence

$$\pi(\alpha([a,b])) = [\pi(\alpha(a)), \pi(\alpha(b))] = e,$$

that is

$$\alpha([a,b]) \in \operatorname{Ker}\pi = F.$$

But $\{[a,b] \mid a \in A^*, b \in B^*\}$ generate F, therefore we get that $\alpha(F) \subset F$. ∎

Proof of Theorem 1

For $x, y \in \Gamma$, denote

$G_1 = \{\alpha \in \mathrm{Aut}(A * B) \mid \alpha(F) = F\}$;

$G_2(x) = \{\alpha \in \mathrm{Aut}(A * B) \mid \alpha(x) \notin \tilde{A} \cup \tilde{B}$ or $\alpha^{-1}(x) \notin \tilde{A} \cup \tilde{B}\}$;

$G_3(x,y) = \{\alpha \in \mathrm{Aut}(A * B) \mid \alpha(x), \alpha(y) \in \tilde{A}$ and $\alpha([x,y]) \notin F$,

or, $\alpha^{-1}(x), \alpha^{-1}(y) \in \tilde{A}$ and $\alpha^{-1}([x,y]) \notin F\}$;

$G_4(x,y) = \{\alpha \in \mathrm{Aut}(A * B) \mid \alpha(x), \alpha(y) \in \tilde{B}$ and $\alpha([x,y]) \notin F$,

or, $\alpha^{-1}(x), \alpha^{-1}(y) \in \tilde{B}$ and $\alpha^{-1}[(x,y]) \notin F\}$.

We begin by infering some consequences of the preceding lemmas.

G_1 is a subgroup of G, having $\overline{F} = \{\overline{f} = \mathrm{Ad}\, f \mid f \in F\}$ as normal subgroup (because $\alpha \overline{f} \alpha^{-1} = \overline{\alpha(f)}$). Since

$$\overline{} : A * B \longrightarrow \mathrm{Aut}(A * B)$$

is one-to-one, $\overline{F}$ is isomorphic to F. Moreover, since

$$\begin{aligned} \alpha \in Z_{G_1}(\overline{F}) &\Longleftrightarrow \alpha \overline{f} \alpha^{-1} = \overline{f} \text{ for all } f \in F \\ &\Longleftrightarrow \overline{\alpha(f)} = \overline{f} \text{ for all } f \in F \\ &\Longleftrightarrow \alpha(f) = f \text{ for all } f \in F, \end{aligned}$$

we get by Lemma 5 that $Z_{G_1}(\overline{F}) = \{e\}$. Consequently, G_1 has as normal subgroup with trivial centralizer the free nonabelian group $\overline{F}$, hence the proof of the theorem of Akemann and Lee shows that:

$$(11) \quad \left\{ \begin{array}{l} \text{for any finite nonvoid set } M \subset G_1^* \text{ and any } \varepsilon > 0, \text{ there is an} \\ \text{averaging process } \theta \text{ with elements of } \overline{F}, \text{ such that} \\ \| \theta(\alpha) \|_{G_1} < \varepsilon, \text{ for all } \alpha \in M. \end{array} \right.$$

Since for all $v \in \Gamma$ and $\alpha \in \mathrm{Aut}(\Gamma)$ one has:

$$\overline{v} \alpha \overline{v}^{-1}(x) = (v \alpha(v)^{-1}) \alpha(x) (v \alpha(v)^{-1})^{-1}, \text{ for all } x \in \Gamma,$$

and since F is a normal subgroup of Γ, we see that for any $x, y \in \Gamma$:

$$(12) \quad \left\{ \begin{array}{l} \alpha \in G_2(x) \Longrightarrow \overline{v} \alpha \overline{v}^{-1} \in G_2(x), \text{ for all } v \in \Gamma; \\ \alpha \in G_k(x,y) \Longrightarrow \overline{v} \alpha \overline{v}^{-1} \in G_k(x,y), \text{ for all } v \in \Gamma, \text{ where } k = 3,4. \end{array} \right.$$

Lemma 15 implies

(13) $$G \setminus G_1 \subset [\bigcup_{x \in A^* \cup B^*} G_2(x)] \cup \bigcup_{(x,y) \in A^* \times B^*} [G_3(x,y) \cup G_4(x,y)].$$

Denote by $\mathcal{G}$ the family of all nonvoid sets of the type $G_2(x)$, $G_3(x,y)$, $G_4(x,y)$ that appear in the union above.

