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# A CLASS OF HOLOMORPHIC VECTOR BUNDLES ON TWO-DIMENSIONAL TORI

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# A class of holomorphic vector bundles on two-dimensional tori

Let $X$ be a nonalgebraic compact complex surface. It is an open problem to decide which topological vector bundles on $X$ admit holomorphic structures. It has been established which topological vector bundles admit filtrable holomorphic structures [B - L]. (Recall that a holomorphic vector bundle $E$ is filtrable iff there exists a chain of coherent subsheaves of $E$

$$0 \subset F_1 \subset \dots \subset F_r = E \quad \text{on } X \quad \text{such that}$$

rank $F_i = i$). But there exist holomorphic vector bundles which are not filtrable, or even not reducible [E-F], [B - L] (A holomorphic vector bundle $E$ is called reducible if it admits a coherent subsheaf $F$ such that $0 < \text{rank } F < \text{rank } E$).

In all the examples appearing in [E-F], [B-L], [S] these bundles also admit filtrable holomorphic structures on their underlying topological types.

Thus, it would be interesting to exhibit larger classes of nonfiltrable or of irreducible bundles.

The purpose of this paper is to prove the following

<u>Proposition</u>. Every (complex) topological vector bundle $E$ on a two-dimensional complex torus $X$ having $c_1(E) \in NS(X)$ and $\Delta(E)=0$ admits some holomorphic structure.

We denoted

$$\Delta(E) = \frac{1}{r}(c_2(E) - \frac{r-1}{2r} c_1(E)^2)$$

where $r$ = rank E.

Using the proposition one finds examples of topological vector bundles admitting holomorphic structures but no filtrable (Corollary 1) or even no reducible structures (Corollary 2).

## 1. Proof of the Proposition

First a

**Lemma**

Let X a two dimensional complex torus, $a \in NS(X)$ and $p$ a prime number with $p \mid \frac{1}{2} a^2$. Then there exists an unramified covering $q: X' \longrightarrow X$ of degree $p$ and $a' \in NS(X')$ such that

$$pa' = q^*(a).$$

**Proof**

Let $\Gamma$ be a lattice generated by $\gamma_1, \ldots, \gamma_4$ in $\mathbb{C}^2$ with $X \cong \mathbb{C}^2/\Gamma$. By Appell-Humbert's theorem (see [M]) it follows that through the natural isomorphism

$$H^2(X, \mathbb{Z}) \cong \mathrm{Alt}^2_{\mathbb{Z}}(\Gamma, \mathbb{Z})$$

NS(X) is isomorphic with $H(\mathbb{C}^2, \Gamma) := \{H \mid H \text{ hermitian form on } \mathbb{C}^2 \text{ with } \operatorname{Im} H(\Gamma \times \Gamma) \subset \mathbb{Z}\}$.

Here $H(\mathbb{C}^2, \Gamma)$ is seen as a subgroup in $\mathrm{Alt}^2_{\mathbb{Z}}(\Gamma, \mathbb{Z})$ associating to each hermitian form H in $H(\mathbb{C}^2, \Gamma)$ its imaginary part $\operatorname{Im} H \in \mathrm{Alt}^2_{\mathbb{Z}}(\Gamma, \mathbb{Z})$ which completely determines H.

Thus we have

$$NS(X) \cong \{A \mid A \text{ hermitian } 2\text{x}2\text{-matrix}$$

with $\mathrm{Im}({}^t\Pi A \bar{\Pi}) \in M_4(\mathbb{Z})\}$

where $\Pi := (\gamma_1, \ldots, \gamma_4)$ is the period matrix ( $\gamma_i$ are column vectors).

We associate to each hermitian 2x2-matrix A in NS(X) the matrices $A_1$, $A_2, A_3$ $M_2(Z)$ giving the decomposition

$$(1) \qquad \begin{pmatrix} A_1 & A_2 \\ {}^t -A_2 & A_3 \end{pmatrix} = \mathrm{Im}({}^t\Pi A \bar{\Pi})$$

Since $\mathrm{Im}({}^t\Pi A \bar{\Pi})$ is skew-symmetric we get the above form and $A_1, A_3$ skew-symmetric.

$$\text{Let } A_1 = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}, \; A_2 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \; A_3 = \begin{pmatrix} 0 & \tau \\ -\tau & 0 \end{pmatrix}.$$

The intersection form on NS(X) corresponds to the exterior product of alternating forms (see [M] ). Computing we find that if $a \in NS(X)$ is represented by A as above, then

$$a^2 = 2(\alpha\delta - \beta\gamma - \theta\tau)$$

(cf. [BF], §2).

The hypothesis of the Lemma becomes:

$$p \mid (\alpha\delta - \beta\gamma - \theta\tau).$$

We shall consider tori X' appearing by factorizing $\mathbb{C}^2$ through lattices obtained by multiplying by p one of $\Gamma$'s generators $\gamma_i$

and preserving the others. The projection $q: X' \longrightarrow X$ will be an unramified covering of degree p. If $\tilde{\Pi}$ is the period matrix thus obtained for X' we will need to get:

$$\mathrm{Im}({}^t\tilde{\Pi} A \bar{\tilde{\Pi}}) \in M_4(p\mathbb{Z}).$$

The element $\frac{1}{p} A \in NS(X')$ would be the looked for a'.

We denote

$$\begin{pmatrix} \tilde{A}_1 & \tilde{A}_2 \\ -{}^t\tilde{A}_2 & \tilde{A}_3 \end{pmatrix} = \mathrm{Im}({}^t\tilde{\Pi} A \bar{\tilde{\Pi}}).$$

Notice that if $\tilde{\Pi}$ is obtained from $\Pi$ multiplying by p column 1 or 2 (resp. 3 or 4) then $\tilde{A}_1 \in M_2(p\mathbb{Z})$ (resp. $\tilde{A}_3 \in M_2(p\mathbb{Z})$) and line (resp. column) 1 or 2 of $\tilde{A}_2$ will take values in $p\mathbb{Z}$.

In order to reach our purpose (i.e. that $\tilde{A}_i \in M_2(p\mathbb{Z})$ for all $i \in \{1,2,3\}$) we will make a suitable base change for $\Gamma$. Another base of $\Gamma$, $(\gamma_i')_{i=\overline{1,4}}$, is related to the previous one by a matrix $M \in M_4(\mathbb{Z})$ with det $M = \pm 1$:

$$\gamma_i' = \sum_j m_{ji} \gamma_j$$

giving the corresponding period matrix

$$\Pi' = \Pi M$$

hence

$$\begin{pmatrix} A_1' & A_2' \\ -{}^tA_2' & A_3' \end{pmatrix} = \mathrm{Im}({}^t\Pi' A \bar{\Pi}') =$$

$$= \mathrm{Im}({}^tM\,{}^t\Pi A \bar{\Pi} M) = {}^tM \begin{pmatrix} A_1 & A_2 \\ -{}^tA_2 & A_3 \end{pmatrix} M.$$

Writing $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a,b,c,d \in M_2(\mathbb{Z})$ we get

$$A_1' = {}^taA_1a - {}^tc\,{}^tA_2a + {}^taA_2e + {}^tcA_3c$$

$$A_2' = {}^taA_1b - {}^tc\,{}^tA_2b + {}^taA_2d + {}^tcA_3d$$

$$A_3' = {}^tbA_1b - {}^td\,{}^tA_2b + {}^tbA_2d + {}^tdA_3d$$

From now on we reduce all computations modulo $p$, all equalities taking place in $\mathbb{Z}_p$.

By assumption $\det A_2 - \tau\theta = 0$.

We distinguish two cases:

I) $\det A_2 \neq 0$

II) $\det A_2 = 0$.

i) $\det A_2 \neq 0$

Then $\theta \neq 0$, $\tau \neq 0$.

It will be enough, considered the form of $\widetilde{A}$, to find $M$ giving one null line (say the first) of $A_2'$ and $A_3' = 0$.