Note that $G_2(x) \neq \phi$ implies that ord x is infinite, because a consequence of Kurosh's Subgroup Theorem ([K]) is that any finite subgroup of $A * B$ is conjugated to a subgroup of A or B (hence if $\mathrm{ord}\,\alpha(x) = \mathrm{ord}\,x$ is finite, then $\alpha(x) \in \tilde{A} \cup \tilde{B}$).

By Lemmas 13 and 14 we infer that

(14) $$\left\{\begin{array}{l} \text{for any } \varepsilon > 0 \text{ and } G_o \in \mathcal{G}, \text{ given a finite nonvoid set } M \subset G_o, \text{ there is} \\ \text{an averaging process } \theta \text{ with elements of } \overline{\Gamma} \text{ such that} \\ \|\theta(\alpha)\|_G < \varepsilon, \text{ for all } \alpha \in M. \end{array}\right.$$

After this preparations, we are ready to prove the theorem. Let $M \subset G^*$ be a finite nonvoid set and $\varepsilon > 0$. By (13), M can be written as a disjoint union

$$M = \bigcup_{i=1}^{N} M_i ; \qquad (N \geq 1)$$

where $M_1 = M \cap G_1 \subset G_1^*$, and for $i = 2,\ldots,N$ there are groups $G_i \in \mathcal{G}$ such that $\phi \neq M_i \subset G_i$.

We shall prove by induction on n, for $n = 1,2,\ldots,N$, the statement:

$(15)_n$ $$\left\{\begin{array}{l} \text{there is an averaging process } \theta_n \text{ with elements of } \overline{\Gamma} \text{ such that} \\ \qquad \|\theta_n(\alpha)\|_G < \varepsilon, \text{ for all } \alpha \in \bigcup_{i=1}^{n} M_i. \end{array}\right.$$

For $n = 1$, this holds due to (11) (if $M_1 = \phi$, we take $\theta_1 = \mathrm{id}$). Assume $(15)_n$ holds for some n, $1 \leq n < N$. Denote

$$M_{(n+1)} := \bigcup \{\mathrm{supp}\,\theta_n(\alpha) \mid \alpha \in M_{n+1}\} .$$

Then $M_{(n+1)}$ is a finite nonvoid set and $M_{(n+1)} \subset G_{n+1}$ due to (12), therefore (14) gives an averaging process $\theta_{(n+1)}$ with elements of $\overline{\Gamma}$ such that

(16) $\qquad \| \theta_{(n+1)}(\alpha) \|_G < \varepsilon$, for all $\alpha \in M_{(n+1)}$.

We define $\theta_{n+1} = \theta_{(n+1)} \circ \theta_n$. Then for $\alpha \in \bigcup_{i=1}^{n} M_i$, one has

$$\| \theta_{n+1}(\alpha) \|_G = \| \theta_{(n+1)}(\theta_n(\alpha)) \|_G \leq \| \theta_n(\alpha) \|_G < \varepsilon$$

and for $\alpha \in M_{n+1}$, one has

$$\| \theta_{n+1}(\alpha) \|_G = \| \theta_{(n+1)}(\theta_n(\alpha)) \|_G < \varepsilon$$

by (16), because $\theta_n(\alpha)$ is a convex combination of elements of $M_{(n+1)}$.

Therefore $(15)_n$ implies $(15)_{n+1}$, hence $(15)_N$ holds, and the theorem is proved. ∎

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