Choose $c=0$, $a=d=I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then

(2) $$A_2' = A_1b + A_2$$

(3) $$A_3' = {}^tbA_1b - {}^tA_2b + {}^tbA_2 + A_3$$

$$A_1b = \theta \begin{pmatrix} b_3 & b_4 \\ -b_1 & -b_2 \end{pmatrix}$$

From (2) and the requirement that the first line of $A_2'$ should be null we find:

(4) $\quad b_3=-\theta^{-1}\alpha \quad , b_4=-\theta^{-1}\beta$ .

If $A_3' = \begin{pmatrix} 0 & \tau' \\ -\tau' & 0 \end{pmatrix}$, (3), (4) and the hypothesis imply

$$\tau' = \theta(b_1b_4-b_2b_3)-\alpha b_2-\gamma b_4+\beta b_1+\delta b_3+\tau = 0$$

Hence $b_1, b_2$ can be arbitrarily chosen.

ii) det $A_2=0$

Then $\theta=0$ or $\tau = 0$.

Assume $\theta=0$ (the case $\tau = 0$ is similar)

Choose $b=c=0$, $a=1$.

Then $A_1'=A_1=0$ and $A_2'=A_2d$.

It is enough to find $d$ such that $A_2'$ has a null column, the first for instance. If

$$d = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} \quad \text{this comes to}$$

$$\begin{cases} \alpha d_1 + \beta d_3 = 0 \\ \gamma d_1 + \delta d_3 = 0 \end{cases}$$

Since det $A_2=0$ this system admits nontrivial solutions $(d_1, d_3)$. Moreover, one can find a solution with one coordinate equal to 1, say $d_1=1$. Then we choose $d_2=0$ and $d_4=1$ and get

$$d = \begin{pmatrix} 1 & 0 \\ d_3 & 1 \end{pmatrix} .$$

Returning to integer values we can choose in both cases (i and ii) representants $M \in M_4(\mathbb{Z})$ with det $M=\pm 1$ for the classes in $M_4(\mathbb{Z}_p)$ found above. One can take, for example all integer representants from $\{0,1,\ldots,p-1\}$. The Lemma is proved.

Now we prove the proposition using induction on the number n of prime factors of r=rang E:

$$r = \prod_{i=1}^{k} p_i^{n_i}, \quad n = \sum_{i=1}^{k} n_i .$$

For n=0 we have r=1 and the statement is true since $c_1(E) \in NS(X)$ by assumption.

Assume now the proposition is true for n. We shall prove it for n+1. Let p be a prime factor of r.

$$0 = \Delta(E) = \frac{1}{r}\left(c_2(E) - \frac{r-1}{2r} c_1(E)^2\right) \quad \text{implies}$$

$p \mid \frac{1}{2} c_1(E)^2$ (the intersection form on NS(X) is even). By the Lemma there exists some unramified covering of degree p $q: X' \to X$ and $a' \in NS(X')$ such that $pa' = q^*(c_1(E))$. X' is again a torus. Consider on X' the topological vector bundle F having rank(F)= $= \frac{r}{p}$, $c_1(F) = a'$, $c_2(F) = \frac{\frac{r}{p} - 1}{2\frac{r}{p}} a'^2 = \frac{\frac{r}{p} - 1}{2r} c_1(E)^2 \in \mathbb{Z}$.

($q^*: H^4(X, Z) \cong Z \to H^4(X', Z) \cong Z$ is the multiplication by p).

Then $\Delta(F) = 0$ and F admits holomorphic structures by the induction hypothesis.

Let $G = \{1, \tau, \ldots, \tau^{p-1}\}$ be the deck-transformation group of X'/X (these are translations of X') and $E' = F \oplus \tau^*(F) \oplus \ldots \oplus (\tau^{p-1})^*(F)$. Then $c_1(E') = pc_1(F) = pa' = q^*(c_1(E))$ ($c_i(\tau^*(F)) = c_i(F)$ since $\tau$ is homotopous to 1), $c_2(E') = \frac{p(p-1)}{2} c_1(F)^2 + pc_2(F) = p^2 \frac{r-1}{r} a'^2$, $\Delta(E') = 0$. One has canonical isomorphisms $E' \to (\tau^m)^*(E')$ compatible with the action of G on X' hence E' induces a holomorphic vector bundle E'' on X such that $q^*E'' = E'$. It follows that $\Delta(E'') = 0$ and $c_1(E'') = c_1(E)$, hence the underlying topological vector bundle of E'' is E, which closes the proof of the proposition.

## 2. Corollaries and Remarks

Let X be a compact complex surface, $a \in NS(X)$ and r a positive integer. We make the following notations:

$$s(r,a) := -\frac{1}{2} \sup_{\mu \in NS(X)} \left(\frac{a}{r} - \mu\right)^2$$

$$t(r,a) := \inf\left\{ \frac{1}{k(r-k)} s(r,ka) \,\middle|\, k=\overline{1,r-1} \right\}$$

When X is nonalgebraic these numbers are non-negative.

<u>Remark 1.</u> For X nonalgebraic and E a filtrable bundle of rank r on it $\Delta(E) \geqslant s(r,c_1(E))$ (see [B-L] § 2 for the proof).

<u>Remark 2.</u> For X nonalgebraic and E a reducible bundle of rank r on it

$$\Delta(E) \geqslant t(r,c_1(E)).$$

<u>Proof.</u>

Let $0 \to E_1 \to E \to E_2 \to 0$ be an exact sequence with $E_i$ coherent sheaves without torsion of ranks $r_i$ and having $c_1(E_i)=a_i$ $(i=1,2)$. Let $a=c_1(E)$. Then $a_1+a_2=a$, $r_1+r_2=r$ and by Riemann-Roch's formula for E, $E_1$ and $E_2$ we find

$$\Delta(E) = \frac{1}{2r}\left(\frac{a^2}{r} - \frac{a_1^2}{r_1} - \frac{a_2^2}{r_2}\right) + \frac{r_1}{r}\Delta(E_1) + \frac{r_2}{r}\Delta(E_2).$$

Since $\Delta(E_i) \geqslant 0$ (see [B-L]) we have

$$\Delta(E) \geqslant \frac{1}{2r}\left(\frac{a^2}{r} - \frac{a_1^2}{r_1} - \frac{a_2^2}{r_2}\right) = -\frac{1}{2r_1r_2}\left(\frac{r_2a}{r} - a_2\right)^2$$

$$\geqslant \frac{1}{r_2(r-r_2)} s(r,r_2a) \geqslant t(r,c_1(E)).$$

Corollary 1

Let X be a non-algebraic 2-torus, r a positive integer, $a \in NS(X)$ such that $r \mid \frac{1}{2} a^2$ and $r^2 \nmid \frac{1}{2} a^2$. Then there exists a topological vector bundle E on X having rank r, $c_1(E)=a$ and $\Delta(E)=0$ admitting holomorphic structures but not filtrable holomorphic structures.

Proof.

We choose the topological vector bundle E having $c_1(E)=a$, $c_2(E)=\frac{r-1}{2r} a^2$. Hence $\Delta(E)=0$ and E admits holomorphic structures by the proposition. Using Remark 1 it will be enough to prove that

$s(r,a)>0$. If this were not so we'd have

$s(r,a) = 0$ i.e.

$$\sup_{\mu \in NS(X)} \left(\frac{a}{r} - \mu\right)^2 = 0, \text{ hence } \left(\frac{a}{r} - \mu\right)^2 = 0$$

for some $\mu$ in NS(X). This implies $a=r\mu+c$ with $c \in NS(X)$ and $c^2=0$. Then c is orthogonal on NS(X) since X is nonalgebraic (examine $(nc+x)^2$ for $n \in \mathbb{Z}$!) . It follows that $a^2=r^2\mu^2$ and $2r^2 \mid a^2$, a contradiction !

Corollary 2

If X is a complex 2-torus and r a positive integer such that NS(X) is cyclic generated by a with $a^2=-2r$, then the topological vector bundle E on X of rank r having $c_1(E)=a$ and $\Delta(E)=0$ admits holomorphic structures but not reducible structures.

Proof. In this case $t(r,a)>0$.

The hypotheses of the Corollaries can be actually fulfilled as is shown by

**Remark 3**

For every positive integer $n$ there exist nonalgebraic 2-tori $X$ having $NS(X)$ cyclic generated by $a$ with $a^2=-2n$.

**Proof**

Let $\Pi=(\Pi_1,\Pi_2)$ with $\Pi_i\in M_2(\mathbb{C})$ $i=1,2$, $\Pi_1=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, $\Pi_2=iP$, $P\in M_2(\mathbb{R})$, $\det P=\frac{1}{n}$, $P=\begin{pmatrix}p & q\\ r & s\end{pmatrix}$ and $\mathrm{rank}_{\mathbb{Q}}(p,q,r,s)=4$.

If $X$ is the 2-torus having the period matrix $\Pi$, $NS(X)$ will be generated by an element $a$ having the associated hermitian matrix

$$A=\begin{pmatrix}0 & in\\ -in & 0\end{pmatrix} \qquad \text{(see [E-F] Appendix).}$$

Hence $a^2=2\det P.\det A=-2n$ (see [B-F] Lemma 2.1).

Moreover $a(X)=0$ since otherwise we would have nonzero elements $b\in NS(X)$ with $b^2\geq 0$.

**Remark 4.** The vector bundles $E$ constructed in the proof of the proposition are of the form $E=\pi_* L$ where $\pi: X'\longrightarrow X$ is an unramified covering of degree $r$ and $L\in Pic(X')$.

In these terms we get the following criterion of reducibility:

**Remark 5.** Let $X$ be a complex 2-torus with $a(X)=0$, $X'\xrightarrow{\pi}X$ a covering of degree $r$, $G$ the covering transformation group, $L\in Pic(X')$, $G'=\{g\in G \mid g^*L\cong L\}$ and $E=\pi_* L$. Then $E$ is reducible if and only if $G'\neq id$. In this case $E=F^{\oplus|G'|}$ with $F$ irreducible. In particular if $r$ is prime:

$$E \text{ reducible} \Leftrightarrow E \text{ filtrable}$$

Proof

If $G' \neq \{id\}$ one considers $\pi' : X' \to X'/G'$, $\pi'' : X'/G' \to X$ and one gets

$$\pi_* L = L'^{\oplus |G'|} \quad \text{where } L' \in Pic(X'/G') \text{ is such that } \pi'^*(L') = L.$$

Hence $\pi_* L = \pi''_* \pi'_* L = (\pi''_* L')^{\oplus G'}$ is reducible.

Conversely, let $G' = \{id\}$ and assume $E$ is reducible. Then take a quotient $C$ of $E$ such that $0 < \text{rank } C < r$ and the exact sequences

$$0 \to K \to E \xrightarrow{p} C \to 0$$

$$0 \to \pi^* K \to \pi^* E \to \pi^* C \to 0$$

$$\pi^* E = \bigoplus_{g \in G} g^* L. \text{ Let } i_g : g^* L \to \pi^* E$$

denote the canonical inclusion.

Then $\pi^* p \circ i_g$ are simultaneously (for $g \in G$) zero or nonzero. If they are zero they induce nontrivial morphisms in $\pi^* K$. In either case, there exists a coherent sheaf without torsion $A$ with $0 < \text{rank } A < r$ and nontrivial morphisms

$$u_g : g^* L \to A, \text{ for all } g \in G.$$

Since $a(X) = 0$, $u_g$ are monomorphisms. Hence $g^* L$ may be seen as subsheaves of $A$. Take $H := \bigoplus_{\substack{g \in G \\ g \neq id}} g^* L$, $v := \sum_{g \neq id} u_g : H \to A$

$$u := \sum_{g \neq id} u_g : \pi^* E \to A, \; u_{id} : L \to A$$

$B := \text{Ker } u \hookrightarrow \pi^* E$, and $p_1 : \pi^* E \to L$, $p_2 : \pi^* E \to H$ the projections.

Since rank $B > 0$ one can assume that $p_1(B) \neq 0$. We have in $A$ the inclusion of subsheaves. $u_{id} \circ p_1(B) \subset v \circ p_2(B)$ inducing a nontrivial morphism $L \to H$ hence a nontrivial morphism $L \to g^* L$ for some $g \neq id$. This is a contradiction.

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